HSMD – Cosmological dynamics of the meshed hypersphere

Cosmological extension of HSM: theory, calibration, and derived outputs

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Contents

Chapter 1

General introduction

1.1 From HSM to HSMD

HSM [1] proposes a mechanical reading of fundamental physics. Its starting point is a compact meshed hypersphere $\mathbb{S}^3$, whose nonlinear internal dynamics can carry localized solitonic-type solutions, massless modes, effective fields, and composite structures.

In this reading, particles are not introduced as elementary point-like objects postulated independently of the support. They are associated with families of localized solutions of the mesh, or with composite structures made of such solutions. Effective fields, masses, charges, and certain structure constants are then read as emergent properties of these solutions and of their closures.

This construction naturally leaves open a global question. If particles and fields are carried by a meshed hypersphere, then the support itself cannot be treated as a passive background. It must have its own dynamics: current radius, elastic response, closure branch, global rotation, and stabilization conditions.

HSMD is this second step. Its objective is not to replace HSM by an independent theory, but to extend HSM toward the global dynamics of the support. The central question then becomes:

> How can the dynamics of a compact meshed hypersphere be read as an effective cosmological dynamics?

1.2 Guiding physical idea

The guiding idea of HSMD is that late-time cosmological expansion can be reformulated as the effective trajectory of the radius $R(t)$ of the meshed hypersphere. The cosmological scale factor is therefore not introduced as an abstract kinematics independent of the support: it is attached to the global evolution of the HSM mesh.

This global evolution is not only radial. A compact hypersphere can also carry a global rotation, closure constraints, and directional effects related to the position of the observer in the geometry. The HSMD program therefore consists in studying simultaneously:

- the late-time homogeneous branch of $R(t)$;
- the global elastic response of the support ;
- the rotational closure ;
- the observational signatures that may result from it.

The radius R_0 and the mesh ℓ_0 used in HSMD are quantities of the present state. They are determined in HSM by the calibration of the fundamental constants, and must not be confused with a natural radius or with a rest state of the structure.

1.3 Soliton genesis and scope of the manuscript

If particles correspond, in HSM, to localized solitonic-type solutions or to composite structures made of such solutions, the question of their appearance becomes a dynamical question: under what conditions does an excitation of the mesh become a stable localized solution?

This question belongs to soliton genesis. It concerns the initial relaxation or quench regime, the formation of the first vibration antinodes, their possible migration toward mass solutions, and then the progressive stabilization of the effective branch coefficients.

The present manuscript does not consume a complete description of this initial regime. It is placed after this phase, in a stabilized homogeneous branch where the relevant global coefficients can be treated as constant at the order under study. Soliton genesis therefore fixes the physical context of HSMD, but the quantitative derivations of the manuscript concern the stabilized late-time dynamics.

1.4 What HSMD seeks to obtain

HSMD seeks to obtain three types of results.

The first is a late-time cosmological branch calibrated on background observables. BAO distances are used to constrain the effective trajectory of the radius without imposing, a priori, an external cosmological calibration. Once this branch has been fixed, the model produces cosmographic outputs: the current acceleration of expansion, the transition redshift toward this accelerated phase, the effective age, the trajectory of $R(t)$, and the structure of the stabilized dynamical core.

The second is a rotational closure. In HSM, the PMNS mixing angles belong to the real internal geometry of neutrino solutions. HSMD adds the possibility that the global rotation of the hypersphere introduces an effective orientation related to our observation domain. The Dirac phase, directional projections, and some dipole-type signatures are then treated as possible readings of this global closure, and not as isolated parameters.

The third is a set of predictions or observational tests: out-of-calibration distances, directional signatures, expected alignments, relations between independent channels, and future tests of cosmological drift. The value of HSMD therefore does not rely on a single numerical adjustment, but on the possibility of relating several families of observables to one and the same geometrical dynamics.

In this reading, the difference between the mean expansion scale inferred from background observables and the apparent local value of the Hubble constant is not introduced as a new free constant. HSMD interprets it as the apparent effect of a local rotational correction, related to the effective angular position of our observation domain in the rotating hypersphere. This reading makes it possible to relate the H_0 tension to the global geometry of the model, while keeping a mean branch calibrated on background observables.

The same rotational closure also leads to comparable directional outputs: a CMB birefringence predicted at the level of a few tenths of a degree, and an HSC/spin signature confronted with the raw slope measured in the active window where the spin axis is detected. These concordances remain flexible and are not presented as isolated proofs; their interest is to test one and the same rotational geometry on several distinct observational channels.

1.5 Method: theory, calibration, predictions

The manuscript strictly distinguishes three levels.

The first level is theoretical. It defines the global dynamics of the meshed hypersphere, the branch quantities, the closure terms, the stabilized regime under study, and the effects of rotation. At this level, no observational value is used as a proof argument.

The second level is calibration. Some observables are used to fix the branch constants needed for comparison: BAO distances, closure parameters, and rotational orientation when it is required. These inputs are explicitly distinguished from the outputs.

The third level is predictive. Once the branch has been calibrated, HSMD compares the outputs of the model with observables that have not been used to fix the corresponding parameters. Each result is then presented with its status: calibration, comparable output, directional test, deferred prediction, or sector left open.

1.6 Scope and falsifiability

In this manuscript, HSMD does not claim to provide a complete history of all regimes of the Universe. It treats the stabilized late-time homogeneous branch of HSM dynamics, and explicitly identifies the sectors that belong to a later step: complete soliton genesis, nonlinear post-quench regime, detailed formation of the mass families, and global stability away from the order under study.

The model is falsifiable because it produces quantitative and geometrical relations between several independent channels: BAO distances, cosmographic parameters, rotational orientation, PMNS Dirac phase, directional signatures, and future tests. A robust incompatibility between these outputs and observations would directly contradict the corresponding closures.

The objective of HSMD is therefore to extend HSM without changing its logic: to start from an explicit mechanical framework, derive equations, identify solutions or branches, close the required constants, and then compare numerical outputs with observations.

Part I

HSMD theory: global dynamics

Chapter 2

HSMD theoretical framework

2.1 Physical object studied: compressed branch, quench, and stabilized closure

HSMD theory studies a homogeneous branch of HSM in which the global radius of the meshed hypersphere becomes a dynamical degree of freedom. The mechanical object considered is a meshed hypersphere arising from a compressed state, then released by a global quench.

Here the quench denotes the effective breakdown of the initial compression equilibrium. After this release, the global radius degree of freedom evolves under the effect of the homogeneous mechanics of the mesh and of the global charges carried by the branch.

The theoretical scope retained here is not the complete post-quench formation phase. The dynamical chain

$$\begin{gathered}\text{quench} \longrightarrow \text{redistribution of the sectors } A_i(a) \longrightarrow \text{soliton genesis}\\ \longrightarrow \text{stabilized homogeneous branch}\end{gathered}$$

is treated as a branch-existence hypothesis. The present chapter takes as its object the effective regime reached once this stabilized branch has already formed.

Among the branch charges, a global rotational component is allowed. This rotation is not created by a perfectly isotropic decompression: it characterizes the stabilized branch studied here. Decompression then modifies its dynamical effect through conservation of the global rotational charge.

Stabilized closure means that, in the domain studied, the homogeneous projections of the different sectors are described by constant coefficients. The earlier phase, in which these coefficients may depend on the internal factor a, belongs to the distinct problem of soliton genesis and is not consumed in the predictions of the present manuscript.

The chain studied in this theoretical part is therefore

$$\begin{gathered}\text{compressed HSM branch} \longrightarrow \text{stabilized-branch hypothesis}\\ \longrightarrow \text{homogeneous dynamics with constant coefficients} \longrightarrow \text{effective branch sectors.}\end{gathered}$$

This section fixes the mechanical object studied, the internal variables, and the stabilized-closure hypothesis.

Definition 2.1 (Internal scale factor)**.** The homogeneous dynamics of HSMD is described by the internal scale factor $a(t)$, defined by

$$R(t) = a(t)R_0. \tag{2.1}$$

The radius R_0 is the reference radius imported from HSM. The condition $a = 1$ only means $R(t) = R_0$. It defines neither an external matching time, nor a release point, nor a dynamical equilibrium of the HSMD branch.

Definition 2.2 (Internal expansion rate)**.** The internal expansion rate is

$$\mathcal{H}(t) := \frac{\dot{a}(t)}{a(t)}. \tag{2.2}$$

Definition 2.3 (Branch release point)**.** The internal factor $a_{\mathrm{rel}} > 0$ denotes the state for which the linear compression of the mesh vanishes in the branch considered. This point is a branch condition of the homogeneous dynamics. It is not identified with 1 without a derivation or an explicit gauge choice.

Hypothesis 2.4 (Stabilized homogeneous branch arising from a quench)**.** The HSMD regime studied in this chapter is a stabilized homogeneous branch of HSM. This branch is defined by the following four regime conditions.

Initial compression. The branch arises from an initial state compressed relative to the release point a_{rel}:

$$0 < a_{\mathrm{init}} < a_{\mathrm{rel}}. \tag{2.3}$$

Triggering by quench. At an internal time t_q, the compressed state ceases to be a locked state and the homogeneous mode $a(t)$ becomes dynamical. The quench fixes an initial branch condition,

$$a(t_q) = a_{\mathrm{init}}, \qquad \dot{a}(t_q) > 0. \tag{2.4}$$

Global rotational charge. The homogeneous branch may carry a conserved global rotational charge, characterized by an internal axis $\widehat{\mathbf{N}}_{\mathrm{rot}}$ and by a global angular velocity Ω_{HS}. This charge is treated as branch data in the homogeneous reduction. Its dynamical effect then evolves with the radius through conservation of the global rotational charge.

Stabilized closure. The homogeneous reduction does not describe the entire post-quench redistribution phase. It describes the regime in which the effective coefficients of the homogeneous dynamics have reached a stabilized closure. If the full form arising from the quench is written with coefficients depending on the internal factor,

$$A_i = A_i(a),$$

the regime studied is the one in which

$$A_i(a) \longrightarrow A_i^{\mathrm{stab}} \tag{2.5}$$

over the branch domain considered.

2.2 Homogeneous mechanical dynamics of the meshed hypersphere

The common kinematic scale imported from HSM is

$$H_\star := H_R := \frac{c_t}{R_0}. \tag{2.6}$$

The definitions of Section 2.1 make it possible to write the homogeneous mechanics of the global radius mode.

The natural relaxation radius of the branch is the radius associated with the point a_{rel} defined in Definition 2.3. We denote

$$R_{\mathrm{nat}} = a_{\mathrm{nat}} R_0, \qquad a_{\mathrm{nat}} := a_{\mathrm{rel}}. \tag{2.7}$$

The reduced variable of the homogeneous mode is then

$$x(t) := \frac{R(t)}{R_{\text{nat}}} = \frac{a(t)}{a_{\text{nat}}}. \tag{2.8}$$

The branch is compressed when $x < 1$, that is, when $R < R_{\text{nat}}$.

The total mesh is assumed to be conserved: when R increases, no new mesh is created. The effective volume density of the mesh therefore decreases with the volume. The global inertia of the homogeneous mode is written on the transported material configuration, and not on a current volume that creates mesh.

The minimal homogeneous energy relation is then

$$\frac{1}{2} I_R \dot{x}^2 + U_{\text{el}}(x) + U_{\text{rot}}(x) + U_{\text{vib}}(x) = E_{\text{tot}}. \tag{2.9}$$

Here I_R is the generalized inertia of the homogeneous radial mode, U_{el} the global elastic compression–relaxation energy, U_{rot} the global rotational energy, and U_{vib} the internal vibrational energy of the mesh.

At minimal order, the elastic energy is

$$U_{\text{el}}(x) = \frac{1}{2} K_{\text{el}} (x-1)^2, \tag{2.10}$$

where K_{el} collects the radial and tangential contributions of the mesh on the reference configuration. Its generalized force is

$$-\frac{dU_{\text{el}}}{dx} = -K_{\text{el}}(x-1).$$

Thus, for $x < 1$, the elastic contribution indeed pushes toward expansion.

The fundamental rotation is not assumed to satisfy $\Omega R =$ constant. In the absence of a specific locking mechanism, the conserved mechanical quantity is the global angular momentum

$$L_{\text{rot}} = I_{\text{rot}}(R)\Omega.$$

For a conserved material mesh, $I_{\text{rot}} \propto R^2$. We therefore write

$$I_{\text{rot}}(x) = I_{\text{rot,nat}} x^2,$$

and

$$U_{\text{rot}}(x) = \frac{L_{\text{rot}}^2}{2I_{\text{rot}}(x)} = U_{\text{rot,nat}} x^{-2}. \tag{2.11}$$

The homogeneous differential equation associated with (2.9) is

$$I_R \ddot{x} = -\frac{d}{dx} \left(U_{\text{el}} + U_{\text{rot}} + U_{\text{vib}} \right). \tag{2.12}$$

By making the two direct mechanical contributions explicit,

$$\ddot{x} = -\omega_{\text{el}}^2 (x-1) + 2\omega_{\text{rot}}^2 x^{-3} - \frac{1}{I_R} \frac{dU_{\text{vib}}}{dx}, \tag{2.13}$$

with

$$\omega_{\text{el}}^2 := \frac{K_{\text{el}}}{I_R}, \qquad \omega_{\text{rot}}^2 := \frac{U_{\text{rot,nat}}}{I_R}.$$

Elastic decompression and rotation at conserved angular momentum are therefore the two direct mechanical drivers of the global expansion. The term U_{vib} describes the internal vibrational content; it is not introduced as an expansion driver until a specific mechanism is derived.

By dividing (2.9) by x^2, one obtains

$$\mathcal{H}^2 = \left(\frac{\dot{x}}{x}\right)^2.$$

The first-integral relation takes the form

$$\mathcal{H}^2(x) = \frac{2}{I_R x^2}\left[E_{\rm tot} - U_{\rm el}(x) - U_{\rm rot}(x) - U_{\rm vib}(x)\right]. \tag{2.14}$$

In the variable $a = a_{\rm nat}x$, the explicit mechanical contributions generate the sectors

$$a^{-4}, \qquad a^{-2}, \qquad a^{-1}, \qquad a^{0}.$$

The localized solitonic projection provides the a^{-3} sector.

The full branch form produced by the quench can be written with effective coefficients depending on the internal factor a,

$$A_i = A_i(a),$$

but the dynamics of this redistribution phase lies outside the scope of the stabilized homogeneous reduction. We therefore apply the stabilized closure (2.5). In this regime, the coefficients are used as constants.

The admissible stabilized homogeneous dynamical function therefore takes the form

$$\boxed{\mathcal{F}_{\rm HS}^{\rm stab}(a) = A_4^{\rm stab}a^{-4} + A_3^{\rm stab}a^{-3} + A_2^{\rm stab}a^{-2} + A_1^{\rm stab}a^{-1} + A_0^{\rm stab}.} \tag{2.15}$$

The coefficient A_1 is the trace of the elastic restoring toward $R_{\rm nat}$. Using (2.10) in (2.14), the elastic contribution to the homogeneous rate is

$$\frac{\Delta H_{\rm el}^2}{H_R^2} = -\frac{\omega_{\rm el}^2}{H_R^2} + 2\frac{\omega_{\rm el}^2}{H_R^2}a_{\rm nat}a^{-1} - \frac{\omega_{\rm el}^2}{H_R^2}a_{\rm nat}^2 a^{-2}. \tag{2.16}$$

We therefore introduce the reduced coefficient of the homogeneous elastic mode

$$\mathcal{B}_{\rm el} := \frac{\omega_{\rm el}^2}{H_R^2} = \frac{\omega_{\rm el}^2 R_0^2}{c_t^2}. \tag{2.17}$$

The a^{-1} sector is then

$$A_1 = 2\,\mathcal{B}_{\rm el}\,a_{\rm nat}. \tag{2.18}$$

The elastic contributions of ranks a^{-2} and a^0 are simultaneously

$$A_{2,\rm el} = -\mathcal{B}_{\rm el}a_{\rm nat}^2, \qquad A_{0,\rm el} = -\mathcal{B}_{\rm el}. \tag{2.19}$$

The same mode can be referred to the natural radius by the coefficient

$$\beta_{\rm el}^{\rm nat} := \frac{\omega_{\rm el}^2 R_{\rm nat}^2}{c_t^2} = \mathcal{B}_{\rm el}a_{\rm nat}^2. \tag{2.20}$$

In this convention,

$$A_1 = \frac{2\beta_{\rm el}^{\rm nat}}{a_{\rm nat}}. \tag{2.21}$$

The two writings are equivalent; the quantity directly reduced on the HSM reference radius is $\mathcal{B}_{\rm el}$.

The coefficient $A_4^{\rm stab}$ represents the stabilized homogeneous sector of rank a^{-4}. Such a rank can arise from a rotational sector at conserved charge, from radiative content, or from a fast vibrational sector, depending on the reduction considered. Before stabilization, this sector could be dynamical and be written more generally as $A_4(a)$. The constant A_4^{stab} of (2.15) denotes only the effective coefficient after stabilized homogeneous closure.

The coefficient A_3^{stab} is the stabilized homogeneous projection of the localized solitonic content in the branch regime studied. Its existence in the homogeneous law reflects the assumption that a stabilized fraction of the post-quench excitation was converted into localized solitonic content before the effective regime described by HSMD. The present chapter does not derive this fraction from soliton-genesis dynamics; it treats it as an effective coefficient of the stabilized branch, used only in this status.

The coefficient A_2^{stab} collects the homogeneous contributions of rank a^{-2}, including the mechanical energy constant, the a^{-2} part of elasticity, and the geometric residuals of the stabilized branch. The coefficient A_0^{stab} is the constant branch-closure sector.

HSM closure of the natural elastic coefficient. The value of $\beta_{\mathrm{el}}^{\mathrm{nat}}$ is determined by the ratio between the homogeneous scalar count of material bonds and the tangential return that fixes the speed c_t in HSM. Let N be the number of material nodes of the homogeneous branch, z the rank 1 coordination, m_{n} the elementary inertial mass associated with a node, k_{e} the axial stiffness of a material bond, and ℓ_{nat} the natural length of a bond when $R = R_{\mathrm{nat}}$.

For a homogeneous dilation $x = R/R_{\mathrm{nat}}$, each material bond satisfies

$$\ell_{ij}(x) = x\ell_{\mathrm{nat}}, \qquad \delta\ell_{ij} = \ell_{\mathrm{nat}}(x-1). \tag{2.22}$$

The number of unoriented bonds is $Nz/2$. The homogeneous elastic energy is therefore

$$U_{\mathrm{el}}(x) = \frac{Nz}{2}\,\frac{1}{2}k_{\mathrm{e}}\ell_{\mathrm{nat}}^2(x-1)^2 = \frac{1}{2}K_{\mathrm{el}}(x-1)^2, \tag{2.23}$$

whence

$$K_{\mathrm{el}} = N\frac{z}{2}\,k_{\mathrm{e}}\ell_{\mathrm{nat}}^2. \tag{2.24}$$

The inertia of the same homogeneous mode, written for the dimensionless variable x, is

$$\frac{1}{2}I_R\dot{x}^2 = \frac{1}{2}\sum_{i=1}^{N} m_{\mathrm{n}}\,(R_{\mathrm{nat}}\dot{x})^2, \qquad I_R = Nm_{\mathrm{n}}R_{\mathrm{nat}}^2. \tag{2.25}$$

It follows that

$$\omega_{\mathrm{el}}^2 = \frac{K_{\mathrm{el}}}{I_R} = \frac{z}{2}\,\frac{k_{\mathrm{e}}}{m_{\mathrm{n}}}\,\frac{\ell_{\mathrm{nat}}^2}{R_{\mathrm{nat}}^2}. \tag{2.26}$$

The HSM tangential closure of the discrete–continuum passage gives, with the return factors $\Xi_{\tan}$ and Ξ_0 defined in HSM as geometric return factors [1, eq. `eq:Xi_defs`], and with the tangential speed defined by [1, eq. `eq:ct2_T_over_rho`],

$$c_t^2 = \frac{k_{\mathrm{e}}}{m_{\mathrm{n}}}\ell_{\mathrm{nat}}^2\Xi_{\tan}, \qquad \Xi_0 = \frac{z}{2}. \tag{2.27}$$

Combining (2.20), (2.26) and (2.27),

$$\beta_{\mathrm{el}}^{\mathrm{nat}} = \frac{\omega_{\mathrm{el}}^2 R_{\mathrm{nat}}^2}{c_t^2} = \frac{\Xi_0}{\Xi_{\tan}}. \tag{2.28}$$

The dependence on the total number of material nodes and on the local bond scale cancels. The coefficient depends only on the local geometric factor that relates the homogeneous count to the tangential channel.

The HSM local isotropic closure of the discrete–continuum passage fixes the mean orientation tensor of the bonds by

$$\left\langle \hat{t}\otimes\hat{t}\right\rangle = \frac{I_3}{3}. \tag{2.29}$$

This relation is taken over here as a local closure already used in HSM; it does not constitute an additional cosmological assumption on the HSMD global branch. If $P_{\tan}$ is the projector onto the local tangent plane, then $P_{\tan}$ projects onto a two-dimensional plane in the local three-dimensional space. It therefore has two unit eigenvalues and one zero eigenvalue, whence

$$\operatorname{tr} P_{\tan} = 2.$$

It follows that

$$\eta_{\tan} = \operatorname{tr}\left(P_{\tan}\frac{I_3}{3}\right) = \frac{\operatorname{tr} P_{\tan}}{3} = \frac{2}{3}. \tag{2.30}$$

Since $\Xi_{\tan} = \Xi_0\eta_{\tan}$ in the HSM tangential closure of the discrete–continuum passage [1, eq. `eq:Xi_defs`], with $\eta_{\tan}$ derived from (2.29), equation (2.28) gives

$$\boxed{\beta_{\rm el}^{\rm nat} = \frac{1}{\eta_{\tan}} = \frac{3}{2}.} \tag{2.31}$$

Consequently, the natural elastic coefficient provides a dimensionless branch coefficient

$$\mathcal{B}_{\rm el}^{\rm LO} := \frac{\beta_{\rm el}^{\rm nat}}{a_{\rm nat}^2} = \frac{3}{2a_{\rm nat}^2}. \tag{2.32}$$

This writing does not identify the global kinematic scale $H_\star = c_t/R_0$ with an effective Hubble constant. It uses only the fact that $\beta_{\rm el}^{\rm nat}$ is a dimensionless ratio: after projection on the homogeneous branch and normalization of the dynamical kernel, the same ratio fixes the relative elastic weight of the restoring term.

In the minimal LO closure of the homogeneous kernel, the restoring coefficient used by the branch dictionary is therefore

$$C_H^{\rm LO} = \mathcal{B}_{\rm el}^{\rm LO} = \frac{3}{2a_{\rm nat}^2}. \tag{2.33}$$

In this convention,

$$A_1 = 2C_H^{\rm LO}a_{\rm nat} = \frac{2\beta_{\rm el}^{\rm nat}}{a_{\rm nat}} = \frac{3}{a_{\rm nat}}. \tag{2.34}$$

The pure elastic part of the other ranks is then

$$A_{2,\rm el} = -\frac{3}{2}, \qquad A_{0,\rm el} = -\frac{3}{2a_{\rm nat}^2}. \tag{2.35}$$

2.3 Localized solutions and solitonic term

HSM admits localized solutions. When a comoving population of such solutions is produced and stabilized, its mean contribution to a hypersphere of volume

$$V_{\mathbb{S}^3}(a) = 2\pi^2 R_0^3 a^3$$

dilutes as a^{-3}, provided that the comoving number and the proper energy at the rank considered are conserved.

Definition 2.5 (Reduced solitonic coefficient)**.** We denote by $A_{\rm sol}$ the reduced coefficient of projection of the localized solitonic population onto the homogeneous mode. Its contribution to $\mathcal{F}$ is

$$\mathcal{F}_{\rm sol}(a) = A_{\rm sol}a^{-3}. \tag{2.36}$$

The coefficient A_{sol} is related to the comoving population of localized solutions. A general reduced form is

$$A_{\mathrm{sol}} = C_{\mathrm{sol}} \frac{N_{\mathrm{sol}}^{\mathrm{com}} E_{\mathrm{sol}}^{\mathrm{LO}}}{2\pi^2 R_0^3\, \rho_{\mathrm{ref}}}, \tag{2.37}$$

where C_{sol} is the homogeneous projection coefficient, $N_{\mathrm{sol}}^{\mathrm{com}}$ the comoving number of localized solutions, $E_{\mathrm{sol}}^{\mathrm{LO}}$ their proper energy at the rank considered, and ρ_{ref} the reference density used in the reduction.

In the case where this population comes from the capture of envelope instabilities, one may write

$$N_{\mathrm{sol}}^{\mathrm{com}} = V_{\mathrm{com}} \frac{\eta_{\mathrm{cap}}}{C_V \xi_f^3}, \tag{2.38}$$

where η_{cap} is the capture yield, ξ_f the freezing width in the slow coordinate, and C_V a geometric coefficient of coherence volume.

The a^{-3} contribution represents the volume average of the localized solutions.

In a detailed closure where the initial compression is explicitly conserved, the rank a^{-3} receives two contributions: the branch compressive projection and the localized solitonic population. The reduced compressive projection Π_ℓ, used below, is defined later in (2.51). One then defines the detailed coefficient

$$A_{3,\mathrm{det}} := 2\Pi_\ell a_{\mathrm{rel}} + A_{\mathrm{sol}}. \tag{2.39}$$

The notation A_3 is reserved for the coefficient effectively consumed in the stabilized homogeneous dictionary. In the late homogeneous reduction, the compressive projection is absorbed into the branch closure, and one sets

$$A_{3,\mathrm{sol}} = A_{3,\mathrm{det}}^{\mathrm{stab}} \qquad \text{or, in the minimal reduction,} \qquad A_{3,\mathrm{sol}} = A_{\mathrm{sol}}^{\mathrm{stab}}. \tag{2.40}$$

Thus, (2.39) defines the coefficient of the detailed closure, whereas the minimal dictionary (2.88) consumes the stabilized coefficient $A_{3,\mathrm{sol}}$.

2.4 Non-dilutive global closure carried by the HSM radial channel

The contribution in a^{-3} defined in (2.36) is a volume average: it measures the solitonic energy referred to the current volume of the hypersphere. It does not, however, exhaust the mechanical effect of this same population on the homogeneous degree of freedom $R(t)$.

The radial channel consumed here is the HSM gravitational-like channel. In HSM, a localized solution of total energy E_{sol} sources the radial mode by an integrated monopolar charge Q_g. The HSM asymptotic reduction gives, in the local validity window of the gravitational-like channel [1, eq. `eq:grav_Qg_Etot_asymptotic_new`],

$$Q_g \simeq \frac{E_{\mathrm{sol}}}{T\kappa_r R_0}. \tag{2.41}$$

The quantities T, κ_r, and R_0 refer respectively to the HSM definitions of the continuum passage [1, eq. `eq:params_effectifs`], of the radial weight [1, eq. `eq:def_kappa_r`], and of the global separation [1, eq. `eq:R0_from_one_ring_Z_micro_symbolic_COMP`].

Here T is the HSM volumetric tension, κ_r the dimensionless radial coefficient, and R_0 the HSM reference radius. Thus Q_g carries the dimension of an effective area, so that $T\kappa_r Q_g$ carries the dimension of a generalized radial force. HSMD imports this upstream HSM result and applies it to the homogeneous degree of freedom $R(t)$.

For a homogeneous variation of the radius,

$$R(t) = R_0 a(t), \qquad \delta R = R_0 \, \delta a,$$

the HSM radial charge exerts a generalized force on the mode R. With the HSM normalization of the radial channel, this force is

$$F_R^{\mathrm{sol}} = T\kappa_r Q_g. \tag{2.42}$$

By combining (2.41) and (2.42), one obtains

$$R_0 F_R^{\mathrm{sol}} = E_{\mathrm{sol}}. \tag{2.43}$$

Thus, the global radial projection of a localized solution consumes the same total energy as its volume projection, but in a non-dilutive form: the mean density dilutes with the volume, whereas the generalized work $R_0 F_R^{\mathrm{sol}}$ is associated with the total action of the solution on the homogeneous mode.

For a stabilized population of localized solutions, one sums (2.43):

$$R_0 F_R^{\mathrm{sol,tot}} = E_{\mathrm{sol}}^{\mathrm{tot}}. \tag{2.44}$$

One then defines the reduced global-closure coefficient by the same homogeneous projection as the one used in (2.37):

$$B_{\mathrm{close}} := C_{\mathrm{sol}} \frac{R_0 F_R^{\mathrm{sol,tot}}}{2\pi^2 R_0^3 \, \rho_{\mathrm{ref}}}. \tag{2.45}$$

Using (2.44), it follows immediately that

$$\boxed{B_{\mathrm{close}} = A_{\mathrm{sol}}.} \tag{2.46}$$

In the minimal branch where the rank a^{-3} is entirely carried by the stabilized solitonic sector, one writes

$$A_{3,\mathrm{sol}} := A_{\mathrm{sol}}. \tag{2.47}$$

The non-dilutive closure then becomes

$$\boxed{B_{\mathrm{close}} = A_{3,\mathrm{sol}}.} \tag{2.48}$$

This equality does not mean that the a^{-3} term stops diluting. It expresses that the same solitonic population has two distinct projections:

$$A_{3,\mathrm{sol}} a^{-3} \quad \text{for the volume average,} \qquad B_{\mathrm{close}} \quad \text{for the global radial work.}$$

The total constant coefficient of the branch therefore receives the combination

$$A_0^{\mathrm{eff}} = -\mathcal{B}_{\mathrm{el}} + A_{3,\mathrm{sol}}, \tag{2.49}$$

where $-\mathcal{B}_{\mathrm{el}}$ is the constant contribution of the elastic recall defined in (2.19).

2.5 Detailed projection and residual in a^{-2}

This section defines the quantities used in the detailed closure. It does not define the minimal homogeneous reduction: it describes a finer lift, useful when the initial compression and the projection residuals are kept explicitly.

We first introduce the homogeneous reference energy

$$E_{\mathrm{ref}} := 2\pi^2 R_0^3 \, \rho_{\mathrm{ref}}. \tag{2.50}$$

The reduced compressive projection is defined by

$$\Pi_\ell := \frac{E_\ell^{\mathrm{comp}}}{E_{\mathrm{ref}}}, \tag{2.51}$$

where E_ℓ^{comp} denotes the branch compressive contribution kept in the detailed closure. The quantity Π_ℓ is therefore a reduced branch condition; it is not a fundamental HSM constant.

The reduced contribution of the total homogeneous energy at order a^{-2} is denoted

$$\mathcal{E}_a := \frac{E_{\mathrm{hom}}^{\mathrm{red}}}{E_{\mathrm{ref}}}, \tag{2.52}$$

where $E_{\mathrm{hom}}^{\mathrm{red}}$ denotes the part of the homogeneous first integral which survives, after division by a^2, at order a^{-2}.

The explicit compressive projection is the function

$$\begin{aligned}\mathcal{F}_{\mathrm{comp}}(a) = &-\mathcal{B}_{\mathrm{el}} + 2\mathcal{B}_{\mathrm{el}} a_{\mathrm{rel}} a^{-1} - \mathcal{B}_{\mathrm{el}} a_{\mathrm{rel}}^2 a^{-2} \\ &- \Pi_\ell a_{\mathrm{rel}}^2 a^{-4} + 2\Pi_\ell a_{\mathrm{rel}} a^{-3} - \Pi_\ell a^{-2} + A_{\mathrm{sol}} a^{-3} + B_{\mathrm{close}}.\end{aligned} \tag{2.53}$$

This writing makes explicit the contributions that are grouped in the coefficients C_i of the detailed closure (2.161).

The residual K_{res} is defined as the reduced projection of order a^{-2} which is not included in $\mathcal{F}_{\mathrm{comp}}$:

$$K_{\mathrm{res}} := \frac{E_{a^{-2}}^{\mathrm{non\ comp}}}{E_{\mathrm{ref}}}. \tag{2.54}$$

It contains the geometric or matching projections that must not be counted twice in the explicit compressive part.

The total effective coefficient of order a^{-2} in the detailed closure is therefore

$$\boxed{K_{\mathrm{eff}} := \mathcal{E}_a - \mathcal{B}_{\mathrm{el}} a_{\mathrm{rel}}^2 - \Pi_\ell + K_{\mathrm{res}}.} \tag{2.55}$$

The corresponding term is

$$\mathcal{F}_K(a) = K_{\mathrm{eff}} a^{-2}. \tag{2.56}$$

In the stabilized minimal branch, the quantities Π_ℓ, $\mathcal{E}_a$, K_{res}, and $\mathcal{F}_{\mathrm{comp}}$ are not used separately. They are absorbed into the stabilized coefficients A_i^{stab}, in accordance with the dictionary (2.88).

2.6 Global rotation

The global rotation is treated as a secondary channel of the initial relaxation. It is not introduced as the primary cause of expansion.

We denote by $\Omega_{\mathrm{HS}}(a)$ the global angular velocity of the branch around its rotation axis. The associated equatorial tangential velocity is

$$v_{\mathrm{rot}}(a) := \Omega_{\mathrm{HS}}(a) R(a). \tag{2.57}$$

An observer located at a geometric latitude ϑ relative to the axis sees only the local tangential projection of this rotation. At inertial LO,

$$v_{\mathrm{loc}}(\vartheta, a) = v_{\mathrm{rot}}(a) \sin\vartheta = \Omega_{\mathrm{HS}}(a) R(a) \sin\vartheta. \tag{2.58}$$

This relation is only the geometric rule "angular velocity $\times$ transverse distance to the axis", with $r_\perp = R(a)\sin\vartheta$.

The homogeneous term associated with the rotation is not the projection seen by one particular observer; it is the average over the section $\mathbb{S}^3$ of the quadratic inertial energy. With the canonical measure on $\mathbb{S}^3$,

$$d\mu_{\mathbb{S}^3} \propto \sin^2\vartheta\, d\vartheta\, d\Omega_2,$$

one obtains

$$\left\langle \sin^2\vartheta \right\rangle_{\mathbb{S}^3} = \frac{\int_0^\pi \sin^4\vartheta\, d\vartheta}{\int_0^\pi \sin^2\vartheta\, d\vartheta} = \frac{3\pi/8}{\pi/2} = \frac{3}{4}. \tag{2.59}$$

The LO inertial energy carries a factor 1/2, so the homogeneous rotational geometric coefficient is therefore

$$\kappa_{\mathrm{rot}}^{\mathrm{LO}} = \frac{1}{2}\left\langle \sin^2\vartheta \right\rangle_{\mathbb{S}^3} = \frac{3}{8}. \tag{2.60}$$

The mean isotropic contribution at quadratic order in $A_{\mathrm{rot}}^{\mathrm{glob}}$ is then

$$C_{\nu\nu 0}^{\mathrm{rot}} = \kappa_{\mathrm{rot}}^{\mathrm{LO}} \left(A_{\mathrm{rot}}^{\mathrm{glob}}\right)^2. \tag{2.61}$$

The LO qualifier here refers to the truncation at quadratic order in the rotational velocity. At this order, the factor $\kappa_{\mathrm{rot}}^{\mathrm{LO}} = 3/8$ is fixed by the exact geometric average $\langle \sin^2\vartheta \rangle_{\mathbb{S}^3} = 3/4$, and not by a numerical fit.

At the reference radius, the dimensionless global rotational amplitude is

$$A_{\mathrm{rot}}^{\mathrm{glob}} := \frac{\Omega_{\mathrm{HS}} R_0}{c_t}. \tag{2.62}$$

The stationary background term carried by the homogeneous LO rotation therefore reads

$$C_{\nu\nu 0}^{\mathrm{rot}} = \kappa_{\mathrm{rot}}^{\mathrm{LO}} \left(A_{\mathrm{rot}}^{\mathrm{glob}}\right)^2 = \frac{3}{8}\left(A_{\mathrm{rot}}^{\mathrm{glob}}\right)^2. \tag{2.63}$$

The transport law is parametrized by

$$v_{\mathrm{rot}}(a) = v_{\mathrm{rot},\star}\, a^{-s_{\mathrm{rot}}/2}, \qquad 0 \le s_{\mathrm{rot}} \le 2. \tag{2.64}$$

The branch $s_{\mathrm{rot}} = 0$ corresponds to tangentially frozen transport. The branch $s_{\mathrm{rot}} = 2$ corresponds to conservation of angular momentum for an effective rigid distribution.

We then define

$$B_{\mathrm{rot}} := \frac{3}{8}\left(\frac{v_{\mathrm{rot},\star}}{c_t}\right)^2, \tag{2.65}$$

and

$$\mathcal{F}_{\mathrm{rot}}^{\mathrm{iso}}(a) = B_{\mathrm{rot}} a^{-s_{\mathrm{rot}}}. \tag{2.66}$$

The anisotropic part is obtained by subtracting the mean:

$$\delta\mathcal{F}_{\mathrm{rot}}(a,\vartheta) = D_{\mathrm{rot}} a^{-s_{\mathrm{rot}}}\left(\sin^2\vartheta - \frac{3}{4}\right), \tag{2.67}$$

with

$$D_{\mathrm{rot}} := \frac{1}{2}\left(\frac{v_{\mathrm{rot},\star}}{c_t}\right)^2. \tag{2.68}$$

The two amplitudes are linked by

$$D_{\mathrm{rot}} = \frac{4}{3} B_{\mathrm{rot}}. \tag{2.69}$$

2.6.1 Geometric flattening associated with a non-rigid rotation

A non-rigid hypersphere carrying a global rotation does not in general preserve an exactly isotropic geometry. As for an ordinary rotating material sphere, the centrifugal constraint distinguishes equatorial directions from directions close to the rotation axis. The first admissible geometric correction is therefore a quadrupolar correction of zero mean.

At first order, we write

$$R(\vartheta, a) = R(a)\left[1 + \epsilon_{\text{flat}}(a)\, Q_2^{(S^3)}(\cos\vartheta)\right], \tag{2.70}$$

where the zonal quadrupole of zero mean on $\mathbb{S}^3$ is defined by

$$Q_2^{(S^3)}(\cos\vartheta) := 4\cos^2\vartheta - 1. \tag{2.71}$$

Indeed, for the canonical zonal measure of $\mathbb{S}^3$, $d\mu_{S^3} \propto \sin^2\vartheta\, d\vartheta$, one has

$$\left\langle \cos^2\vartheta \right\rangle_{\mathbb{S}^3} = \frac{1}{4}, \qquad \left\langle Q_2^{(S^3)} \right\rangle_{\mathbb{S}^3} = 0. \tag{2.72}$$

The quantity $\epsilon_{\text{flat}}(a)$ measures only the amplitude of the geometric flattening in this zonal quadrupolar channel. It is not a new background coordinate: its angular mean is zero and it therefore does not modify the isotropic branch defined by the mean branch function.

The geometric source of this flattening is the anisotropic part of the rotational term. Using

$$v_{\text{rot}}(a) = \Omega_{\text{HS}}(a)R(a),$$

the characteristic centrifugal constraint is of order

$$\Omega_{\text{HS}}^2(a)R(a) = \frac{v_{\text{rot}}^2(a)}{R(a)}. \tag{2.73}$$

Thus, even in the tangentially frozen branch

$$v_{\text{rot}}(a) = \Omega_{\text{HS}}(a)R(a) = \text{constant},$$

the local constraint that deforms the geometry varies as $1/R(a)$. The constancy of $\Omega_{\text{HS}}R$ therefore fixes the mean rotation term, but it does not impose constancy of the anisotropic response of the geometry.

This separation may be summarized by

$$\epsilon_{\text{flat}}(a) \propto \frac{\Omega_{\text{HS}}^2(a)R(a)}{K_{\text{geom}}(a)} = \frac{v_{\text{rot}}^2(a)}{R(a)K_{\text{geom}}(a)}, \tag{2.74}$$

where $K_{\text{geom}}(a)$ denotes the effective geometric stiffness of the flattening mode. Equation (2.74) is not used here to fix numerically a quantity of the observed Universe: it indicates only that global rotation naturally produces a quadrupolar geometric residual.

This residual is precisely the object that cannot be absorbed into the isotropic projection of the mean branch. It must be carried by the anisotropic closure sector.

2.7 Homogeneous dynamic equation and branch dictionary

Purpose of the section. The homogeneous HSMD dynamics is written as an effective law for the internal scale factor

$$a(t) := \frac{R(t)}{R_0}, \qquad a(t_0) = 1, \tag{2.75}$$

where R_0 is the current radius of the hypersphere. The natural radius of the homogeneous branch is denoted by

$$R_{\rm nat} = a_{\rm nat} R_0. \tag{2.76}$$

The quantity $a_{\rm nat}$ is a global branch condition. It is not to be identified with R_0, which characterizes the current state.

We introduce the reduced variable

$$x(t) := \frac{R(t)}{R_{\rm nat}} = \frac{a(t)}{a_{\rm nat}}. \tag{2.77}$$

The minimal homogeneous sector is described by a first integral

$$\frac{1}{2} I_x \dot{x}^2 + \frac{1}{2} K_{\rm el}(x-1)^2 = E_{\rm tot}, \tag{2.78}$$

where I_x is the generalized inertia of the effective homogeneous mode, $K_{\rm el}$ its restoring stiffness, and $E_{\rm tot}$ the total energy of the homogeneous branch.

This writing does not assume that the mobilized inertia is that of the whole global hypersphere. The homogeneous mode that enters $H(z)$ is the causally consistent effective mode of the observed branch.

2.7.1 Causal closure of the homogeneous restoring term

Since the propagation of mechanical perturbations is bounded by the effective speed c_t, the inertially coherent domain during an expansion time has size

$$L_H \simeq \frac{c_t}{H_0}. \tag{2.79}$$

We parametrize the effective closure of the restoring term by the dimensionless coefficient C_H, which enters the homogeneous branch dictionary. The passage between the natural mechanical coefficient $\mathcal{B}_{\rm el}^{\rm LO}$ and the normalized branch coefficient is written

$$C_H := \mathcal{N}_H \, \mathcal{B}_{\rm el}^{\rm LO}, \qquad \mathcal{B}_{\rm el}^{\rm LO} = \frac{\beta_{\rm el}^{\rm nat}}{a_{\rm nat}^2}. \tag{2.80}$$

The factor $\mathcal{N}_H$ is a branch-normalization coefficient: it collects the causal-domain and homogeneous-geometry corrections that are not developed at LO order. It is not an HSM fundamental constant.

The minimal LO closure used by the branch dictionary chooses the canonical normalization

$$\mathcal{N}_H^{\rm LO} = 1, \qquad C_H^{\rm LO} = \mathcal{B}_{\rm el}^{\rm LO} = \frac{3}{2a_{\rm nat}^2}. \tag{2.81}$$

This closure does not set $H_\star = H_0$. It only fixes the choice of branch normalization in which the natural elastic ratio $\beta_{\rm el}^{\rm nat} = 3/2$ is used without an additional correction at LO order. A beyond-LO derivation will have to replace $\mathcal{N}_H = 1$ by an explicit response of the effective homogeneous domain.

2.7.2 Dynamic normalization and branch coefficients

The first integral (2.78) gives

$$H^2 = \frac{\dot{x}^2}{x^2} = \frac{2}{I_x x^2}\left[E_{\rm tot} - \frac{1}{2} K_{\rm el}(x-1)^2\right]. \tag{2.82}$$

In the normalized homogeneous core, with the branch coefficient C_H defined by (2.80), the elastic contribution gives

$$-C_H + 2C_H a_{\text{nat}} a^{-1} - C_H a_{\text{nat}}^2 a^{-2}. \tag{2.83}$$

The total-energy term contributes at order a^{-2}. The condition $E_{\text{HS}}^2(1) = 1$, applied to the complete equation including the stabilized and fast contents, fixes this total-energy term and avoids introducing an additional independent coefficient.

We denote by $A_{3,\text{sol}}$ the current contribution of the stabilized or solitonic content, and by A_4 the contribution of order a^{-4}, arising from radiative, rotational, or fast vibrational sectors when they are present. We define

$$B_0 := 1 - 2A_{3,\text{sol}} - A_4. \tag{2.84}$$

After inclusion of the non-dilutive closure (2.48), the normalized homogeneous sector then reads

$$E_{\text{hom}}^2(a) = (A_{3,\text{sol}} - C_H) + 2C_H a_{\text{nat}} a^{-1} + [B_0 + C_H(1 - 2a_{\text{nat}})]\, a^{-2}. \tag{2.85}$$

One immediately checks that

$$E_{\text{hom}}^2(1) = 1 - A_{3,\text{sol}} - A_4. \tag{2.86}$$

The complete equation is written in the form

$$E_{\text{HS}}^2(a) = A_4 a^{-4} + A_3 a^{-3} + A_2 a^{-2} + A_1 a^{-1} + A_0. \tag{2.87}$$

The minimal HSMD dictionary is

$$A_0 = A_{3,\text{sol}} - C_H, \tag{2.88}$$
$$A_1 = 2C_H a_{\text{nat}}, \tag{2.89}$$
$$A_2 = 1 - 2A_{3,\text{sol}} - A_4 + C_H(1 - 2a_{\text{nat}}), \tag{2.90}$$
$$A_3 = A_{3,\text{sol}}, \tag{2.91}$$
$$A_4 = A_4. \tag{2.92}$$

In this dictionary, $A_{3,\text{sol}}$ denotes the stabilized coefficient effectively used at order a^{-3}. It can arise from the detailed projection (2.39), or from the minimal reduction $A_{3,\text{sol}} = A_{\text{sol}}^{\text{stab}}$. The dictionary therefore does not use Π_ℓ and A_{sol} separately: it uses their single stabilized projection $A_{3,\text{sol}}$.

Thus,

$$A_0 + A_1 + A_2 + A_3 + A_4 = 1, \tag{2.93}$$

and therefore

$$E_{\text{HS}}^2(1) = 1. \tag{2.94}$$

The coefficients A_i are therefore not free parameters. They are produced by the physical branch variables

$$\Theta_{\text{init}} = (a_{\text{nat}}, C_H, A_{3,\text{sol}}, A_4)\,, \tag{2.95}$$

to which are added the structural constants inherited from HSM, in particular R_0^{HSM}, ℓ_0^{HSM}, ρ, T, c_t, and the effective normalizations associated with localized solutions. These constants are not determined in the HSMD theoretical part; they are taken as structural constants of the HSM base describing the present state.

2.7.3 Expansion law and observables

With

$$a = \frac{1}{1+z}, \tag{2.96}$$

the dynamic law becomes

$$E^2_{\rm HS}(z) = A_4(1+z)^4 + A_3(1+z)^3 + A_2(1+z)^2 + A_1(1+z) + A_0. \tag{2.97}$$

The function $E_{\rm HS}$ is a normalized function:

$$E_{\rm HS}(0) = 1. \tag{2.98}$$

The associated Hubble function is therefore written

$$H_{\rm HS}(z) = H_0 E_{\rm HS}(z). \tag{2.99}$$

This writing does not mean that $H_0 = H_\star$. The scale $H_\star = c_t/R_0$ defined in (2.6) is the internal kinematic scale imported from HSM. The observable rate of the stabilized branch is obtained after normalizing the internal dynamic function:

$$\boxed{H_0 = H_\star \sqrt{\mathcal{F}^{\rm stab}_{\rm br}(a=1)}.} \tag{2.100}$$

Equivalently,

$$E^2_{\rm HS}(a) = \frac{\mathcal{F}^{\rm stab}_{\rm br}(a)}{\mathcal{F}^{\rm stab}_{\rm br}(1)}. \tag{2.101}$$

The dictionary (2.88) already works with the normalized coefficients for which $E^2_{\rm HS}(1) = 1$. The internal scale $H_\star$ therefore remains the branch's own kinematic scale; it is not replaced by an external Hubble constant.

The distances below are reconstructed on the local light cone. They do not describe the whole hypersphere, but only the portion accessible up to the redshift considered. This portion is characterized by the angular coordinate

$$0 \le \chi \le \chi(z),$$

on S^3. For a domain bounded by $z_{\rm max}$, the associated angular size is therefore

$$\chi_{\rm max} := \chi(z_{\rm max}).$$

When $\chi_{\rm max} \ll 1$, the transverse projection is locally close to the flat limit. When $\chi_{\rm max}$ becomes non-negligible, the global closure of S^3 enters explicitly through the $\sin\chi$ factor.

The global curvature of the hypersphere enters the projection of distances, and not as a dynamic coefficient added arbitrarily at order a^{-2}. We define

$$\chi(z) = \frac{c_t}{R_0} \int_0^z \frac{dz'}{H_{\rm HS}(z')}. \tag{2.102}$$

The transverse distance is then

$$D_M(z) = R_0 \sin\chi(z). \tag{2.103}$$

The Hubble distance is

$$D_H(z) = \frac{c_t}{H_{\rm HS}(z)}. \tag{2.104}$$

The isotropic BAO distance is

$$D_V(z) = \left[z D^2_M(z) D_H(z) \right]^{1/3}. \tag{2.105}$$

The luminosity distance and the distance modulus are

$$D_L(z) = (1+z)D_M(z), \tag{2.106}$$

$$\mu(z) = 5\log_{10}\left(\frac{D_L(z)}{10\,\mathrm{pc}}\right). \tag{2.107}$$

The deceleration is given by

$$q(z) = -1 + \frac{1+z}{2E_{\mathrm{HS}}^2(z)}\frac{dE_{\mathrm{HS}}^2}{dz}. \tag{2.108}$$

At the present time, the dictionary (2.88) gives

$$q_0 = -\frac{A_{3,\mathrm{sol}}}{2} + A_4 + C_H(1-a_{\mathrm{nat}}). \tag{2.109}$$

This relation is not used to define the theory; it serves as a kinematic check of the branch obtained after fixing the global conditions. In particular, for $C_H > 0$, $A_{3,\mathrm{sol}} \geq 0$, and $A_4 \geq 0$, the domain

$$a_{\mathrm{nat}} > 1 \tag{2.110}$$

gives an elastic contribution with an accelerating sign at the present time,

$$C_H(1-a_{\mathrm{nat}}) < 0.$$

It corresponds to

$$R_{\mathrm{nat}} > R_0. \tag{2.111}$$

The mechanical reading of this domain is that of a branch still compressed with respect to its natural radius, whose effective homogeneous restoring term contributes to the current accelerated expansion. Other sign balances remain possible if the other terms of the kinematic check compensate the sum differently.

Later matching rule. The theoretical part does not fix the global conditions of the realized branch. It defines the internal chain

$$\Theta_{\mathrm{init}} \longrightarrow A_i^{\mathrm{HS}} \longrightarrow E_{\mathrm{HS}}(z),$$

where

$$\Theta_{\mathrm{init}} = (a_{\mathrm{nat}}, C_H, A_{3,\mathrm{sol}}, A_4)$$

denotes the physical branch variables. The coefficients A_0, A_1, A_2, A_3, A_4 are therefore not introduced as independent fundamental constants: they are produced by the dictionary (2.88).

The normalized function E_{HS} makes it possible to construct matching quantities such as $H(z)$, $D_H(z)$, $D_M(z)$, $D_V(z)$, and $\mu(z)$. These derived quantities do not enter the definition of the autonomous theory of the branch.

Status of the stabilized closure. The polynomial form in powers of a^{-1} used here describes the homogeneous branch under the stabilized-closure hypothesis 2.4. This hypothesis defines the scope of the present chapter: the coefficients A_i^{stab} are the effective coefficients of the branch after stabilization, not the transient functions $A_i(a)$ of the formation phase.

The dynamic transition from the compressed post-quench state to a stabilized branch, including the redistribution of sectors and soliton genesis, is not used as a computational ingredient in the downstream sections. It constitutes the existence hypothesis of the stabilized branch. HSMD predictions must therefore be read conditionally on this effective branch.

The coefficient $A_{3,\mathrm{sol}}^{\mathrm{stab}}$ denotes the stabilized homogeneous projection of the solitonic content, while A_4^{stab}, when it is kept in the homogeneous law, denotes the stabilized remnant of the sector of order a^{-4}.

2.8 Rotational geometric closures

Purpose of the section. This section describes the geometric closure associated with a global rotational branch charge. It separates the isotropic contribution, which may enter the homogeneous dynamics, from the anisotropic residual, which is consumed only by directional channels.

2.8.1 Geometric support and rotational field

The spatial section is represented by the unit sphere

$$\mathbb{S}^3 = \left\{ \mathbf{x} \in \mathbb{R}^4 \, ; \, \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{R}^4} = 1 \right\},$$

and by its physical immersion

$$\mathbf{X}(t, \mathbf{x}) = R_0 a(t) \, \mathbf{x}. \tag{2.112}$$

A global branch rotation is described by an antisymmetric endomorphism

$$\mathsf{A}_{\mathrm{HS}}(t) \in \mathfrak{so}(4), \qquad \mathsf{A}_{\mathrm{HS}}^T = -\mathsf{A}_{\mathrm{HS}}. \tag{2.113}$$

The associated rotational velocity field is

$$\mathbf{v}_{\mathrm{rot}}(t, \mathbf{x}) = R_0 a(t) \, \mathsf{A}_{\mathrm{HS}}(t) \mathbf{x}. \tag{2.114}$$

This velocity is tangent to $\mathbb{S}^3$, since

$$\langle \mathbf{x}, \mathsf{A}_{\mathrm{HS}} \mathbf{x} \rangle_{\mathbb{R}^4} = 0$$

by antisymmetry of A_{HS}.

2.8.2 Rotational amplitude and geometric motif

The mean quadratic norm of the generator defines the rotational amplitude

$$\Omega_{\mathrm{HS}}^2(t) := \left\langle |\mathsf{A}_{\mathrm{HS}}(t) \mathbf{x}|_{\mathbb{R}^4}^2 \right\rangle_{\mathbb{S}^3}. \tag{2.115}$$

When $\Omega_{\mathrm{HS}}(t) \neq 0$, the normalized generator is introduced as

$$\mathsf{J}_{\mathrm{HS}}(t) := \frac{\mathsf{A}_{\mathrm{HS}}(t)}{\Omega_{\mathrm{HS}}(t)}. \tag{2.116}$$

It satisfies

$$\left\langle |\mathsf{J}_{\mathrm{HS}}(t) \mathbf{x}|_{\mathbb{R}^4}^2 \right\rangle_{\mathbb{S}^3} = 1. \tag{2.117}$$

The scalar motif associated with rotation is

$$Q_{\mathrm{rot}}(t, \mathbf{x}) := |\mathsf{J}_{\mathrm{HS}}(t) \mathbf{x}|_{\mathbb{R}^4}^2, \tag{2.118}$$

and its anisotropic residual is

$$\delta Q_{\mathrm{rot}}(t, \mathbf{x}) := Q_{\mathrm{rot}}(t, \mathbf{x}) - \langle Q_{\mathrm{rot}}(t, \cdot) \rangle_{\mathbb{S}^3}. \tag{2.119}$$

With the normalization (2.117),

$$\delta Q_{\mathrm{rot}}(t, \mathbf{x}) = Q_{\mathrm{rot}}(t, \mathbf{x}) - 1, \qquad \langle \delta Q_{\mathrm{rot}}(t, \cdot) \rangle_{\mathbb{S}^3} = 0. \tag{2.120}$$

2.8.3 Isotropic projection and anisotropic residual

For any scalar quantity $X(\mathbf{x})$ defined on $\mathbb{S}^3$, we write

$$\mathcal{P}_0[X] := \langle X \rangle_{\mathbb{S}^3}, \qquad \mathcal{P}_\perp[X] := X - \mathcal{P}_0[X]. \tag{2.121}$$

The normalized rotational closure is

$$\mathcal{K}_{\mathrm{rot}}(t,\mathbf{x}) := \frac{R_0^2 \Omega_{\mathrm{HS}}^2(t)}{c_t^2} Q_{\mathrm{rot}}(t,\mathbf{x}). \tag{2.122}$$

Its isotropic part is

$$\overline{\mathcal{K}}_{\mathrm{rot}}(t) := \mathcal{P}_0[\mathcal{K}_{\mathrm{rot}}(t,\cdot)] = \frac{R_0^2 \Omega_{\mathrm{HS}}^2(t)}{c_t^2}, \tag{2.123}$$

and its anisotropic residual is

$$\delta\mathcal{K}_{\mathrm{rot}}(t,\mathbf{x}) := \mathcal{P}_\perp[\mathcal{K}_{\mathrm{rot}}(t,\cdot)] = \frac{R_0^2 \Omega_{\mathrm{HS}}^2(t)}{c_t^2} \delta Q_{\mathrm{rot}}(t,\mathbf{x}). \tag{2.124}$$

It satisfies

$$\langle \delta\mathcal{K}_{\mathrm{rot}}(t,\cdot) \rangle_{\mathbb{S}^3} = 0. \tag{2.125}$$

The isotropic contribution to the effective geometric coefficient is defined by

$$\Omega_{\mathcal{K}}^{\mathrm{rot}} := \Xi_{\mathcal{K}} \overline{\mathcal{K}}_{\mathrm{rot}}(t_{\mathrm{ref}}), \tag{2.126}$$

where $\Xi_{\mathcal{K}}$ is the projection sign chosen by the scalar closure convention. The total contribution of geometric rank decomposes as

$$\Omega_{\mathcal{K}} = \Omega_{\mathcal{K}}^{(0)} + \Omega_{\mathcal{K}}^{\mathrm{rot}}. \tag{2.127}$$

2.8.4 Directional residual and observer

The position of the observer is denoted by

$$\mathbf{x}_{\mathrm{obs}} \in \mathbb{S}^3. \tag{2.128}$$

A local line-of-sight direction is a unit tangent vector

$$\mathbf{n} \in T_{\mathbf{x}_{\mathrm{obs}}}\mathbb{S}^3, \qquad \|\mathbf{n}\| = 1. \tag{2.129}$$

The corresponding geodesic is

$$\gamma_{\mathbf{n}}(\chi) = \cos\chi\, \mathbf{x}_{\mathrm{obs}} + \sin\chi\, \mathbf{n}. \tag{2.130}$$

The directional residual along the line of sight is

$$\delta\Omega_{\mathcal{K}}^{\mathrm{rot}}(\chi,\mathbf{n}) := \Xi_{\mathcal{K}}\, \delta\mathcal{K}_{\mathrm{rot}}(t, \gamma_{\mathbf{n}}(\chi)). \tag{2.131}$$

It has zero mean over local directions:

$$\left\langle \delta\Omega_{\mathcal{K}}^{\mathrm{rot}}(\chi,\mathbf{n}) \right\rangle_{\mathbf{n}} = 0. \tag{2.132}$$

At first order, it induces an effective directional modulation

$$H^2(z,\mathbf{n}) = H_{\mathrm{br}}^2(z) + H_0^2(1+z)^2 \delta\Omega_{\mathcal{K}}^{\mathrm{rot}}(z,\mathbf{n}). \tag{2.133}$$

2.8.5 Directional transfer function for distances

The directional residual is written in normalized form as

$$\delta\mathcal{F}_{\rm rot}(z,\hat{\mathbf{n}}) = D_{\rm rot}(1+z)^2\mathcal{M}_{\rm rot}(\hat{\mathbf{n}}), \qquad \langle\mathcal{M}_{\rm rot}\rangle = 0. \tag{2.134}$$

The effective directional function is then

$$E^2(z,\hat{\mathbf{n}}) = E^2_{\rm br}(z) + \delta\mathcal{F}_{\rm rot}(z,\hat{\mathbf{n}}). \tag{2.135}$$

At first order,

$$\frac{\delta E}{E} = \frac{1}{2}\frac{\delta\mathcal{F}_{\rm rot}}{E^2_{\rm br}(z)}. \tag{2.136}$$

The integrated distance perturbation is

$$\frac{\delta D_L}{D_L}(z,\hat{\mathbf{n}}) = -\frac{1}{2}D_{\rm rot}\,K_H(z)\mathcal{M}_{\rm rot}(\hat{\mathbf{n}}), \tag{2.137}$$

with

$$K_H(z) := \frac{\displaystyle\int_0^z \frac{(1+z')^2}{E^3_{\rm br}(z')}\,dz'}{\displaystyle\int_0^z \frac{dz'}{E_{\rm br}(z')}}. \tag{2.138}$$

Local estimators of H_0. A local estimator of H_0, built from a sample $\mathcal{S}$, measures a quantity weighted by the directions, redshifts, and effective weights of the sample. We define

$$K_H(\mathcal{S}) := \langle K_H(z_i)\,\mathcal{M}_{\rm rot}(\hat{\mathbf{n}}_i)\rangle_{\mathcal{S}}\,. \tag{2.139}$$

For a redshift window,

$$K_H(z_{\rm min}, z_{\rm max}; \mathcal{S}) := \langle K_H(z_i)\,\mathcal{M}_{\rm rot}(\hat{\mathbf{n}}_i)\rangle_{\mathcal{S},\ z_{\rm min}\le z_i\le z_{\rm max}}\,. \tag{2.140}$$

The directional correction associated with a survey estimator is written

$$\frac{H_0^{\rm loc}(\mathcal{S}) - H_{\rm ref}}{H_{\rm ref}} = \frac{D_{\rm rot}}{2}K_H(\mathcal{S}), \tag{2.141}$$

and, for a window,

$$H_0^{\rm loc}([z_{\rm min}, z_{\rm max}]) = H_{\rm ref}\left[1 + \frac{D_{\rm rot}}{2}K_H(z_{\rm min}, z_{\rm max}; \mathcal{S})\right]. \tag{2.142}$$

These equations describe the directional transfer of an explicit sample. The motif $\mathcal{M}_{\rm rot}$ is constructed with zero mean over the full $\mathbb{S}^3$: it therefore measures a directional contrast, not a new homogeneous component of expansion.

The $H_0^{\rm loc}$ channel used below relies on a local scalar reduction at LO rank. The global mean of the directional motif is absorbed into the reference branch $H_{\rm ref}$, whereas the local observer consumes the quadratic projection of the radius onto the rotation plane. We therefore introduce the effective local response

$$\mathcal{M}_H^{\rm LO}(\chi_{\rm obs}) = \cos^2\chi_{\rm obs}. \tag{2.143}$$

The compact local reading of the squared rate is therefore written

$$\frac{\left(H_0^{\rm loc,LO}\right)^2}{H^2_{\rm ref}} = 1 + D_{\rm rot}\mathcal{M}_H^{\rm LO}(\chi_{\rm obs}) = 1 + D_{\rm rot}\cos^2\chi_{\rm obs}. \tag{2.144}$$

This relation is an effective LO closure for the local reading. It is not presented as the complete derivation of a survey estimator from the kernel $K_H(\mathcal{S})$. The full connection would depend on the mask, the weights, the redshift distribution, and the operational definition of the local estimator. In the present document, the kernel K_H carries the status of the directional transfer, whereas (2.144) defines the local scalar prescription used for the $H_0^{\rm loc}$ comparison.

2.8.6 Local projection of the observer

The local projection depends on the observer's position, on the intrinsic rotation axis, and on the channel used. For a linear directional channel, we write

$$\mathcal{S}_X = C_X \sin\chi_{\text{obs}}, \tag{2.145}$$

where C_X is the response coefficient of channel X. This relation defines an effective channel amplitude; by itself it does not fix the latitude χ_{obs}. Symmetrized axial channels may carry a different quadratic response; the spin/HSC channel is treated separately at LO rank in Subsection 2.8.8.

2.8.7 Rotational completion of the PMNS phase channel

The PMNS sector of the HSM baseline is treated, at the published rank, as an effective real reduction: it fixes the three mixing angles, but does not close the Dirac phase. More precisely, at the rank where the internal geometry of the neutrino-like composite is treated without a dynamic global orientation, the effective mixing matrix can be taken as real. The Jarlskog invariant associated with the Dirac sector,

$$J_{\text{CP}} = \text{Im}\Big(U_{e1}U_{\mu 2}U^*_{e2}U^*_{\mu 1}\Big), \tag{2.146}$$

is then zero. In the standard parametrization,

$$J_{\text{CP}} = c_{12}s_{12}c_{23}s_{23}c^2_{13}s_{13}\sin\delta_{\text{CP}}. \tag{2.147}$$

For non-degenerate mixing angles, the real LO rank therefore imposes

$$\sin\delta_{\text{CP}} = 0, \qquad \delta_{\text{CP}} = 0 \text{ or } \pi \quad (\text{modulo } 2\pi). \tag{2.148}$$

A non-trivial Dirac phase therefore cannot be produced by the real LO reduction of the PMNS sector alone. It must be carried by an additional dynamic orientation.

In HSMD, this orientation is provided by the global rotation of the hypersphere. We denote by

$$A^{\text{glob}}_{\text{rot}} = \frac{\Omega_{\text{HS}}R^{\text{HSM}}_0}{c_t} \tag{2.149}$$

the dimensionless global rotation amplitude. For a domain located at the effective angular position χ_{obs} relative to the rotation plane, the radius projected onto that plane is

$$R_\perp = R^{\text{HSM}}_0 \cos\chi_{\text{obs}}. \tag{2.150}$$

The LO rotational phase accumulated over a complete internal cycle $\psi \in [0, 2\pi]$ of the PMNS channel is then the projected holonomy

$$\delta^{\text{rot}}_{\text{CP}} = \oint_{\Gamma_{\text{PMNS}}} \frac{\Omega_{\text{HS}}R_\perp}{c_t}\,d\psi = \int_0^{2\pi} A^{\text{glob}}_{\text{rot}}\cos\chi_{\text{obs}}\,d\psi. \tag{2.151}$$

Since $A^{\text{glob}}_{\text{rot}}$ and χ_{obs} are constant over the LO internal cycle considered, one obtains

$$\delta^{\text{rot}}_{\text{CP}} = 2\pi A^{\text{glob}}_{\text{rot}}\cos\chi_{\text{obs}}. \tag{2.152}$$

The factor 2π therefore comes from integration over a complete cycle of the internal phase coordinate, whereas the factor $\cos\chi_{\text{obs}}$ comes from projecting the rotation radius onto the rotation plane.

This relation does not mean that a PMNS phase directly measures a global cosmological coordinate. At the theoretical level, it defines the projected rotational phase carried by a branch of the rotating hypersphere. No numerical inversion of this relation is performed in the theory part: there, χ_{obs} remains a free effective angular position of the domain considered.

2.8.8 LO response of the spin/HSC channel

The spin channel is considered as an axial channel for reading the global rotation. Let $\widehat{\mathbf{N}}_{\mathrm{rot}}$ be the rotation axis and $\widehat{\mathbf{r}}_{\mathrm{obs}}$ the effective radial direction of the observing domain on the hypersphere. The angular position χ_{obs} is measured relative to the rotation plane. With

$$\Pi_\perp = \mathbf{1} - \widehat{\mathbf{N}}_{\mathrm{rot}}\widehat{\mathbf{N}}_{\mathrm{rot}}^T, \tag{2.153}$$

one has

$$\|\Pi_\perp \widehat{\mathbf{r}}_{\mathrm{obs}}\| = \cos\chi_{\mathrm{obs}}. \tag{2.154}$$

After azimuthal averaging around the rotation axis, a term linear in $\Pi_\perp \widehat{\mathbf{r}}_{\mathrm{obs}}$ would depend on the choice of azimuthal phase and would not define a scalar channel amplitude. The first non-zero scalar invariant of the spin channel at LO rank is therefore

$$\mathcal{M}_{\mathrm{spin}}^{\mathrm{LO}} = \|\Pi_\perp \widehat{\mathbf{r}}_{\mathrm{obs}}\|^2 = \cos^2\chi_{\mathrm{obs}}. \tag{2.155}$$

The slope measured by a projected chirality estimator also carries an azimuthal loss. Denoting by φ the local azimuthal phase around the axis, the projection average is

$$\langle|\cos\varphi|\rangle = \frac{1}{2\pi}\int_0^{2\pi} |\cos\varphi|\, d\varphi = \frac{2}{\pi}. \tag{2.156}$$

The deprojected amplitude of the spin channel is defined by

$$\mathcal{M}_{\mathrm{spin}}^{\mathrm{LO}} = \langle|\cos\varphi|\rangle\, \mathcal{S}_{\mathrm{HSC}}^{\mathrm{LO}}. \tag{2.157}$$

Combining (2.155)–(2.157), one obtains

$$\mathcal{S}_{\mathrm{HSC}}^{\mathrm{LO}} = \frac{\pi}{2}\cos^2\chi_{\mathrm{obs}}. \tag{2.158}$$

The rotational slope predicted in the spin channel is then

$$\beta_{\mathrm{HSC}}^{\mathrm{pred}} = A_{\mathrm{rot}}^{\mathrm{glob}}\, \mathcal{S}_{\mathrm{HSC}}^{\mathrm{LO}} = A_{\mathrm{rot}}^{\mathrm{glob}}\frac{\pi}{2}\cos^2\chi_{\mathrm{obs}}. \tag{2.159}$$

2.8.9 Detailed closure with explicit compression

The detailed closure explicitly keeps the terms associated with the initial compression, the geometric residual of rank a^{-2}, and the structural re-ascent of the branch. The dynamic function is

$$\mathcal{F}_{\mathrm{det}}(a) = C_4 a^{-4} + C_3 a^{-3} + C_2 a^{-2} + C_1 a^{-1} + C_0. \tag{2.160}$$

The retained coefficients are related to the upstream structural quantities by

$$\begin{aligned} &C_4 = -\Pi_\ell a_{\mathrm{rel}}^2, && C_3 = 2\Pi_\ell a_{\mathrm{rel}} + A_{\mathrm{sol}}, && C_2 = K_{\mathrm{eff}}, \\ &C_1 = 2\mathcal{B}_{\mathrm{el}} a_{\mathrm{rel}}, && C_0 = -\mathcal{B}_{\mathrm{el}} + B_{\mathrm{close}}. \end{aligned} \tag{2.161}$$

The detailed closure imposes a finer re-ascent toward the internal coefficients of the branch. Its real domain is therefore controlled by the condition

$$\mathcal{F}_{\mathrm{det}}(a) \geq 0. \tag{2.162}$$

2.8.10 Matching functionals associated with reductions

The matching functionals do not define the theory; they transform an internal branch S into quantities computable from $\mathcal{F}_{\rm br}$. For a reduction S, the normalized rate is

$$H_S(z_{\rm th}) = H_{\rm ref}^{(S)} \sqrt{\frac{F_S(z_{\rm th})}{F_S(0)}}, \tag{2.163}$$

the internal radial distance is

$$D_{H,S}^{\rm th}(z_{\rm th}) = \frac{c_t}{H_S(z_{\rm th})}, \tag{2.164}$$

and the quasi-flat transverse distance is

$$D_{M,S}^{\rm th}(z_{\rm th}) \simeq \frac{c_t}{H_{\rm ref}^{(S)}} \int_0^{z_{\rm th}} \frac{dz'}{E_S(z')}. \tag{2.165}$$

For a length rule $r_\star$, the distance ratios used in BAO-type matchings are defined by

$$\frac{D_{H,S}^{\rm th}}{r_\star}, \qquad \frac{D_{M,S}^{\rm th}}{r_\star}, \qquad \frac{D_{V,S}^{\rm th}}{r_\star}, \qquad D_{V,S}^{\rm th} := \left[z_{\rm th} \left(D_{M,S}^{\rm th} \right)^2 D_{H,S}^{\rm th} \right]^{1/3}. \tag{2.166}$$

2.9 Theoretical matching functionals

The following quantities are theoretical functionals associated with a closure reduction S. They consume no external numerical value. In this section, $\mathcal{F}_{\rm br}$ denotes an admissible dynamic function, for instance an LO reduction or a detailed closure.

Definition 2.6 (Internal quantities at the matching point)**.** The internal radius and the internal expansion rate at the matching point are

$$R_{\rm ref} := a_{\rm ref} R_0, \qquad H_{\rm ref}^{(S)} := \mathcal{H}_S(a_{\rm ref}). \tag{2.167}$$

These quantities belong to the internal matching of the branch.

Definition 2.7 (Normalized expansion function)**.** For a reduction S, the normalized expansion function is

$$E_S(a) := \frac{\mathcal{H}_S(a)}{\mathcal{H}_S(a_{\rm ref})} = \sqrt{\frac{\mathcal{F}_S(a)}{\mathcal{F}_S(a_{\rm ref})}}. \tag{2.168}$$

In the theoretical redshift variable,

$$E_S(z_{\rm th}) = \sqrt{\frac{\mathcal{F}_S(a_{\rm ref}/(1+z_{\rm th}))}{\mathcal{F}_S(a_{\rm ref})}}. \tag{2.169}$$

The internal angular distance along a geodesic is

$$\chi_S(a; a_{\rm ref}) = \int_a^{a_{\rm ref}} \frac{da'}{a'^2 \sqrt{\mathcal{F}_S(a')}}. \tag{2.170}$$

The theoretical radial distance is

$$D_{H,S}^{\rm th}(a) = \frac{c_t}{\mathcal{H}_S(a)} = \frac{R_0}{\sqrt{\mathcal{F}_S(a)}}. \tag{2.171}$$

The theoretical transverse distance on $\mathbb{S}^3$ is

$$D_{M,S}^{\rm th}(a) = a_{\rm ref} R_0 \sin[\chi_S(a; a_{\rm ref})] \,. \tag{2.172}$$

In the quasi-flat regime, this expression reduces to

$$D^{\mathrm{th}}_{M,S}(z_{\mathrm{th}}) \simeq \frac{c_t}{H^{(S)}_{\mathrm{ref}}} \int_0^{z_{\mathrm{th}}} \frac{dz'}{E_S(z')}. \tag{2.173}$$

The theoretical luminosity distance is

$$D^{\mathrm{th}}_{L,S}(a) = \frac{a_{\mathrm{ref}}}{a} D^{\mathrm{th}}_{M,S}(a). \tag{2.174}$$

Finally, the internal time separating two states a_1 and a_2 is

$$\Delta t_S(a_1 \to a_2) = \frac{1}{H_R} \int_{a_1}^{a_2} \frac{da}{a\sqrt{\mathcal{F}_S(a)}}. \tag{2.175}$$

2.10 Logical status of the coefficients

HSMD coefficients do not all have the same logical status. The distinction used in this part is the following: quantities imported from HSM, branch conditions, homogeneous projection coefficients, closure coefficients, and internal matching rules.

Table 2.1: Logical status of the quantities and coefficients used in the HSMD theory part.

Quantity	Definition or role	Logical status
$H_\star$	$H_R = c_t/R_0$	kinematic scale imported from HSM
$\mathcal{B}_{\mathrm{el}}$	(2.17)	reduced coefficient of the homogeneous elastic mode
$\beta^{\mathrm{nat}}_{\mathrm{el}}$	(2.20)	same mode referred to the natural radius
$\mathcal{F}_S(a)$	dynamic function of a reduction S	theoretical closure
a_{ref}	Definition 2.6	internal matching point
$H^{(S)}_{\mathrm{ref}}$	$\mathcal{H}_S(a_{\mathrm{ref}})$	internal rate at matching
$E_S(z_{\mathrm{th}})$	normalized expansion function	functional of the reduction S
$A_{3,\mathrm{det}}$	(2.39)	detailed projection of rank a^{-3}
$A_{3,\mathrm{sol}}$	(2.40)	stabilized coefficient consumed by the minimal dictionary
K_{eff}	(2.55)	homogeneous projection of rank a^{-2}
A_2^{LO}	(2.178)	geometric or kinematic residual retained at rank a^{-2} in the LO reduction
A_0^{LO}	(2.178)	effective constant term of the LO branch reduction
$C^{\mathrm{rot}}_{\nu\nu 0}$	(2.179)	rotational specialization of the LO constant term
C_4	$-\Pi_\ell a^2_{\mathrm{rel}}$, with Π_ℓ defined by (2.51)	structural coefficient of the detailed closure
C_3	$A_{3,\mathrm{det}}$, defined by (2.39)	structural coefficient of the detailed closure
C_2	K_{eff}, defined by (2.55)	residual to be connected without double counting
C_1	$2\mathcal{B}_{\mathrm{el}} a_{\mathrm{rel}}$	structural coefficient of the detailed closure
C_0	$-\mathcal{B}_{\mathrm{el}} + B_{\mathrm{close}}$	asymptotic coefficient of the detailed closure
Π_ℓ	(2.51)	reduced compressive projection of branch
a_{rel}	release point	branch condition
A_{sol}	localized population projection	solitonic projection
B_{close}	global closure	non-dilutive closure term
$r_\star$	(2.166)	distance normalization, outside branch dynamics

Rule of use. Reduction coefficients must not be confused with fundamental constants. In the theory part, they define an admissible closure of the HSMD equation. The matching rules, including $r_\star$, do not modify the dynamic branch; they serve to form ratios of computed quantities.

2.11 Synthesis

The HSMD theoretical dynamics is organized into three levels.

Common level. The starting point is the common differential equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_\star^2\,\mathcal{F}_{\rm br}(a), \tag{2.176}$$

where $\mathcal{F}_{\rm br}$ denotes the branch function resulting from an admissible micro–macro closure reduction and where $H_\star$ is defined by (2.6).

Mechanical reference closure. The complete homogeneous mechanical form retained at the theoretical level is

$$\mathcal{F}_{\rm HS}(a) = A_4 a^{-4} + A_3 a^{-3} + A_2 a^{-2} + A_1 a^{-1} + A_0. \tag{2.177}$$

The sectors A_1 and A_4 are associated respectively with elastic restoring and global rotation. The sector A_3 corresponds to the localized solitonic content. The sector A_2 gathers kinematic and geometric contributions and possible branch projections in a^{-2}. The sector A_0 is the constant term of branch closure.

The LO branch reduction does not require the introduction of independent cosmological-type coefficients. It is written directly in the HSMD dictionary as the restriction of (2.177) to the sectors retained at late homogeneous rank:

$$\mathcal{F}_{\rm br}^{\rm LO}(a) = A_{3,\rm sol}\,a^{-3} + A_2^{\rm LO}\,a^{-2} + A_1^{\rm LO}\,a^{-1} + A_0^{\rm LO}. \tag{2.178}$$

The coefficient $A_{3,\rm sol}$ denotes the stabilized dilutive rank of the branch. The coefficient $A_2^{\rm LO}$ gathers the geometric or kinematic residual retained at rank a^{-2}. The coefficient $A_1^{\rm LO}$ denotes the effective elastic-restoring rank retained by the homogeneous LO closure of the branch. The term $A_0^{\rm LO}$ denotes the effective constant term of the branch. When this constant term is carried by the global homogeneous rotation, it specializes as

$$A_0^{\rm LO} = C_{\nu\nu 0}^{\rm rot} = \kappa_{\rm rot}^{\rm LO}\left(A_{\rm rot}^{\rm glob}\right)^2, \tag{2.179}$$

In the homogeneous LO branch closure, the rank $A_1 a^{-1}$ is explicitly retained. The LO notation used here therefore does not mean that the rank a^{-1} is abandoned; it means that only the stabilized sectors of the late branch are retained, with $A_4 = 0$ in the main branch studied.

Detailed structural closure. The reduction with explicit compression is

$$\mathcal{F}_{\rm det}(a) = C_4 a^{-4} + C_3 a^{-3} + C_2 a^{-2} + C_1 a^{-1} + C_0. \tag{2.180}$$

Its real branch is subject to the condition (2.162).

Matching functionals. For any reduction S, one defines

$$E_S(z_{\rm th}) = \sqrt{\frac{F_S(z_{\rm th})}{F_S(0)}}, \tag{2.181}$$

then

$$D_{H,S}^{\rm th}(z_{\rm th}) = \frac{c_t}{H_{\rm ref}^{(S)}\,E_S(z_{\rm th})}, \tag{2.182}$$

and, in the quasi-flat regime,

$$D^{\rm th}_{M,S}(z_{\rm th}) \simeq \frac{c_t}{H^{(S)}_{\rm ref}} \int_0^{z_{\rm th}} \frac{dz'}{E_S(z')}. \tag{2.183}$$

The logical chain of the theory is

$$\begin{aligned} \text{HSM constants and HSMD global conditions} &\longrightarrow \text{closure reduction} \\ &\longrightarrow \text{dynamic branch} \\ &\longrightarrow \text{matching functionals} \\ &\longrightarrow \text{external ratios and tests.} \end{aligned} \tag{2.184}$$

Part II

Branch calibration and late-time confrontations

Chapter 3

Observational branch calibration

3.1 Calibration scope and consumed HSM constants

The theoretical chapter defined the HSMD mechanical object: a stabilized homogeneous branch of the meshed hypersphere, described by an effective dynamics in powers of the internal scale factor. The present chapter does not modify this theoretical dynamics. It calibrates the branch coordinates that allow this dynamics to be confronted with the cosmological observables of the Universe.

The calibration therefore concerns the effective parameters of the stabilized homogeneous branch, together with the observational scales required for the comparison with measured distances and expansion rates. It does not recalibrate the fundamental constants of HSM.

The HSM constants explicitly consumed in this observational projection are:

$$c_t = c_{\rm lum} = 2.99792458 \times 10^8\ \mathrm{m\,s^{-1}}, \qquad R_0^{\rm HSM} = 1.50841 \times 10^{29}\ \mathrm{m}. \tag{3.1}$$

The observational convention $c_t = c_{\rm lum}$ is that of the HSM landing, with

$$c_{\rm lum} = 299\,792\,458\ \mathrm{m\,s^{-1}},$$

the exact SI value of the speed of light in vacuum [2]. In HSM, the tangential speed is defined by $c_t^2 = T/\rho$ [1, eq. `eq:ct2_T_over_rho`], then identified with $c_{\rm lum}$ in the observational landing.

The global radius $R_0^{\rm HSM}$ is the value closed in HSM by the micro-local one-ring separation [1, eq. `eq:R0_from_one_ring_Z_micro_symbolic_COMP`]. When critical densities or equivalent mass densities are formed, one also consumes the effective gravitational constant identified in HSM,

$$G_{\rm eff} = G_N = 6.67430 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}, \tag{3.2}$$

whose numerical value is the 2022 CODATA/NIST value of the Newtonian constant of gravitation [3]. In HSMD, $G_{\rm eff}$ is identified with this constant by the HSM LO gravitation-like closure [1, eq. `eq:grav_GN_final_new`], with the corresponding observational closure [1, eq. `eq:OBS_GN_effective_LO_calib_Qr`]. These constants are HSM inputs that are already fixed. They are not adjusted in this chapter.

The stabilized branch consumed by the calibration is written in the form

$$F_{\rm HS}^{\rm stab}(z) = A_4^{\rm stab}(1+z)^4 + A_3^{\rm stab}(1+z)^3 + A_2^{\rm stab}(1+z)^2 + A_1^{\rm stab}(1+z) + A_0^{\rm stab}. \tag{3.3}$$

This writing corresponds to the stabilized homogeneous closure defined in the theoretical part. The normalized rate used in the observables is

$$E_{\rm HS}^2(z) = \widehat{A}_4(1+z)^4 + \widehat{A}_3(1+z)^3 + \widehat{A}_2(1+z)^2 + \widehat{A}_1(1+z) + \widehat{A}_0, \tag{3.4}$$

with

$$\widehat{A}_i := \frac{A_i^{\mathrm{stab}}}{F_{\mathrm{HS}}^{\mathrm{stab}}(0)}, \qquad \widehat{A}_0 = 1 - \widehat{A}_1 - \widehat{A}_2 - \widehat{A}_3 - \widehat{A}_4. \tag{3.5}$$

In the remainder of this chapter, the coefficients denoted A_i designate the normalized coefficients $\widehat{A}_i$ of (3.4). This local convention avoids overloading the calibration expressions.

The calibration selects the global coordinates of the mean branch compatible with the retained observational set. The logical chain consumed in this chapter is

$$\boxed{\Theta_{\mathrm{HSM}} + \Theta_{\mathrm{branche}}^{\mathrm{stab}} \longrightarrow E_{\mathrm{HS}}(z) \longrightarrow \text{cosmological observables} \longrightarrow \text{calibrated branch coordinates.}} \tag{3.6}$$

Derived quantities such as q_0, the transition redshift, the effective age, or the ratio $a_{\mathrm{nat}} = R_{\mathrm{nat}}/R_0^{\mathrm{HSM}}$ are not injected as selection data. They are calculated only after the branch selection.

3.2 Observables used

The branch Hubble rate is

$$H_{\mathrm{br}}(z) = H_0 E_{\mathrm{HS}}(z). \tag{3.7}$$

The BAO distances are calculated by

$$I(z) := \int_0^z \frac{dz'}{E_{\mathrm{HS}}(z')}, \tag{3.8}$$

$$\frac{D_H(z)}{r_\star} = \frac{c_t}{H_0 r_\star}\frac{1}{E_{\mathrm{HS}}(z)}, \qquad \frac{D_M(z)}{r_\star} = \frac{c_t}{H_0 r_\star} I(z), \tag{3.9}$$

$$\frac{D_V(z)}{r_\star} = \left[z\left(\frac{D_M(z)}{r_\star}\right)^2 \frac{D_H(z)}{r_\star}\right]^{1/3}. \tag{3.10}$$

The directly constrained BAO combination is

$$\alpha_{\mathrm{BAO}} := \frac{c_t}{H_0 r_\star}. \tag{3.11}$$

After fitting H_0 with cosmic chronometers, the associated BAO scale is

$$r_\star = \frac{c_t}{\alpha_{\mathrm{BAO}} H_0}. \tag{3.12}$$

Type Ia supernovae are compared with the shape of the luminosity distance

$$\mu_{\mathrm{th}}(z) = 5\log_{10}\big[(1+z)I(z)\big] + \mathcal{M}, \tag{3.13}$$

where $\mathcal{M}$ is the additive magnitude offset, analytically profiled with the full covariance of the Pantheon+ set. This channel fixes the shape of $D_L(z)$, without separately imposing H_0 when $\mathcal{M}$ is profiled.

The SNe subsample consumed by (3.18) is defined by the following selection on the Pantheon+ file:

$$\boxed{\text{calibrator} = 0, \qquad z_{\mathrm{HD}} > 0.01.} \tag{3.14}$$

After applying (3.14), the SNe vector contains

$$N_{\mathrm{SN}} = 1580 \tag{3.15}$$

supernovae. The covariance C_{SN} used in (3.17)–(3.18) is the submatrix extracted from the full Pantheon+ covariance according to this same selection and in the same order as the distance-modulus vector.

The observational set consumed in this section includes the combined DESI DR2 BAO measurements, the Pantheon+ set with full covariance, and the cosmic chronometers used for the $H(z)$ channel [4, 5, 6, 7, 8, 9].

3.3 Objective function

For the BAO vector $\mathbf{y}_{\rm BAO}$ and its covariance $C_{\rm BAO}$, we define

$$\chi^2_{\rm BAO} = (\mathbf{y}^{\rm th}_{\rm BAO} - \mathbf{y}^{\rm obs}_{\rm BAO})^T C^{-1}_{\rm BAO} (\mathbf{y}^{\rm th}_{\rm BAO} - \mathbf{y}^{\rm obs}_{\rm BAO}). \tag{3.16}$$

For supernovae, the offset $\mathcal{M}$ is eliminated by analytic minimization. If $\mathbf{1}$ denotes the unit vector and $\Delta(\mathcal{M}=0)$ the residual before offset, then

$$\mathcal{M}_{\rm prof} = -\frac{\mathbf{1}^T C^{-1}_{\rm SN} \Delta(0)}{\mathbf{1}^T C^{-1}_{\rm SN} \mathbf{1}}. \tag{3.17}$$

The corresponding contribution is

$$\chi^2_{\rm SN} = \Delta(\mathcal{M}_{\rm prof})^T C^{-1}_{\rm SN} \Delta(\mathcal{M}_{\rm prof}). \tag{3.18}$$

For cosmic chronometers,

$$\chi^2_{\rm CC} = \sum_i \left[\frac{H_0 E_{\rm HS}(z_i) - H_i^{\rm obs}}{\sigma_{H,i}}\right]^2. \tag{3.19}$$

The total objective function is

$$\boxed{\chi^2_{\rm tot} = \chi^2_{\rm BAO} + \chi^2_{\rm SN} + \chi^2_{\rm CC}.} \tag{3.20}$$

The selection further imposes positivity of $E^2_{\rm HS}(a)$ on the past branch probed by the data.

3.4 Effective constant residue and theoretical constraint

The constant coefficient A_0 consumed by the late-time homogeneous kernel represents the effective constant residue visible to late-time cosmological observables. In HSMD, this residue is not a free parameter devoid of structure: it measures the balance between the global closure carried by the solitonic radial channel and the homogeneous elastic restoring term,

$$A_0^{\rm eff} = B_{\rm close} - \mathcal{B}_{\rm el}. \tag{3.21}$$

The radial closure gives

$$B_{\rm close} = A_{3,\rm sol}, \tag{3.22}$$

whereas the homogeneous elastic restoring term is written as

$$\mathcal{B}_{\rm el} = \frac{3}{2a^2_{\rm nat}}. \tag{3.23}$$

The coordinate A_1 is related to the natural branch by

$$A_1 = \frac{3}{a_{\rm nat}}. \tag{3.24}$$

Eliminating $a_{\rm nat}$, one obtains the structural constraint

$$\boxed{A_0^{\rm th} = A_{3,\rm sol} - \frac{A_1^2}{6}.} \tag{3.25}$$

The role of the profiled scan is therefore not to replace this constraint. It is to test it: A_0 is temporarily relaxed, $(H_0, r_\star, A_1, A_{3,\rm sol})$ are then refitted, and one finally checks where the relation (3.25) is satisfied.

The vector of fitted coordinates in the profiled scan is

$$\Theta_{\rm cal} = (\alpha_{\rm BAO}, H_0, A_1, A_{3,\rm sol})\,. \tag{3.26}$$

For a general profiled scan, the kernel is written as

$$E^2_{\rm HS}(z) = A_0 + A_1(1+z) + A_2(1+z)^2 + A_{3,\rm sol}(1+z)^3, \tag{3.27}$$

with

$$A_2 = 1 - A_0 - A_1 - A_{3,\rm sol}. \tag{3.28}$$

For each imposed value of A_0, the parameters $(H_0, r_\star, A_1, A_{3,\rm sol})$ are refitted, and A_2 is then deduced from (3.28).

The exact-compensation point

$$A_0 = 0 \tag{3.29}$$

remains a control point of the profiled scan. The central theoretical profile is the self-consistent point satisfying (3.25).

3.5 Exact-compensation control point

The minimization of (3.20) is performed on the profiled kernel (3.27), with the coordinate set (3.26). The recalibration gives

$$\boxed{\begin{aligned} \alpha_{\rm BAO} &= 30.120, \\ H_0 &= 67.7516\ \mathrm{km\,s^{-1}\,Mpc^{-1}}, \\ r_\star &= 146.908\ \mathrm{Mpc}, \\ A_1 &= 1.207331, \\ A_3 &= 0.403711. \end{aligned}} \tag{3.30}$$

The corresponding normalized coefficients are

$$\boxed{\begin{aligned} A_0 &= 0, \\ A_1 &= 1.207331, \\ A_2 &= -0.611042, \\ A_3 &= 0.403711, \\ A_4 &= 0. \end{aligned}} \tag{3.31}$$

These values satisfy the normalization

$$A_0 + A_1 + A_2 + A_3 + A_4 = 1. \tag{3.32}$$

The point $A_0 = 0$ is retained only as an exact-compensation control point of the profiled scan. It does not define the central value of the calibration: the calibration provides an admissible family in A_0, and the self-consistent representative point is $A_0^\star \simeq 0.270$. The decomposition of the contributions to the objective function is

$$\chi^2_{\rm BAO} = 8.10, \qquad \chi^2_{\rm SN} = 1387.32, \qquad \chi^2_{\rm CC} = 14.95, \tag{3.33}$$

and

$$\chi^2_{\rm tot} = 1410.37. \tag{3.34}$$

3.6 Profiled scan of the constant residue A_0

In order to test the structural constraint (3.25), a profiled scan of the constant residue A_0 is performed. For each fixed value of A_0, the parameters $(H_0, r_\star, A_1, A_{3,\mathrm{sol}})$ are refitted, whereas A_2 is fixed by the normalization $E_{\mathrm{HS}}(0) = 1$.

The free scan shows that the data allow a family of branches. The free statistical minimum lies around $A_0 \simeq 0.225$. The theoretical constraint

$$A_0 = A_{3,\mathrm{sol}} - \frac{A_1^2}{6}$$

selects instead a self-consistent point located at

$$A_0^\star \simeq 0.270, \qquad \Delta\chi^2 \simeq 0.009.$$

The scan therefore does not replace the derivation: it tests it, and the derived point is statistically indistinguishable from the free minimum.

Table 3.1: Profiled scan of the effective constant residue A_0. For each row, $(H_0, r_\star, A_1, A_{3,\mathrm{sol}})$ are refitted and $A_2 = 1 - A_0 - A_1 - A_{3,\mathrm{sol}}$. The column A_0^{th} gives the derived value $A_{3,\mathrm{sol}} - A_1^2/6$.

A_0	H_0	A_1	A_2	$A_{3,\mathrm{sol}}$	A_0^{th}	$\Delta\chi^2$
−0.500	67.440	2.161	−1.165	0.503	−0.275	3.374
−0.300	67.567	1.780	−0.944	0.464	−0.064	1.780
−0.100	67.691	1.398	−0.722	0.424	0.098	0.686
0.000	67.752	1.207	−0.611	0.404	0.161	0.330
0.100	67.812	1.016	−0.500	0.384	0.212	0.103
0.225	67.885	0.777	−0.361	0.359	0.258	0.000
0.270	67.910	0.692	−0.312	0.350	0.270	0.009
0.300	67.929	0.634	−0.278	0.344	0.277	0.037
0.500	68.042	0.251	−0.055	0.304	0.293	0.497
0.800	68.204	−0.324	0.280	0.244	0.227	2.191
1.000	68.307	−0.708	0.504	0.204	0.120	3.999

The zone $\Delta\chi^2 < 1$ provides the observational thickness of the family of admissible profiles. On the displayed grid, it induces in particular the range

$$A_0 \in [-0.100,\, 0.500],$$

which must not be read as a symmetric uncertainty, but as a scan envelope. It also induces a range of possible values for

$$a_{\mathrm{nat}} = \frac{3}{A_1}.$$

The self-consistent point $A_0 = A_0^{\mathrm{th}}$ provides the central LO representative of this family, for which

$$a_{\mathrm{nat}}^\star = \frac{3}{0.692} \simeq 4.34.$$

This point does not cancel the scan envelope; it only provides the central value used when downstream outputs are displayed as representative LO values.

3.7 Cosmographic check of the point $A_0 = 0$

The deceleration parameter associated with the profiled kernel (3.27) is

$$q(z) = -1 + \frac{1+z}{2E_{\rm HS}^2(z)} \frac{dE_{\rm HS}^2}{dz}. \tag{3.35}$$

For the control point $A_0 = 0$, the present value is

$$q_0 = -1 + \frac{1}{2}\left(A_1 + 2A_2 + 3A_3\right). \tag{3.36}$$

The transition redshift z_t is defined by

$$q(z_t) = 0. \tag{3.37}$$

With the values of (3.30), one obtains the cosmographic check associated with the point $A_0 = 0$,

$$\boxed{q_0 = -0.402, \qquad z_t = 0.729.} \tag{3.38}$$

The effective age associated with this compensated kernel is

$$t_0 = 13.585 \text{ Gyr}. \tag{3.39}$$

These values are not the central outputs consumed downstream. They document only the control point $A_0 = 0$. The LO central values are displayed in the predictive part at the self-consistent point $A_0^\star \simeq 0.270$, with the scan envelope associated with the family admissible in A_0.

3.8 Auxiliary directional calibration

The homogeneous background calibration does not consume the directional channels. The directional outputs of the predictions chapter nevertheless use a small auxiliary set of channel quantities, distinct from the HSM constants and distinct from the homogeneous coordinates of the late closed branch.

The global rotational amplitude used in these channels is traced from an auxiliary BAO shape calibration, prior to the local directional calculation. This calibration concerns the effective dilutive coefficient of the homogeneous branch written as

$$E^2(z) = \Omega_{\rm eff}^{\rm shape}(1+z)^3 + \Omega_{\mathcal{K}}^{\rm shape}(1+z)^2 + C_{\nu\nu 0}^{\rm shape}. \tag{3.40}$$

In the auxiliary flat branch,

$$\Omega_{\mathcal{K}}^{\rm shape} = 0, \qquad E(0)^2 = 1, \tag{3.41}$$

hence

$$C_{\nu\nu 0}^{\rm shape} = 1 - \Omega_{\rm eff}^{\rm shape}. \tag{3.42}$$

The auxiliary BAO shape-only calibration of the reduced flat branch gives the shape estimator

$$\Omega_{\rm eff}^{\rm shape,fit} = 0.2975 \pm 0.008. \tag{3.43}$$

In the LO directional chains, the rounded central value consumed is

$$\Omega_{\rm eff}^{\rm shape} = 0.30. \tag{3.44}$$

This quantity has the status of an auxiliary BAO shape calibration. It is not identified with the coefficients of the complete homogeneous kernel $(A_0, A_1, A_2, A_3, A_4)$, which are fitted in a distinct closure. The downstream outputs that depend on it must therefore be read as LO central values, until the full covariance of the auxiliary calibration is propagated.

One therefore deduces, by the sum rule of the auxiliary flat branch,

$$C_{\nu\nu 0}^{\rm shape} = 1 - \Omega_{\rm eff}^{\rm shape} = 0.70. \tag{3.45}$$

The conversion of the stationary background term into a global rotational amplitude uses the LO rotational lemma of the Theoretical Part, (2.60)–(2.63). We define

$$C_{\nu\nu 0}^{\rm shape} = \kappa_{\rm rot}^{\rm LO}\left(A_{\rm rot}^{\rm glob}\right)^2, \qquad A_{\rm rot}^{\rm glob} := \frac{\Omega_{\rm HS} R_0^{\rm HSM}}{c_{\rm lum}}. \tag{3.46}$$

The coefficient $\kappa_{\rm rot}^{\rm LO}$ is therefore not calibrated in this section: it is fixed geometrically by the exact inertial average on $\mathbb{S}^3$, computed in (2.59)–(2.60). The qualifier LO refers to the truncation at quadratic order in the rotational speed, not to the geometric value 3/8 itself:

$$\kappa_{\rm rot}^{\rm LO} = \frac{3}{8}. \tag{3.47}$$

Thus

$$A_{\rm rot}^{\rm glob} = \sqrt{\frac{C_{\nu\nu 0}^{\rm shape}}{\kappa_{\rm rot}^{\rm LO}}} = \sqrt{\frac{8}{3} C_{\nu\nu 0}^{\rm shape}}. \tag{3.48}$$

With (3.45), one obtains

$$\boxed{A_{\rm rot}^{\rm glob} = \sqrt{\frac{8}{3} \times 0.70} \simeq 1.366.} \tag{3.49}$$

The shape statistical uncertainty associated with (3.43), propagated only at this level of the auxiliary chain, gives

$$\sigma\left(A_{\rm rot}^{\rm glob}\right) \simeq \frac{4}{3A_{\rm rot}^{\rm glob}}\,\sigma\left(\Omega_{\rm eff}^{\rm shape,fit}\right) \simeq 0.008. \tag{3.50}$$

This uncertainty does not replace a full propagation toward the PMNS, CMB, HSC and local H_0 channels; it only fixes the statistical scale of the auxiliary shape calibration.

The chain consumed by the directional outputs is therefore

$$\Omega_{\rm eff}^{\rm shape} = 0.30, \qquad C_{\nu\nu 0}^{\rm shape} = 0.70, \qquad A_{\rm rot}^{\rm glob} \simeq 1.366. \tag{3.51}$$

These values are the LO central values consumed by the directional channels. They do not constitute a full propagation of the uncertainties of the auxiliary calibration toward the PMNS, CMB, HSC and local H_0 outputs.

The coefficient $\Omega_{\rm eff}^{\rm shape}$ must not be identified with the coefficient $A_{3,\rm sol}$ of the complete closed fit. The first is an auxiliary BAO shape estimator in a reduced flat branch ($\Omega_{\mathcal{K}}^{\rm shape} = 0$, $C_{\nu\nu 0}^{\rm shape} = 1 - \Omega_{\rm eff}^{\rm shape}$). The second belongs to the complete stabilized kernel $(A_0, A_1, A_2, A_3, A_4)$, where the ranks $a^0, a^{-1}, a^{-2}, a^{-3}$ are fitted jointly. Their numerical proximity indicates coherence at rank a^{-3}, but they are not consumed as one and the same quantity.

The anisotropic rotational contrast consumed by the local outputs is deduced from the theoretical definition of $D_{\rm rot}$:

$$D_{\rm rot} = \frac{1}{2}\left(A_{\rm rot}^{\rm glob}\right)^2. \tag{3.52}$$

With (3.49), one obtains

$$D_{\rm rot} = 0.933. \tag{3.53}$$

The effective latitude of the observer is fixed by the PMNS phase channel, according to the theoretical closure (2.152). At the order used here, the Dirac phase consumed for the central box is treated as a fiducial reading point of the local projection of the global rotation:

$$\delta_{\rm CP}^{\rm ref} = 2\pi A_{\rm rot}^{\rm glob} \cos\chi_{\rm obs}. \tag{3.54}$$

This relation gives

$$\cos\chi_{\rm obs} = \frac{\delta_{\rm CP}^{\rm ref}}{2\pi A_{\rm rot}^{\rm glob}}. \tag{3.55}$$

For the central directional boxes, the fiducial reading point consumed is

$$\delta_{\rm CP}^{\rm ref} = 197^\circ, \qquad A_{\rm rot}^{\rm glob} = 1.366.$$

This point must not be identified with a unique central value of NuFIT 6.0 [10]. It serves as an internal fiducial point for the LO central values; the dependence on the consumed value of $\delta_{\rm CP}$ is documented in the sensitivity table of the predictive part. One then obtains the LO central value

$$\boxed{\chi_{\rm obs} \simeq 66.4^\circ, \qquad \sin\chi_{\rm obs} \simeq 0.916, \qquad \cos\chi_{\rm obs} \simeq 0.400.} \tag{3.56}$$

The variation induced by the choice of the $\delta_{\rm CP}$ reading point and by $A_{\rm rot}^{\rm glob}$ is not propagated in this box; the sensitivity to $\delta_{\rm CP}$ is documented in the predictive part, and the full directional propagation is left to later directional tests.

The relation (3.54) does not calibrate the local value of H_0. It only closes the angular projection $\cos\chi_{\rm obs}$. The local value of H_0 is then a derived output of this projection.

For an observer located at the effective latitude $\chi_{\rm obs}$ relative to the rotation plane, the radius projected onto this plane is

$$R_\perp = R_0 \cos\chi_{\rm obs}. \tag{3.57}$$

The local geometric factor that projects the quadratic rotational term is therefore

$$M_{\rm loc} = \left(\frac{R_\perp}{R_0}\right)^2 = \cos^2\chi_{\rm obs}. \tag{3.58}$$

At LO order, the local reading of H_0 uses the effective prescription defined in theory by (2.144). The global mean is carried by the calibrated branch $H_{\rm ref}^{\rm cal}$, while the local correction consumes the quadratic projection $\cos^2\chi_{\rm obs}$. The primary correction therefore acts on the squared rate:

$$\frac{\left(H_0^{\rm loc,pred}\right)^2}{\left(H_{\rm ref}^{\rm cal}\right)^2} = 1 + D_{\rm rot} M_{\rm loc} = 1 + D_{\rm rot}\cos^2\chi_{\rm obs}. \tag{3.59}$$

This relation is an LO effective prescription for the local reading, not the complete reconstruction of a directional survey estimator. Passing from H^2 to H then gives

$$\boxed{H_0^{\rm loc,pred} = H_{\rm ref}^{\rm cal}\sqrt{1 + D_{\rm rot}\cos^2\chi_{\rm obs}}.} \tag{3.60}$$

At first order in $D_{\rm rot}\cos^2\chi_{\rm obs}$,

$$\frac{H_0^{\rm loc,pred}}{H_{\rm ref}^{\rm cal}} - 1 \simeq \frac{D_{\rm rot}}{2}\cos^2\chi_{\rm obs}. \tag{3.61}$$

The factor 1/2 in the relative correction to H_0 is therefore not a fitted coefficient: it comes from passing from the quadratic correction on H^2 to the linearized correction on H.

Numerically, with the LO central values

$$H_{\rm ref}^{\rm cal} = 67.91~{\rm km\,s^{-1}\,Mpc^{-1}}, \qquad D_{\rm rot} = 0.933, \qquad \chi_{\rm obs} \simeq 66.4^\circ,$$

one obtains

$$M_{\rm loc} = \cos^2\chi_{\rm obs} \simeq 0.160, \tag{3.62}$$

then

$$\boxed{H_0^{\mathrm{loc,pred}} \simeq 72.8\ \mathrm{km\,s^{-1}\,Mpc^{-1}}.} \tag{3.63}$$

These two values are LO central outputs of the local projection. They do not contain the full propagation of the directional uncertainties.

The apparent local value of the SH0ES/Pantheon+ channel [11],

$$H_0^{\mathrm{loc,obs}} = 73.04 \pm 1.04\ \mathrm{km\,s^{-1}\,Mpc^{-1}},$$

is therefore compatible with the output of the local quadratic projection, without consuming the SH0ES/Pantheon+ value in the angular closure.

The logical status is the following. Under the HSMD rotational hypothesis, the PMNS channel provides a projection $\cos\chi_{\mathrm{obs}}$. This projection is then carried into the local geometry through $R_\perp = R_0 \cos\chi_{\mathrm{obs}}$, which gives $M_{\mathrm{loc}} = \cos^2\chi_{\mathrm{obs}}$. The comparison of $H_0^{\mathrm{loc,pred}}$ with the apparent local value then constitutes a cross-test of the PMNS–rotation closure.

Channel	Fixed or predicted quantity	Status
PMNS	$\chi_{\mathrm{obs}} \simeq 66.4^\circ$	LO central value of angular calibration
Global rotation	$D_{\mathrm{rot}} = 0.933$	LO central value of quadratic projection
Local projection	$M_{\mathrm{loc}} = \cos^2\chi_{\mathrm{obs}} \simeq 0.160$	LO central value of latitude geometry
Local H_0 channel	$H_0^{\mathrm{loc,pred}} \simeq 72.8\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$	derived LO central output

Table 3.2: Derived local closure. PMNS fixes the effective latitude, the global rotation fixes D_{rot}, and the projection $R_\perp/R_0 = \cos\chi_{\mathrm{obs}}$ determines M_{loc}.

The directional confrontations associated with this projection are treated in the predictions chapter.

3.9 Numerical traceability table

Element	Status	Traceability chain
DESI DR2 BAO set	CALIBRATED	[4, 5], (3.16)
Pantheon+ SNe Ia set	CALIBRATED	[6, 7], (3.14)–(3.18)
Cosmic chronometers	CALIBRATED	[8, 9], (3.19)
$H_0 = 67.7516\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$	CONTROL POINT $A_0 = 0$	(3.26), (3.20), (3.30)
$r_\star = 146.908$ Mpc	CONTROL POINT $A_0 = 0$	(3.12), (3.30)
$A_0 = 0$	EXACT COMPENSATION POINT TESTED	(3.29), 3.1
$A_0 \simeq 0.225$	FREE STATISTICAL MINIMUM OF THE SCAN	3.1
$A_0^\star \simeq 0.270$	SELF-CONSISTENT REPRESENTATIVE POINT	(3.25), 3.1
$A_1 = 1.207331$	CONTROL POINT $A_0 = 0$	(3.26), (3.30)
$A_2 = -0.611042$	CONTROL POINT $A_0 = 0$, TRACED FROM NORMALIZATION	(3.28), (3.31)
$A_3 = 0.403711$	CONTROL POINT $A_0 = 0$	(3.26), (3.30)
$q_0 = -0.402$	COSMOGRAPHIC CHECK AT POINT $A_0 = 0$	(3.36), (3.38)
$z_t = 0.729$	COSMOGRAPHIC CHECK AT POINT $A_0 = 0$	(3.37), (3.38)
$t_0 = 13.585$ Gyr	COSMOGRAPHIC CHECK AT POINT $A_0 = 0$	(3.39)
$\Omega_{\mathrm{eff}}^{\mathrm{shape,fit}} = 0.2975 \pm 0.008$	REPRODUCIBLE AUXILIARY BAO SHAPE-ONLY CALIBRATION	(3.43)
$\Omega_{\mathrm{eff}}^{\mathrm{shape}} = 0.30$	ROUNDED LO VALUE CONSUMED BY THE DIRECTIONAL CHANNELS	(3.44)
$C_{\nu\nu 0}^{\mathrm{shape}} = 0.70$	TRACED from $\Omega_{\mathrm{eff}}^{\mathrm{shape}}$	(3.42), (3.45)
$A_{\mathrm{rot}}^{\mathrm{glob}} \simeq 1.366$	TRACED from the auxiliary BAO shape-only calibration	(3.46), (3.49), (3.50)
$D_{\mathrm{rot}} = 0.933$	TRACED from $A_{\mathrm{rot}}^{\mathrm{glob}}$	(3.52), (3.53)
$\chi_{\mathrm{obs}} \simeq 66.4^\circ$	PMNS LO CENTRAL VALUE; DIRECTIONAL UNCERTAINTY NOT PROPAGATED	(3.54)–(3.56)
$M_{\mathrm{loc}} \simeq 0.160$	LO CENTRAL VALUE OF THE PROJECTION $\cos^2\chi_{\mathrm{obs}}$	(3.57)–(3.62)
$H_0^{\mathrm{loc,pred}} \simeq 72.8\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$	LO CENTRAL OUTPUT OF THE LOCAL QUADRATIC PROJECTION	(3.59)–(3.63)

3.10 Conclusion

The late-time calibration does not fix a unique value of the effective constant residue A_0. It provides an admissible family of stabilized profiles, corresponding to different balances between homogeneous restoring and global closure. On the displayed grid, the zone $\Delta\chi^2 < 1$ gives the scan envelope

$$A_0 \in [-0.100,\, 0.500].$$

This envelope is not a symmetric uncertainty and must not be written with a $\pm$ symbol.

The point

$$A_0 = 0$$

is only the exact compensation check. It is kept to trace the limit where the effective constant residue vanishes, but it is not the central value of the calibration. The free statistical minimum of the scan lies near

$$A_0 \simeq 0.225.$$

The structural constraint

$$A_0 = A_{3,\mathrm{sol}} - \frac{A_1^2}{6}$$

selects, for its part, the self-consistent point

$$A_0^\star \simeq 0.270, \qquad \Delta\chi^2 \simeq 0.009.$$

This point is statistically indistinguishable from the free minimum and provides the representative point used for the LO central values in the predictive part. Downstream outputs that depend on the E_{HS} kernel must therefore be read as central values at the point $A_0^\star$, accompanied by the admissible scan envelope in A_0.

Part III

Derived outputs and predictions

Chapter 4

Observable post-calibration predictions

4.1 Post-calibration scope

The late-time calibration profiles the effective constant residue A_0. This residue is not a free parameter devoid of structure: in HSMD, it measures the balance between radial solitonic closure and the homogeneous elastic restoring term,

$$A_0 = B_{\text{close}} - \mathcal{B}_{\text{el}}. \tag{4.1}$$

The radial closure gives

$$B_{\text{close}} = A_{3,\text{sol}}, \tag{4.2}$$

whereas the homogeneous elastic restoring term is written

$$\mathcal{B}_{\text{el}} = \frac{3}{2a_{\text{nat}}^2}. \tag{4.3}$$

With

$$A_1 = \frac{3}{a_{\text{nat}}}, \tag{4.4}$$

one obtains the theoretical constraint

$$\boxed{A_0^{\text{th}} = A_{3,\text{sol}} - \frac{A_1^2}{6}.} \tag{4.5}$$

The profiled scan of A_0 is therefore used as a test of this constraint: for each imposed value of A_0, the parameters $(H_0, r_\star, A_1, A_{3,\text{sol}})$ are refitted, and one then checks whether the relation (4.5) is satisfied. The self-consistent point, defined by $A_0 = A_0^{\text{th}}$, lies at

$$\boxed{A_0^\star \simeq 0.270, \qquad A_1^\star \simeq 0.692, \qquad A_{3,\text{sol}}^\star \simeq 0.350, \qquad H_0^\star \simeq 67.91\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}.} \tag{4.6}$$

This profile is almost indistinguishable from the free minimum of the scan:

$$\Delta\chi^2 \simeq 0.009.$$

It constitutes the representative self-consistent point used for the LO central values. It does not reduce the calibration to a single branch: the points of the scan in A_0 define an admissible envelope of stabilized profiles.

In what follows, a downstream quantity that depends on $E_{\rm stab}(z)$, and therefore on the coefficients $A_i^{\rm stab}$, must be read at two levels:

$$\text{central value at the point } A_0^\star \quad \text{and} \quad \text{scan envelope over } \Delta\chi^2 < 1.$$

The envelopes are written as intervals $[x_{\rm min}, x_{\rm max}]$. They are not symmetric statistical uncertainties and must not be written with a $\pm$ symbol.

The free scan of A_0 also induces a possible range of values for

$$a_{\rm nat} = \frac{3}{A_1}.$$

The LO central value

$$a_{\rm nat}^\star = \frac{3}{A_1^\star} \simeq 4.34$$

corresponds to the self-consistent point (4.5).

In the stabilized regime, the kernel used in this chapter is

$$E^2_{\rm stab}(z) = A_4^{\rm stab}(1+z)^4 + A_{3,\rm sol}^{\rm stab}(1+z)^3 + A_2^{\rm stab}(1+z)^2 + A_1^{\rm stab}(1+z) + A_0^{\rm stab}. \tag{4.7}$$

The present normalization imposes

$$A_0^{\rm stab} = 1 - A_1^{\rm stab} - A_2^{\rm stab} - A_{3,\rm sol}^{\rm stab} - A_4^{\rm stab}. \tag{4.8}$$

At the representative self-consistent point, the stabilized remnant of the a^{-4} sector is not required by the calibration:

$$A_4^{\rm stab} = 0. \tag{4.9}$$

This relation is not a theoretical cancellation of the a^{-4} sector. It only means that its independent late-time remnant is negligible in the stabilized regime confronted with background observables.

At the representative self-consistent point, one uses

$$H_{\rm ref}^{\rm cal} \simeq 67.91\ {\rm km\,s^{-1}\,Mpc^{-1}}, \qquad r_\star^{\rm cal} \simeq 146.9\ {\rm Mpc}. \tag{4.10}$$

and

$$A_0^{\rm stab} = 0.270, \qquad A_1^{\rm stab} = 0.692, \qquad A_2^{\rm stab} = -0.312, \qquad A_{3,\rm sol}^{\rm stab} = 0.350, \qquad A_4^{\rm stab} = 0. \tag{4.11}$$

These values satisfy

$$A_0^{\rm stab} + A_1^{\rm stab} + A_2^{\rm stab} + A_{3,\rm sol}^{\rm stab} + A_4^{\rm stab} \simeq 1.$$

They also satisfy

$$A_0^{\rm stab} \simeq A_{3,\rm sol}^{\rm stab} - \frac{(A_1^{\rm stab})^2}{6}.$$

The point $A_0 = 0$ and the free best-fit point $A_0 \simeq 0.225$ are kept as checks of the scan. They show that the theoretical constraint falls within the statistically admissible region of the calibration, and practically at the minimum of the scan.

The stabilized coordinate $A_{3,\rm sol}^{\rm stab} = 0.350$ must not be confused with the auxiliary estimator $\Omega_{\rm eff}^{\rm shape} \simeq 0.30$ used in the shape-only calibration. The former belongs to the complete stabilized kernel $(A_0, A_1, A_2, A_3, A_4)$, whereas the latter is a BAO shape estimator in a reduced flat branch.

The Hubble radial distance, the quasi-flat transverse distance, and the volume distance are

$$D_H(z) = \frac{c_{\text{lum}}}{H_{\text{stab}}(z)}, \tag{4.12}$$

$$D_M(z) \simeq c_{\text{lum}} \int_0^z \frac{dz'}{H_{\text{stab}}(z')}, \tag{4.13}$$

$$D_V(z) = \left[z\, D_M^2(z) D_H(z)\right]^{1/3}. \tag{4.14}$$

The same distances define the usual BAO ratios,

$$\frac{D_H(z)}{r_\star^{\text{cal}}}, \qquad \frac{D_M(z)}{r_\star^{\text{cal}}}, \qquad \frac{D_V(z)}{r_\star^{\text{cal}}}.$$

These ratios are not presented here as independent predictions: the BAO channel already enters the global calibration that fixes $r_\star^{\text{cal}}$, $H_{\text{ref}}^{\text{cal}}$, and the coefficients A_i^{stab}. Their observational confrontation therefore belongs to the calibration chapter.

4.2 Derived kinematic quantities

The deceleration parameter of the stabilized branch is defined by

$$q(z) = -1 + (1+z)\frac{d\ln H_{\text{stab}}}{dz}. \tag{4.15}$$

Injecting (4.7), one obtains

$$q(z) = -1 + \frac{4A_4^{\text{stab}}(1+z)^4 + 3A_{3,\text{sol}}^{\text{stab}}(1+z)^3 + 2A_2^{\text{stab}}(1+z)^2 + A_1^{\text{stab}}(1+z)}{2E_{\text{stab}}^2(z)}. \tag{4.16}$$

At the representative self-consistent point (4.11), this gives

$$q_{0,\star}^{\text{HSMD}} = q(0)\big|_{A_0^\star} \simeq -0.441. \tag{4.17}$$

The transition redshift, defined by $q(z_t) = 0$, satisfies

$$4A_4^{\text{stab}}(1+z_t)^4 + 3A_{3,\text{sol}}^{\text{stab}}(1+z_t)^3 + 2A_2^{\text{stab}}(1+z_t)^2 + A_1^{\text{stab}}(1+z_t) = 2E_{\text{stab}}^2(z_t). \tag{4.18}$$

With (4.11), one obtains the LO central value

$$z_{t,\star}^{\text{HSMD}} \simeq 0.699. \tag{4.19}$$

The effective age associated with the stabilized regime is given by

$$t_0^{\text{HSMD}} = \frac{1}{H_{\text{ref}}^{\text{cal}}} \int_0^1 \frac{da}{aE_{\text{stab}}(a)}. \tag{4.20}$$

With (4.10) and (4.11), this integral gives the LO central value

$$t_{0,\star}^{\text{HSMD}} \simeq 13.642 \text{ Gyr}. \tag{4.21}$$

The profiled scan of A_0 induces an envelope of values for any quantity that depends on the kernel E_{stab}. On the calibration grid belonging to $\Delta\chi^2 < 1$, one obtains the following envelope:

The main kinematic values of the present chapter are therefore LO central values at the self-consistent point $A_0^\star$, accompanied by the scan envelope when the quantity depends on the kernel E_{stab}.

Table 4.1: Kinematic envelope induced by the profiled scan of A_0. The brackets denote a scan envelope, not a symmetric statistical uncertainty. The row $A_0^\star$ provides the LO central value used in the text.

Quantity	Central value at point $A_0^\star$	Envelope $\Delta\chi^2 < 1$
A_0	0.270	[−0.100, 0.500]
$H_{\rm ref}^{\rm cal}$ [km s^{-1} Mpc^{-1}]	67.910	[67.691, 68.042]
A_1	0.692	[0.251, 1.398]
A_2	−0.312	[−0.722, −0.055]
$A_{3,\rm sol}$	0.350	[0.304, 0.424]
$a_{\rm nat} = 3/A_1$	4.34	[2.15, 11.95]
q_0	−0.441	[−0.474, −0.387]
z_t	0.699	[0.671, 0.740]
t_0 [Gyr]	13.642	[13.562, 13.695]

The integral (4.20) must not be interpreted as a complete resolution of the initial phase in which the stabilized regime is established. It gives the age associated with the effective continuation of the stabilized kernel down to very small scale factors.

If the integration is started only after effective entry into the stabilized regime, for instance from a scale factor of order 10^{-3}, the resulting age is

$$t_0 - t_{\rm stab} = \frac{1}{H_{\rm ref}^{\rm cal}} \int_{10^{-3}}^{1} \frac{da}{aE_{\rm stab}(a)}. \tag{4.22}$$

The difference with (4.20) is below the Myr for the self-consistent branch. Thus, the late-time estimate $t_0^{\rm HSMD} \simeq 13.642\,\mathrm{Gyr}$ is insensitive to the precise cutoff used to mark entry into the stabilized regime, as long as this cutoff remains far before the late-time observational scales.

4.2.1 Observational determination of the spin axis

The mean branch fixes the background quantities. Directional observables then test the local projection of rotational residues, without recalibrating the global branch coordinates. To determine the directional axis used below, a spin channel is preferred, because the observable is signed and does not pass through an integrated distance reconstruction.

The channel used relies on objects from the Hyper Suprime-Cam Subaru Strategic Program and on an external classification of galactic chirality [12, 13]. A recent independent analysis of the same type of data finds no significant evidence for spin anisotropy [14]. The HSC/spin channel is therefore kept here as a downstream directional confrontation, not as a calibration of the background or of the global rotation.

One considers a catalogue of objects with a line-of-sight direction $\widehat{\mathbf{s}}_i$, a redshift z_i, and a discrete rotation orientation

$$A_i \in \{-1, +1\}.$$

In a given redshift window, one fits

$$A_i = a\,(\widehat{\mathbf{s}}_i \cdot \widehat{\mathbf{N}}) + b, \tag{4.23}$$

where $\widehat{\mathbf{N}}$ is the searched-for directional axis and where b absorbs a possible global classification imbalance.

The scan over redshift windows highlights an active zone in

$$0.10 \le z < 0.16,$$

containing $N \simeq 4.7 \times 10^3$ objects. This zone is not defined by amplitude agreement with HSMD, but by the free detection of a stable spin axis. In this window, the free fit gives an axis close to

$$(\mathrm{RA}, \mathrm{Dec}) \simeq (293.1^\circ, -61.9^\circ).$$

Thus, for the amplitude measurement on a fixed axis, the retained spin axis is

$$\boxed{(\mathrm{RA}, \mathrm{Dec})_{\mathrm{spin}} \simeq (293^\circ, -62^\circ).} \tag{4.24}$$

This axis is a rounded central value of the HSC directional diagnostic. Its directional uncertainty is not propagated into the derived outputs.

On the same sample, the fixed axis

$$(\mathrm{RA}, \mathrm{Dec}) = (293.45^\circ, -61.85^\circ)$$

gives the local raw slope

$$a_{\mathrm{brute}} = 0.2804 \pm 0.0551, \qquad Z_{\mathrm{loc}} \simeq 5.1. \tag{4.25}$$

The quantity Z_{loc} is a local significance, evaluated in the active window and on the fixed axis. It is not corrected for the look-elsewhere factor associated with the search over window, axis, and redshift selection. It must therefore not be read as a global catalogue detection. This raw slope serves only as a local diagnostic for the presence of the signal on the fixed axis. It is not used to calibrate χ_{obs}, nor to determine H_0^{loc}.

The HSC channel is treated as a third weak downstream confrontation. It does not provide a global catalogue detection: the available global analyses are compatible with the absence of a net signal. The retained confrontation is more modest. It concerns the active window in which the spin axis is freely detected, then the raw slope measured on the fixed axis. The observable compared to the HSMD prediction is therefore the local output of the HSC estimator applied to this window and this axis,

$$a_{\mathrm{act}} = 0.2804 \pm 0.0551.$$

An injection–recovery control applied to the same active window, the same fixed axis, and the same estimator gives, for the injected amplitude $\beta_{\mathrm{inj}} = 0.344$,

$$a_{\mathrm{rec}} = 0.342955, \qquad B_{\mathrm{rec}} = \frac{a_{\mathrm{rec}}}{\beta_{\mathrm{inj}}} = 0.996963.$$

In a stochastic binary injection with the same expectation, one obtains

$$a_{\mathrm{rec}} = 0.342955 \pm 0.055686.$$

The estimator response is therefore compatible with unity for this configuration, which justifies confronting the HSMD prediction with the active raw slope. The HSC channel fixes neither χ_{obs}, nor the local amplitude consumed in the correction of H_0. It only tests, downstream, the directional compatibility and the active slope of the chirality signal.

4.3 Rotation speed of the hypersphere and position of the observer

The purpose of this section is to gather the rotational quantities used by the observable directional confrontations. We distinguish the axis projected on the sky, the global angular speed of the hypersphere, and the effective angular position of our observational domain in the hypersphere. These quantities do not define a second fundamental expansion parameter. The apparent local value of H_0 is derived below from the local quadratic projection of the global rotation, while the HSC/spin channel is treated as a directional confrontation on the raw slope in the active window where the spin axis is detected.

The HSC/spin channel reveals, in a stable redshift band, a directional axis projected on the sky,

$$\widehat{\mathbf{N}}_{\rm spin}^{\rm HSC} \simeq (293^\circ, -62^\circ), \tag{4.26}$$

as indicated in (4.24). This axis is not used as a branch calibration. It provides a directional confrontation: HSMD predicts that a chirality/spin channel must not be isotropic, but must exhibit an axial structure depending on redshift.

We then introduce the normalized global rotational amplitude

$$A_{\rm rot}^{\rm glob} := \frac{\Omega_{\rm HS} R_0^{\rm HSM}}{c_{\rm lum}}. \tag{4.27}$$

In the directional channels, this quantity is consumed as an auxiliary coordinate calibrated in (3.49):

$$A_{\rm rot}^{\rm glob} \simeq 1.366. \tag{4.28}$$

This quantity does not denote a BAO distance. It normalizes the amplitude of the global rotation of the hypersphere in the directional channels. Every downstream rotational quantity is recalculated from $A_{\rm rot}^{\rm glob}$, without introducing an independent rotational closure coefficient.

The corresponding global angular speed is therefore

$$\Omega_{\rm HS} = A_{\rm rot}^{\rm glob} \frac{c_{\rm lum}}{R_0^{\rm HSM}}. \tag{4.29}$$

With the calibrated value $A_{\rm rot}^{\rm glob} = 1.366$, and with the HSM constants recalled in the calibration in (3.1),

$$R_0^{\rm HSM} = 1.50841 \times 10^{29}\ \text{m}, \qquad c_{\rm lum} = 2.99792458 \times 10^8\ \text{m}\,\text{s}^{-1},$$

we obtain

$$\boxed{\Omega_{\rm HS} \simeq 2.72 \times 10^{-21}\ \text{s}^{-1}.} \tag{4.30}$$

The global rotational speed associated with the hyperspherical radius is then

$$v_{\rm rot}^{\rm glob} = \Omega_{\rm HS} R_0^{\rm HSM} = A_{\rm rot}^{\rm glob} c_{\rm lum} \simeq 1.366\, c_{\rm lum} \simeq 4.10 \times 10^8\ \text{m}\,\text{s}^{-1}. \tag{4.31}$$

This speed is not a local matter speed in a flat space and does not define a signal propagation speed. It characterizes the global rotation of the hyperspherical geometry in the embedding, that is, a background coordinate normalized by $R_0^{\rm HSM}$. Local observables do not directly consume $v_{\rm rot}^{\rm glob}$, but projections of the global rotational amplitude on the axis and on the position of our observational domain. There is therefore no local causal constraint of the type $v_{\rm rot}^{\rm glob} < c_{\rm lum}$ on this global quantity.

The effective angular position of our observational domain relative to the global rotational axis is inherited from the PMNS calibration established in the calibration chapter, (3.54)–(3.56). To keep a local chain of references in this chapter, we recall the calibrated value:

$$\boxed{\chi_{\rm obs} \simeq 66.4^\circ, \qquad \sin\chi_{\rm obs} \simeq 0.916, \qquad \cos\chi_{\rm obs} \simeq 0.400.} \tag{4.32}$$

Before using the PMNS channel as a rotational closure, two levels must be distinguished. In HSM, the PMNS mixing angles θ_{12}, θ_{13} and θ_{23} belong to the actual internal geometry of the neutrino-like solutions. At the rank where this internal geometry is treated without a dynamic global orientation, the effective mixing matrix may be taken as real. In that case, the Dirac sector is CP-conserving.

Indeed, for a real PMNS matrix, the Jarlskog invariant

$$J_{\mathrm{CP}} = \mathrm{Im}\Big(U_{e1}U_{\mu 2}U_{e2}^{*}U_{\mu 1}^{*}\Big)$$

is zero. In the standard parametrization,

$$J_{\mathrm{CP}} = c_{12}s_{12}c_{23}s_{23}c_{13}^{2}s_{13}\sin\delta_{\mathrm{CP}}.$$

Since the three mixing angles are nonzero, $J_{\mathrm{CP}} = 0$ imposes

$$\sin\delta_{\mathrm{CP}} = 0, \qquad \delta_{\mathrm{CP}} = 0 \text{ or } \pi \quad (\text{modulo } 2\pi).$$

Thus, the real internal geometry alone of the PMNS triplet does not generate a nontrivial Dirac phase. Current measurements of δ_{CP}, however, remain insufficient to isolate a unique value independent of the choice of mass ordering and experimental parametrization. The rotational closure is therefore treated here as a testable dynamic consequence of the HSMD framework, to be confronted with future measurements of δ_{CP}.

In HSMD, this orientation is supplied by the global rotation of the hypersphere. The generic closure relation associates an observed Dirac phase with a rotational phase projected on the effective angular position of our observational domain:

$$\delta_{\mathrm{CP}}^{\mathrm{obs}} = 2\pi A_{\mathrm{rot}}^{\mathrm{glob}} \cos\chi_{\mathrm{obs}}.$$

For the central LO values displayed in this chapter, we consume the fiducial reading point

$$\delta_{\mathrm{CP}}^{\mathrm{ref}} = 197^{\circ}.$$

This fiducial point is not a unique central value attributed to NuFIT 6.0; the dependence on the consumed phase is documented by the sensitivity table below. The closure relation fixes the projection $\cos\chi_{\mathrm{obs}}$ through the HSMD rotational closure. It is not counted here as an independent prediction of δ_{CP}. By contrast, the projection thus obtained is then consumed by three downstream confrontations: H_0^{loc}, through $\cos^2\chi_{\mathrm{obs}}$, CMB birefringence, through $\sin\chi_{\mathrm{obs}}$, and the HSC/spin channel in the redshift window where the spin axis is freely detected. The PMNS–rotation closure is therefore tested as a multi-channel overconstraint, and not by the H_0 channel alone.

The HSC/spin channel provides a third downstream confrontation of the PMNS–rotation closure. It is not used to fix χ_{obs}, nor to determine H_0^{loc}. The confrontation is not made on the global mean of the HSC catalogue, which mixes distinct redshift regimes. It is made on the active window where a spin axis is freely detected and remains aligned with the expected rotation axis.

The retained active window is

$$0.100 \le z < 0.160. \tag{4.33}$$

In this window, the free fit recovers an axis close to

$$(\mathrm{RA}, \mathrm{Dec})_{\mathrm{libre}} \simeq (293.1^{\circ}, -61.9^{\circ}), \tag{4.34}$$

in agreement with the fixed axis used for the amplitude measurement,

$$(\mathrm{RA}, \mathrm{Dec})_{\mathrm{rot}} = (293.45^{\circ}, -61.85^{\circ}).$$

This step defines the active window by directional stability, and not by amplitude agreement with the HSMD prediction.

On the fixed axis, the slope measured in the active window is

$$a_{\mathrm{act}} = 0.2804 \pm 0.0551. \tag{4.35}$$

The raw slope a_{act} is the direct output of the estimator applied to the active window and to the fixed axis. In parallel, diagnostics by redshift windows are built in order to control the stability of the spin-channel response. In a window w, the catalogued chirality is fitted on the fixed axis $\widehat{\mathbf{N}}_{\mathrm{spin}}$:

$$s_i = b_w + a_w x_i + \varepsilon_i, \qquad x_i = \widehat{\mathbf{s}}_i \cdot \widehat{\mathbf{N}}_{\mathrm{spin}}, \qquad i \in w. \tag{4.36}$$

The coefficient b_w absorbs the residual monopolar imbalance of the window, while a_w is the raw directional slope.

The effective response factor $B_{\mathrm{eff},w}$ is obtained, for each window w, by the same injection–recovery procedure as that defined in (4.41), applied with the same fixed axis, the same window mask, and the same estimator as (4.36). It is not a fundamental constant of HSMD: it summarizes the geometric response of the estimator in the considered window. The diagnostic corrected amplitude is defined by

$$\beta^{\mathrm{diag}}_{\mathrm{HSC},w} = \frac{a_w}{B_{\mathrm{eff},w}}, \qquad \sigma^{\mathrm{diag}}_{\beta,w} = \frac{\sigma_{a,w}}{|B_{\mathrm{eff},w}|}. \tag{4.37}$$

The stable set of windows used for this diagnostic contains $N_{\mathrm{win}} = 26$ windows. It is defined by

$$\mathcal{W}_{\mathrm{stab}} = \{[0.105745,\ 0.132580)\} \cup \bigcup_{\Delta z \in \{0.04, 0.05, 0.06, 0.07, 0.08\}} \{[z_0(\Delta z) + 0.01k,\ z_0(\Delta z) + 0.01k + \Delta z)\}_{k=0}^{4}, \tag{4.38}$$

with

$$z_0(0.04) = 0.080, \quad z_0(0.05) = 0.075, \quad z_0(0.06) = 0.070, \quad z_0(0.07) = 0.065, \quad z_0(0.08) = 0.060. \tag{4.39}$$

The first window of (4.38) corresponds to the fourth bin of the quantile-6 cut; the other twenty-five windows are the sliding windows of width Δz and step 0.01.

The diagnostic synthesis is then

$$(a_w, \sigma_{a,w}, B_{\mathrm{eff},w})_{w \in \mathcal{W}_{\mathrm{stab}}} \longrightarrow \beta^{\mathrm{diag}}_{\mathrm{HSC},w} = \frac{a_w}{B_{\mathrm{eff},w}} \longrightarrow \beta^{\mathrm{diag}}_{\mathrm{HSC}} = 0.336004 \pm 0.016628, \qquad N_{\mathrm{win}} = 26. \tag{4.40}$$

The stability of this synthesis is controlled by

$$\chi^2_{\mathrm{stab}} = 24.3255, \qquad \mathrm{dof} = 25, \qquad \chi^2_{\mathrm{stab}}/\mathrm{dof} = 0.973.$$

These diagnostics document the effective response of the HSC channel, but they are not used to recalibrate the PMNS–rotation closure. To avoid turning $B_{\mathrm{eff},w}$ into a hidden theoretical constant, the main confrontation uses the active raw slope, controlled by injection–recovery, rather than an amplitude corrected by a B_{eff} factor.

An injection–recovery control is performed on the same active window, the same fixed axis, and the same estimator. A synthetic signal of the form

$$\mathbb{E}[s_i] = \beta_{\mathrm{inj}} x_i, \qquad x_i = \widehat{r}_i \cdot \widehat{n}_{\mathrm{rot}},$$

is injected, and the slope is then recovered by the same fit

$$s_i = b + a x_i.$$

The measured response coefficient is

$$B_{\mathrm{rec}} = \frac{a_{\mathrm{rec}}}{\beta_{\mathrm{inj}}}. \tag{4.41}$$

For the injected amplitude corresponding to the central prediction, $\beta_{\rm inj} = 0.344$, the deterministic test gives

$$a_{\rm rec} = 0.344, \qquad B_{\rm rec} = 1.$$

In the stochastic binary test with the same expectation, recovery gives

$$B_{\rm rec} = \frac{0.342955}{0.344} \simeq 0.996963. \tag{4.42}$$

The deviation from unity is negligible compared with the slope uncertainty of the active window. Thus, for this estimator and this window, the raw slope $a_{\rm act}$ can be confronted directly with the predicted amplitude.

The HSMD prediction does not consume HSC. It uses only $A_{\rm rot}^{\rm glob}$, fixed by the global calibration, and $\chi_{\rm obs}$, fixed by the PMNS channel. It consumes the LO response of the spin channel derived in the theory part, (2.158)–(2.159). We therefore obtain

$$\beta_{\rm HSC}^{\rm pred} = A_{\rm rot}^{\rm glob} \frac{\pi}{2} \cos^2 \chi_{\rm obs}. \tag{4.43}$$

The factor $\pi/2$ is not fitted on HSC: it comes from the LO azimuthal deprojection of the spin channel, while the factor $\cos^2 \chi_{\rm obs}$ comes from the projected quadratic invariant of the same channel. With the central LO values

$$A_{\rm rot}^{\rm glob} = 1.366, \qquad \chi_{\rm obs} \simeq 66.4^\circ,$$

we obtain

$$\boxed{\beta_{\rm HSC}^{\rm pred} \simeq 0.344.} \tag{4.44}$$

This value is the central LO output of the spin/HSC channel. It does not contain the full propagation of directional uncertainties. The relevant confrontation is therefore

$$\boxed{\beta_{\rm HSC}^{\rm pred} = 0.344 \quad \text{vs} \quad a_{\rm act} = 0.2804 \pm 0.0551.} \tag{4.45}$$

The figures displayed in the box retain the central LO values used in the ratio and significance calculations. The textual syntheses and sensitivity tables round them to 0.20° and 0.34. These roundings are not error bars: the full propagation of directional uncertainties is not performed at this stage. The discrepancy is about 1.2σ. The HSC channel therefore does not provide an exact concordance, but an active-zone compatibility independent of the PMNS–rotation closure.

The published logical chain is unidirectional: PMNS fixes $\chi_{\rm obs}$, then HSMD predicts the expected HSC amplitude in the window where the spin axis is detected. The global catalogue mean, which is weaker, is not the observable predicted here, because it mixes the active window with slices where the axis is not stable or where the signal changes regime.

4.3.1 Apparent local value of H_0

The apparent local value of H_0 is derived from the local quadratic projection of the global rotation. At latitude $\chi_{\rm obs}$, the projected radius in the rotation plane is

$$R_\perp = R_0 \cos \chi_{\rm obs}, \tag{4.46}$$

whence

$$M_{\rm loc} = \left(\frac{R_\perp}{R_0}\right)^2 = \cos^2 \chi_{\rm obs}. \tag{4.47}$$

At LO rank, the local prediction uses the effective local-reading prescription introduced in THEO and calibrated in CALI. The calibrated branch fixes the mean $H_{\rm ref}^{\rm cal}$, while the local rotational correction consumes the quadratic projection $\cos^2\chi_{\rm obs}$. The primary correction therefore acts on the squared rate:

$$\frac{\left(H_0^{\rm loc,pred}\right)^2}{\left(H_{\rm ref}^{\rm cal}\right)^2} = 1 + D_{\rm rot}M_{\rm loc} = 1 + D_{\rm rot}\cos^2\chi_{\rm obs}. \tag{4.48}$$

This relation is used as an effective LO prescription for the local reading; it does not replace the complete directional transfer of a survey estimator based on $K_H(\mathcal{S})$. Thus,

$$\boxed{H_0^{\rm loc,pred} = H_{\rm ref}^{\rm cal}\sqrt{1 + D_{\rm rot}\cos^2\chi_{\rm obs}}.} \tag{4.49}$$

The linearized form is

$$\frac{H_0^{\rm loc,pred}}{H_{\rm ref}^{\rm cal}} - 1 \simeq \frac{D_{\rm rot}}{2}\cos^2\chi_{\rm obs}. \tag{4.50}$$

The factor 1/2 comes from passing from H^2 to H, and not from an adjusted response coefficient.

With the central LO values

$$H_{\rm ref}^{\rm cal} = 67.91\ {\rm km\,s^{-1}\,Mpc^{-1}}, \qquad D_{\rm rot} = 0.933, \qquad \chi_{\rm obs} \simeq 66.4^\circ,$$

one obtains

$$M_{\rm loc} \simeq 0.160, \tag{4.51}$$

and then

$$\boxed{H_{0,\star}^{\rm loc,pred} \simeq 72.8\ {\rm km\,s^{-1}\,Mpc^{-1}}.} \tag{4.52}$$

If $D_{\rm rot}$ and $\chi_{\rm obs}$ are kept at their central LO values, the $\Delta\chi^2 < 1$ envelope induced by the scan of A_0 through $H_{\rm ref}^{\rm cal}$ is

$$H_0^{\rm loc,pred} \in [\,72.58,\ 72.95\,]\ {\rm km\,s^{-1}\,Mpc^{-1}}. \tag{4.53}$$

This envelope is not a complete statistical uncertainty: it isolates only the dependence on the A_0 profile in the background kernel. This output is to be compared with the local value published by SH0ES/Pantheon+ [11], used here as an external observational benchmark:

$$H_0^{\rm loc,obs} = 73.04 \pm 1.04\ {\rm km\,s^{-1}\,Mpc^{-1}}.$$

The HSC/spin channel does not enter this closure. In particular, no $\beta_{\rm HSC}$ amplitude is consumed to determine $H_0^{\rm loc,pred}$. The Pantheon+SH0ES angular mask is not consumed as a closure kernel; it can only serve for independent residual checks.

The choice $M_{\rm loc} = \cos^2\chi_{\rm obs}$ is not an adjustment of the local value of H_0. It follows from the geometric convention according to which $\chi_{\rm obs}$ is a latitude measured relative to the rotation plane: the projected radius is $R_\perp = R_0\cos\chi_{\rm obs}$, and the rotational contribution to the squared rate is proportional to $R_\perp^2$.

The PMNS closure enters upstream: it identifies the observed Dirac phase with the rotational projection $\delta_{\rm CP}^{\rm obs} = 2\pi A_{\rm rot}^{\rm glob}\cos\chi_{\rm obs}$. This relation fixes $\cos\chi_{\rm obs}$, but by itself does not validate the rotational closure. The first downstream test is the apparent local value of H_0, predicted without consuming the SH0ES/Pantheon+ value. The second downstream test, independent of $H_0^{\rm loc}$, is CMB birefringence. The third downstream test is the HSC/spin channel, not as a global catalogue mean, but as the raw slope in the active window where the spin axis is freely detected. The three outputs consume the same projection $\chi_{\rm obs}$ from the PMNS channel.

4.3.2 Cosmic birefringence: second downstream output of the PMNS closure

Puzzle. Some CMB analyses indicate a possible global rotation of the polarization plane, with an amplitude of order a few tenths of a degree. The observational status depends on the data combinations and on the controls of instrumental calibration. In HSMD, this observable provides an independent geometric test of the rotational projection extracted from the PMNS channel.

HSMD predictive chain. The effective latitude of our observational domain is fixed by the PMNS channel, according to (3.54)– (3.56), then recalled in (4.32). CMB birefringence is then a second downstream output of the same PMNS–rotation closure: it consumes the same $\chi_{\rm obs}$ as $H_0^{\rm loc}$, but through $\sin\chi_{\rm obs}$ instead of $\cos^2\chi_{\rm obs}$.

At LO rank, HSMD associates a geometric rotation of polarization with the projection of the global rotation on the line of sight. For a source at redshift z, one writes

$$\beta(\widehat{n}) \simeq \mathcal{W}(z)\, A_{\rm rot}^{\rm glob}\, \sin\chi_{\rm obs}\, \cos\Theta(\widehat{n}, \widehat{n}_{\rm rot}),$$

where $A_{\rm rot}^{\rm glob}$ is defined in (4.27), where Θ is the angle between the observed direction and the HSMD rotation axis, and where $\mathcal{W}(z)$ is the conformal transfer factor.

For the CMB, the stabilized transfer factor is introduced as

$$\mathcal{W}_{\rm CMB}^{\rm stab} \equiv \frac{c_{\rm lum}}{R_0^{\rm HSM}} \int_0^{z_*} \frac{dz}{H_{\rm ref}^{\rm cal} E_{\rm stab}(z)}. \tag{4.54}$$

The recombination redshift consumed in (4.54) is

$$z_* = 1089, \tag{4.55}$$

the usual reference value of CMB decoupling in the standard cosmological framework [15]. In this channel, z_* fixes only the upper bound of the conformal transfer; it is not used to calibrate the HSMD homogeneous kernel.

The expected mean rotation is then

$$\beta_{\rm CMB}^{\rm pred} = \mathcal{W}_{\rm CMB}^{\rm stab} A_{\rm rot}^{\rm glob} \sin\chi_{\rm obs}. \tag{4.56}$$

With the stabilized kernel (4.7), one obtains

$$\mathcal{W}_{\rm CMB}^{\rm stab} \simeq 2.79 \times 10^{-3}. \tag{4.57}$$

Using (4.28) and (4.32), this gives

$$\beta_{\rm CMB}^{\rm pred} \simeq (2.79 \times 10^{-3})(1.366)(0.916) = 3.49 \times 10^{-3}\ \text{rad}.$$

The central LO value is therefore

$$\boxed{\beta_{\rm CMB}^{\rm pred} \simeq 0.200^\circ.} \tag{4.58}$$

This value does not contain the complete propagation of directional uncertainties.

Since $\sin\chi_{\rm obs} \leq 1$, the same relation imposes a central LO geometric ceiling:

$$\boxed{\beta_{\rm CMB}^{\rm max} = \mathcal{W}_{\rm CMB}^{\rm stab} A_{\rm rot}^{\rm glob} \simeq 0.218^\circ.} \tag{4.59}$$

Since $\mathcal{W}_{\rm CMB}^{\rm stab}$ depends on the kernel $E_{\rm stab}$, this output is weakly sensitive to the A_0 profile. Over the $\Delta\chi^2 < 1$ envelope, keeping $A_{\rm rot}^{\rm glob}$ and $\chi_{\rm obs}$ at their central LO values, one obtains

$$\beta_{\rm CMB}^{\rm pred} \in [\,0.194^\circ,\, 0.205^\circ\,]. \tag{4.60}$$

The half-width of this envelope, of order 0.006°, is much smaller than the observational uncertainty currently available on this channel. The geometric ceiling (4.59) must therefore be read as a central LO ceiling, itself weakly dependent on the A_0 profile.

This ceiling makes the CMB channel falsifiable: a future stabilized measurement significantly above (4.59) would put this rotational closure in difficulty.

Current extractions of cosmic birefringence give central values of order a few tenths of a degree, for example

$$\beta_{\rm CMB}^{\rm obs} = 0.35^\circ \pm 0.14^\circ \quad [16],$$

and

$$\beta_{\rm CMB}^{\rm obs} = 0.30^\circ \pm 0.11^\circ \quad [17].$$

The prediction (4.58) is therefore compatible with these measurements at the current level of uncertainty, but lies below their central values. The CMB channel thus provides an independent but still flexible concordance: it strengthens the PMNS–rotation closure if future measurements converge toward $\simeq 0.2^\circ$, and would put it under tension if they stabilized significantly above the ceiling (4.59).

The chain is non-circular with respect to CMB birefringence: $\beta_{\rm CMB}^{\rm obs}$ is not used to determine $\chi_{\rm obs}$. The effective latitude is fixed by the PMNS channel, then injected into the CMB transfer law (4.56). The same projection therefore feeds two independent downstream observables:

$$\chi_{\rm obs} \longrightarrow H_0^{\rm loc,pred} \quad \text{and} \quad \chi_{\rm obs} \longrightarrow \beta_{\rm CMB}^{\rm pred}.$$

Consumed $\delta_{\rm CP}$	$\chi_{\rm obs}$	$H_0^{\rm loc,pred}$	$\beta_{\rm CMB}^{\rm pred}$	$\beta_{\rm HSC}^{\rm pred}$
150°	72.2°	70.8 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$	0.21°	0.20
170°	69.8°	71.6 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$	0.21°	0.26
197° fiducial	66.4°	72.8 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$	0.20°	0.34
220°	63.4°	74.0 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$	0.20°	0.43
250°	59.4°	75.7 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$	0.19°	0.56

Table 4.2: Joint sensitivity of the three downstream outputs to the PMNS phase consumed in the closure $\delta_{\rm CP}^{\rm obs} = 2\pi A_{\rm rot}^{\rm glob}\cos\chi_{\rm obs}$. The values use the central LO values $A_{\rm rot}^{\rm glob} = 1.366$, $D_{\rm rot} = 0.933$, $H_{\rm ref}^{\rm cal} = 67.91\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\mathcal{W}_{\rm CMB}^{\rm stab} = 2.79\times 10^{-3}$, and $\beta_{\rm HSC}^{\rm pred} = A_{\rm rot}^{\rm glob}(\pi/2)\cos^2\chi_{\rm obs}$. They are not a statistical propagation of the NuFIT uncertainty on $\delta_{\rm CP}$, but a central sensitivity table. The point 197° is the fiducial point retained for the central LO boxes, not a unique central value attributed to NuFIT 6.0. The relevant HSC comparison is made with the active raw slope $a_{\rm act} = 0.2804 \pm 0.0551$, measured on the fixed axis in the window where the axis is freely detected, and not with the global mean of the catalogue.

4.4 Joint synthesis of the PMNS–rotation closure

The PMNS–rotation closure introduces a single effective angular position of our observation domain, $\chi_{\rm obs}$, relative to the global rotation axis. The projection consumed by the Dirac phase is $\cos\chi_{\rm obs}$, fixed by

$$\delta_{\rm CP}^{\rm obs} = 2\pi A_{\rm rot}^{\rm glob}\cos\chi_{\rm obs}.$$

The three downstream confrontations do not readjust this projection. They consume the same value of $\chi_{\rm obs}$, but they do not provide three independent angular levers. Two geometric dependences enter: a $\cos^2\chi_{\rm obs}$ branch, common to the apparent local value of H_0 and to the HSC/spin channel,

$$H_0^{\rm loc} \propto \sqrt{1 + D_{\rm rot}\cos^2\chi_{\rm obs}}, \qquad \beta_{\rm HSC}^{\rm pred} \propto A_{\rm rot}^{\rm glob}\cos^2\chi_{\rm obs},$$

and a $\sin\chi_{\rm obs}$ branch, carried by the CMB birefringence,

$$\beta_{\rm CMB} \quad \propto \quad A_{\rm rot}^{\rm glob}\sin\chi_{\rm obs}.$$

The closure is therefore tested by three downstream datasets, but by only two distinct angular dependences. The HSC channel does not provide a new lever on $\chi_{\rm obs}$ relative to $H_0^{\rm loc}$; it provides an independent observational confrontation of the same $\cos^2\chi_{\rm obs}$ branch in a different directional channel.

At the fiducial point $\delta_{\rm CP}^{\rm ref} \simeq 197°$, the associated central LO outputs are

$$H_0^{\rm loc,pred} \simeq 72.8\ {\rm km\,s^{-1}\,Mpc^{-1}}, \qquad \beta_{\rm CMB}^{\rm pred} \simeq 0.20°, \qquad \beta_{\rm HSC}^{\rm pred} \simeq 0.34.$$

These values must not be read as a full propagation of the NuFIT uncertainty on $\delta_{\rm CP}$, nor as a full propagation of the axis or latitude uncertainty. The point 197° is a fiducial reading point; the other plausible values of $\delta_{\rm CP}$ are documented in the sensitivity table. These outputs are the central LO evaluation of the PMNS–rotation closure. They are confronted respectively with

$$H_0^{\rm loc,obs} \simeq 73.04 \pm 1.04\ {\rm km\,s^{-1}\,Mpc^{-1}},$$
$$\beta_{\rm CMB}^{\rm obs} \simeq 0.30° \pm 0.11°,$$

and, for the HSC/spin channel, with the active raw slope

$$a_{\rm act} = 0.2804 \pm 0.0551$$

measured on the fixed axis in the window where the spin axis is freely detected.

As an indication, neglecting correlations between channels and using the displayed observational uncertainties, the normalized deviations are

$$\Delta_{H_0} \simeq 0.2\sigma, \qquad \Delta_{\rm CMB} \simeq 1.0\sigma, \qquad \Delta_{\rm HSC} \simeq 1.2\sigma.$$

The joint compatibility can therefore be summarized as

$$\chi^2_{\rm joint} \simeq (0.2)^2 + (1.0)^2 + (1.2)^2 \simeq 2.5, \tag{4.61}$$

for three downstream confrontations. This quantity is not presented as a full statistical fit: it serves only to make explicit that the three channels remain simultaneously compatible with the same projection $\chi_{\rm obs}$. It is not presented as a measure of preference over competing models either: it establishes an internal multi-channel compatibility of the PMNS–rotation closure, not a statistical detection of this closure.

The status of this closure is therefore the following. It is not a hard proof of the identification $\delta_{\rm CP}^{\rm obs} = 2\pi A_{\rm rot}^{\rm glob}\cos\chi_{\rm obs}$, because the three channels remain observationally flexible and because the joint compatibility does not yet discriminate this closure against competing models. On the other hand, it is no longer a mere coincidence on $H_0^{\rm loc}$: the same projection extracted from the PMNS channel feeds two angular branches, $\cos^2\chi_{\rm obs}$ and $\sin\chi_{\rm obs}$, tested by three distinct datasets. The PMNS–rotation closure is thus a multi-channel consequence, coherent and falsifiable through future improvement of any one of these three confrontations.

4.5 Uncertainty status of the post-calibration outputs

The numerical values presented in this chapter are post-calibration outputs. The coefficient A_0 is the constant closure residue, structurally defined as the balance between radial closure and homogeneous elastic restoring. The calibration does not fix a unique value of A_0: it provides an admissible family of profiles. The central values are displayed at the self-consistent point

$A_0^\star$. When an observable depends sensitively on A_0, an envelope from the profiled scan is also provided, written as an interval $[x_{\min}, x_{\max}]$.

These scan intervals are not symmetric statistical uncertainties. They must not be denoted with the symbol $\pm$. The symbol $\pm$ is reserved for explicitly cited observational uncertainties, for example an external measurement or a measured slope with its standard error.

A full statistical propagation would require propagating the covariance of the global calibration over the parameters

$$H_{\mathrm{ref}}^{\mathrm{cal}}, \qquad r_\star^{\mathrm{cal}}, \qquad A_i^{\mathrm{stab}},$$

and then reevaluating each derived observable. This full propagation is not included in the numerical scope presented here. In contrast, for outputs sensitive to A_0, the envelope of the profiled scan is explicitly propagated.

We therefore distinguish four statuses:

- central LO values, computed at the representative point $A_0^\star$;
- outputs sensitive to A_0, given with a scan envelope $[x_{\min}, x_{\max}]$;
- directional outputs, which also depend on the axis, the rotational amplitude and the observer position;
- prospective or structural outputs, for which the central value indicates a theoretical target or a model scale, without a full statistical confrontation.

Thus, when an observational comparison is mentioned, it must be understood as a post-calibration confrontation of a branch or envelope, not as a new full statistical fit.

4.6 Traceability table for observable predictions

Quantity	Status	Calculation chain
$q_{0,\star}^{\mathrm{HSMD}} \simeq -0.441$, envelope $q_0 \in [-0.474,\ -0.387]$	Central LO value at the point $A_0^\star$, with scan envelope $\Delta\chi^2 < 1$	(4.15)–(4.17), 4.1
$z_{t,\star}^{\mathrm{HSMD}} \simeq 0.699$, envelope $z_t \in [0.671,\ 0.740]$	Central LO value at the point $A_0^\star$, with scan envelope $\Delta\chi^2 < 1$	(4.18)–(4.19), 4.1
$t_{0,\star}^{\mathrm{HSMD}} \simeq 13.642$ Gyr, envelope $t_0 \in [13.562,\ 13.695]$ Gyr	Central LO value at the point $A_0^\star$, with scan envelope $\Delta\chi^2 < 1$	(4.20)–(4.21), 4.1
$\widehat{\mathbf{N}}_{\mathrm{spin}}^{\mathrm{HSC}} \simeq (293°, -62°)$	Observed directional axis; rounded central value, axis uncertainty not propagated into the derived outputs	(4.23)–(4.24)
$\Omega_{\mathrm{HS}} \simeq 2.72 \times 10^{-21}\ \mathrm{s}^{-1}$	Output derived from the directional calibration	(3.49), (4.29)–(4.30)
$v_{\mathrm{rot}}^{\mathrm{glob}} \simeq 1.366\, c_{\mathrm{lum}} \simeq 4.10 \times 10^8\ \mathrm{m\,s^{-1}}$	Central embedding value; not interpreted as a signal speed	(4.29)–(4.31)
$\chi_{\mathrm{obs}} \simeq 66.4°$, from $\delta_{\mathrm{CP}}^{\mathrm{ref}} \simeq 197°$	Fiducial PMNS reading point for χ_{obs}; the other plausible values of δ_{CP} are covered by the sensitivity table	(3.54)–(4.32)
HSC/spin channel: $\beta_{\mathrm{HSC}}^{\mathrm{pred}} \simeq 0.344$ vs $a_{\mathrm{act}} = 0.2804 \pm 0.0551$	Third downstream confrontation in the active window; $\beta_{\mathrm{HSC}}^{\mathrm{pred}}$ is a central LO value, while $a_{\mathrm{act}} = 0.2804 \pm 0.0551$ is the raw slope measured with its standard error; freely recovered axis and raw slope on fixed axis, without correction by B_{eff} in the main confrontation	(2.155)–(2.159), (4.23)–(4.25), (4.33)–(4.42), (4.43)–(4.45)
$H_{0,\star}^{\mathrm{loc,pred}} \simeq 72.8\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, conditional envelope $H_0^{\mathrm{loc,pred}} \in [72.58,\ 72.95]\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$	Local output derived from the quadratic projection; envelope displayed while keeping D_{rot} and χ_{obs} at the central LO values	(4.48)–(4.53)
$\beta_{\mathrm{CMB}}^{\mathrm{pred}} \simeq 0.200°$, A_0 envelope: $[0.194°,\ 0.205°]$	Second central LO downstream output of the PMNS–rotation closure; central LO ceiling $\beta_{\mathrm{CMB}}^{\mathrm{max}} \simeq 0.218°$; directional uncertainty not propagated	(4.54)–(4.60)

The quantities in this table are post-calibration outputs that can be related to current observables or to explicit directional checks. The BAO ratios associated with the distances D_H, D_M and D_V are not presented here as independent predictions: their observational confrontation belongs to the calibration chapter.

The low-redshift SNe Ia channel is not listed as an independent prediction in this table, because no distance-modulus residual is evaluated here on a specified catalogue. It only provides a secondary check protocol for the local projection.

Chapter 5

Prospective predictions and structural outputs

This chapter gathers the outputs that can be computed after calibration and that, in the current observational state, do not constitute a direct confrontation with the same status as the preceding channels. Some are prospective, because they target future measurements; others are structural, because they express a scale or a dimensional translation imposed by the stabilized kernel.

5.0.1 Effective trajectory of the hyperspherical radius in the stabilized regime

The hyperspherical radius is related to the scale factor by

$$R(t) = R_0^{\mathrm{HSM}}\, a(t). \tag{5.1}$$

In the stabilized regime, the trajectory $R(t)$ is therefore determined by the solution $a(t)$ associated with the kernel (4.7).

By definition, the stabilized expansion rate is

$$H_{\mathrm{stab}}(a) = H_{\mathrm{ref}}^{\mathrm{cal}} E_{\mathrm{stab}}(a) = \frac{\dot{a}}{a}. \tag{5.2}$$

With (5.2), one obtains

$$\dot{a} = a\, H_{\mathrm{ref}}^{\mathrm{cal}} E_{\mathrm{stab}}(a). \tag{5.3}$$

The trajectory is given implicitly by

$$t - t_{\mathrm{stab}} = \frac{1}{H_{\mathrm{ref}}^{\mathrm{cal}}} \int_{a_{\mathrm{stab}}}^{a(t)} \frac{da'}{a' E_{\mathrm{stab}}(a')}. \tag{5.4}$$

The radius then follows

$$R(t) = R_0^{\mathrm{HSM}}\, a(t), \tag{5.5}$$

where $a(t)$ is defined by (5.4).

The coefficient A_0^{stab} represents the effective constant residue of the late-time kernel:

$$A_0^{\mathrm{stab}} = B_{\mathrm{close}} - \mathcal{B}_{\mathrm{el}}.$$

In the stabilized branch, the solitonic radial closure is carried by

$$B_{\mathrm{close}} = A_{3,\mathrm{sol}}^{\mathrm{stab}},$$

and the homogeneous elastic restoring by

$$\mathcal{B}_{\rm el} = \frac{3}{2a_{\rm nat}^2} = \frac{(A_1^{\rm stab})^2}{6}.$$

One therefore obtains

$$\boxed{A_0^{\rm stab} = A_{3,\rm sol}^{\rm stab} - \frac{(A_1^{\rm stab})^2}{6}.} \tag{5.6}$$

With the self-consistent branch (4.11),

$$A_{3,\rm sol}^{\rm stab} - \frac{(A_1^{\rm stab})^2}{6} = 0.350 - \frac{0.692^2}{6} \simeq 0.270,$$

which reproduces the calibrated constant residue. The profiled scan of A_0 therefore does not replace this relation: it tests it by temporarily relaxing A_0. The fact that the self-consistent point is located at $\Delta\chi^2 \simeq 0.009$ from the free minimum indicates that this structural constraint is compatible with the calibration data.

As a function of the scale factor, the stabilized kernel is written

$$E_{\rm stab}^2(a) = A_0^{\rm stab} + A_1^{\rm stab}a^{-1} + A_2^{\rm stab}a^{-2} + A_{3,\rm sol}^{\rm stab}a^{-3} + A_4^{\rm stab}a^{-4}. \tag{5.7}$$

In the minimal late-time branch where $A_4^{\rm stab} = 0$, a possible future turnaround is defined by the first positive zero greater than $a = 1$:

$$E_{\rm stab}^2(a_{\rm max}) = 0, \qquad a_{\rm max} > 1. \tag{5.8}$$

If $A_0^{\rm stab} < 0$, the kernel becomes negative at sufficiently large scale factor and such a zero exists within the scope of the stabilized kernel, provided the branch remains in its domain of validity. If $A_0^{\rm stab} = 0$, the constant term does not force a turnaround at this order. If $A_0^{\rm stab} > 0$, no asymptotic turnaround is produced by the constant term.

Thus, in this section, a possible $a_{\rm max}$ is not an independent input. It is a possible output of the stabilized kernel after selecting an admissible branch

$$\left(H_{\rm ref}^{\rm cal}, r_\star^{\rm cal}, A_0^{\rm stab}, A_1^{\rm stab}, A_2^{\rm stab}, A_{3,\rm sol}^{\rm stab}\right).$$

At the point $A_0^{\rm stab} = 0$ of the scan, no future turnaround is forced by a negative constant term at minimal order. For an excess profile $A_0^{\rm stab} > 0$, the constant term instead contributes to maintaining the asymptotic expansion within the domain of validity of the stabilized kernel. The prospective predictions must therefore be read as discrimination tests between admissible profiles of the profiled scan.

5.0.2 Current equivalent mass density of the stabilized a^{-3} sector

In HSMD, the organized non-geometric content is carried, in the stabilized regime, by the matter-like term of the kernel. Using (4.7), this term is

$$E_{\rm stab}^2(a) \supset A_{3,\rm sol}^{\rm stab}a^{-3}.$$

The coefficient $A_{3,\rm sol}^{\rm stab}$ is a coordinate calibrated by the background observables. The density derived below is therefore an output of the calibrated kernel.

The current critical density associated with the branch scale is

$$\rho_{\rm crit,0}^{\rm HSMD} = \frac{3(H_{\rm ref}^{\rm cal})^2}{8\pi G}. \tag{5.9}$$

With $H_{\mathrm{ref}}^{\mathrm{cal}}$ given by (4.10), one obtains

$$\rho_{\mathrm{crit},0}^{\mathrm{HSMD}} \simeq 8.66 \times 10^{-27}\ \mathrm{kg\,m^{-3}}. \tag{5.10}$$

The effective density of the organized non-geometric content is therefore

$$\rho_{\mathrm{cont},0}^{\mathrm{HSMD}} = A_{3,\mathrm{sol}}^{\mathrm{stab}} \rho_{\mathrm{crit},0}^{\mathrm{HSMD}}. \tag{5.11}$$

The coefficient $A_{3,\mathrm{sol}}^{\mathrm{stab}}$ does not denote the whole set of HSM solitonic solutions. It denotes the stabilized part of the solitonic content whose energy density dilutes as a^{-3}, that is, as matter-like content. The classification criterion is therefore not the solitonic nature of the object, but its cosmological dilution law. Radiative excitations, even when they are described by HSM solutions, belong to the a^{-4} dilution sector when their energy is redshifted by the expansion.

With $A_{3,\mathrm{sol}}^{\mathrm{stab}}$ given by (4.11), the central LO value is

$$\boxed{\rho_{a^{-3},0,\star}^{\mathrm{HSMD}} \simeq 3.03 \times 10^{-27}\ \mathrm{kg\,m^{-3}}.} \tag{5.12}$$

On the $\Delta\chi^2 < 1$ envelope of the profiled scan in A_0, the same translation gives

$$\rho_{a^{-3},0}^{\mathrm{HSMD}} \in [\,2.64,\ 3.65\,] \times 10^{-27}\ \mathrm{kg\,m^{-3}}. \tag{5.13}$$

In energy density, the central LO value is

$$\boxed{u_{\mathrm{cont},0,\star}^{\mathrm{HSMD}} = u_{a^{-3},0,\star}^{\mathrm{HSMD}} \simeq 2.72 \times 10^{-10}\ \mathrm{J\,m^{-3}}.} \tag{5.14}$$

The corresponding scan envelope is

$$u_{a^{-3},0}^{\mathrm{HSMD}} \in [\,2.38,\ 3.28\,] \times 10^{-10}\ \mathrm{J\,m^{-3}}. \tag{5.15}$$

This density is the current effective density associated with the stabilized part of the non-geometric content carried by the a^{-3} sector. It is obtained by dimensional translation of the calibrated coefficient $A_{3,\mathrm{sol}}^{\mathrm{stab}}$, via

$$\rho_{\mathrm{cont},0}^{\mathrm{HSMD}} = A_{3,\mathrm{sol}}^{\mathrm{stab}} \rho_{\mathrm{crit},0}^{\mathrm{HSMD}}.$$

It does not constitute, in this chapter, a genesis derivation of the solitonic content from a quench phase. In particular, the upstream chain

$$\text{quench} \longrightarrow \text{solitonic capture} \longrightarrow A_{3,\mathrm{sol}}^{\mathrm{stab}}$$

is not used as a closure in the present chapter. The value $\rho_{\mathrm{cont},0}^{\mathrm{HSMD}}$ must therefore be read as a late-time output of the calibrated stabilized kernel, not as a matter density derived directly from the quench dynamics.

This quantity also does not identify the a^{-3} sector with the total energy injected during a possible pre-stabilized phase. It does not introduce a dark energy fluid: in HSMD, the constant term of the branch belongs to the geometric and dynamical closure, not to an independent material reservoir.

External comparison. For comparison, the Planck 2018 analysis in the base ΛCDM framework gives approximately [15]

$$H_0^{\mathrm{Pl18}} \simeq 67.4\ \mathrm{km\,s^{-1}\,Mpc^{-1}}, \qquad \Omega_m^{\mathrm{Pl18}} \simeq 0.315.$$

The corresponding critical density is

$$\rho_{\mathrm{crit},0}^{\mathrm{Pl18}} = \frac{3(H_0^{\mathrm{Pl18}})^2}{8\pi G} \simeq 8.53 \times 10^{-27}\ \mathrm{kg\,m^{-3}},$$

hence

$$\rho_{m,0}^{\mathrm{Pl18}} = \Omega_m^{\mathrm{Pl18}} \rho_{\mathrm{crit},0}^{\mathrm{Pl18}} \simeq 2.69 \times 10^{-27}\ \mathrm{kg\,m^{-3}}.$$

The HSMD effective density of the a^{-3} sector is therefore higher than this reference value, which reflects the fact that $A_{3,\mathrm{sol}}^{\mathrm{stab}}$ is not identified term by term with the parameter Ω_m of a ΛCDM model.

5.0.3 Spectroscopic redshift drift

The stabilized kernel (4.7) gives the temporal drift of the cosmological redshift. For a comoving source and a comoving observer, the relation $1 + z = a(t_0)/a(t_\mathrm{e})$, combined with null radial propagation, gives the Sandage–Loeb relation [18, 19]

$$\dot{z} \equiv \frac{dz}{dt_0} = (1+z)H_{\mathrm{ref}}^{\mathrm{cal}} - H_{\mathrm{stab}}(z). \tag{5.16}$$

The spectroscopic drift expressed as an apparent velocity per unit observer time is then

$$\frac{\Delta v(z)}{\Delta t_0} = \frac{c_{\mathrm{lum}}\,\dot{z}}{1+z} = c_{\mathrm{lum}} H_{\mathrm{ref}}^{\mathrm{cal}} \left[1 - \frac{E_{\mathrm{stab}}(z)}{1+z}\right], \tag{5.17}$$

where $E_{\mathrm{stab}}(z)$ is defined by (4.7). This observable does not depend on $r_\star^{\mathrm{cal}}$ and consumes no directional observable. It does not enter the global calibration of the stabilized branch. It is therefore a post-calibration kinematic prediction: its observational confrontation belongs to future measurements of spectroscopic redshift drift, of Sandage–Loeb type.

In units $\mathrm{cm\,s^{-1}\,yr^{-1}}$, we first display the central LO value at the self-consistent point $A_0^\star$ of (4.11):

z	$E_{\mathrm{stab}}(z)$	$E_{\mathrm{stab}}(z)/(1+z)$	Δv [$\mathrm{cm\,s^{-1}\,yr^{-1}}$]
1.0	1.791	0.895	+0.218
2.0	2.998	0.999	+0.001
3.0	4.522	1.130	−0.272
4.0	6.299	1.260	−0.541
5.0	8.294	1.382	−0.796

Because this observable depends directly on the shape of $E_{\mathrm{stab}}(z)$, it must also be read with the envelope obtained from the profiled scan of A_0. On the scan points belonging to $\Delta\chi^2 < 1$, one obtains:

Table 5.1: Central LO value and scan envelope in A_0 for the Sandage–Loeb drift. Brackets indicate a scan envelope, not a symmetric statistical uncertainty.

z	Central value at the point $A_0^\star$	Envelope $\Delta\chi^2 < 1$
1.0	+0.218	$[+0.216, +0.219]$
2.0	+0.001	$[-0.005, +0.004]$
3.0	−0.272	$[-0.307, -0.251]$
4.0	−0.541	$[-0.610, -0.499]$
5.0	−0.796	$[-0.898, -0.732]$

This envelope does not reduce the admissible range of A_0. On the contrary, it shows that the Sandage–Loeb drift is a prospective target capable of later discriminating the admissible profiles of the scan in A_0, once direct redshift-drift measurements become available.

The nontrivial sign change is obtained by solving

$$E_{\mathrm{stab}}(z_\times) = 1 + z_\times. \tag{5.18}$$

With $x = 1 + z_\times$, this condition becomes

$$A_4^{\mathrm{stab}} x^4 + A_{3,\mathrm{sol}}^{\mathrm{stab}} x^3 + A_2^{\mathrm{stab}} x^2 + A_1^{\mathrm{stab}} x + A_0^{\mathrm{stab}} = x^2. \tag{5.19}$$

With the coefficients of the representative self-consistent point (4.11), the nontrivial root of this equation gives

$$z_\times \simeq 2.005. \tag{5.20}$$

Thus, at the self-consistent point $A_0^\star$, the HSMD kernel predicts a positive drift below $z \simeq 2.005$, then a negative drift in the high-redshift domain. The scan envelope in A_0 slightly shifts the amplitudes and the neighborhood of the sign change. Table 5.1 therefore provides a central LO observational target accompanied by its scan envelope, to be compared with future redshift-drift measurements expressed in $\mathrm{cm\,s^{-1}\,yr^{-1}}$.

5.0.4 Global geometric curvature of the hypersphere

The coefficient A_2^{stab} of the stabilized kernel (4.7) is an effective coordinate of the $(1+z)^2$ sector. It belongs to the late-time homogeneous closure selected by the calibration and must not be confused with the global geometric curvature associated with the finite radius R_0^{HSM} of the hypersphere.

The dimensionless quantity that measures the global geometric curvature in Hubble units is defined by

$$\varepsilon_{\mathrm{curv}}^{\mathrm{HS}} := \left(\frac{c_{\mathrm{lum}}}{R_0^{\mathrm{HSM}} H_{\mathrm{ref}}^{\mathrm{cal}}} \right)^2. \tag{5.21}$$

This expression compares the current Hubble scale $c_{\mathrm{lum}}/H_{\mathrm{ref}}^{\mathrm{cal}}$ with the global geometric radius R_0^{HSM}. The form of the ratio is the usual measure of a curvature scale expressed in Hubble units; its predictive content here comes from the fact that R_0^{HSM} is not a free FLRW curvature parameter, but a geometric scale inherited from the HSM/HSMD construction.

This quantity is structural: it is nonzero as soon as R_0^{HSM} is finite. It does not replace the effective coefficient A_2^{stab} of the $(1+z)^2$ sector, which belongs to the calibrated stabilized homogeneous kernel. It may be compared with an apparent curvature expressed in Hubble units, but it is not introduced as a free Ω_k parameter of a FLRW model.

The numerical substitution gives

$$H_{\mathrm{ref}}^{\mathrm{cal}} \simeq 67.91\ \mathrm{km\,s^{-1}\,Mpc^{-1}} = \frac{67.91 \times 10^3}{3.085677581 \times 10^{22}}\ \mathrm{s^{-1}} \simeq 2.20081 \times 10^{-18}\ \mathrm{s^{-1}}. \tag{5.22}$$

With

$$R_0^{\mathrm{HSM}} = 1.50841 \times 10^{29}\ \mathrm{m}, \tag{5.23}$$

one obtains

$$\frac{c_{\mathrm{lum}}}{R_0^{\mathrm{HSM}} H_{\mathrm{ref}}^{\mathrm{cal}}} = \frac{2.99792458 \times 10^8}{(1.50841 \times 10^{29})(2.20081 \times 10^{-18})} \simeq 9.0306 \times 10^{-4}, \tag{5.24}$$

then

$$\boxed{\varepsilon_{\mathrm{curv}}^{\mathrm{HS}} \simeq 8.16 \times 10^{-7}.} \tag{5.25}$$

The physical reading is therefore the following: HSMD inherits from HSM a global geometry of finite radius, which forbids strict identification with an infinite Euclidean geometry. The corresponding geometric effect is nevertheless of order 10^{-6} in the radius ratio relevant to the Hubble scale.

This prediction does not modify the evaluations derived from the stabilized kernel (4.7). The quantity $\varepsilon_{\rm curv}^{\rm HS}$ measures the global geometric curvature associated with the finite radius of the hypersphere; it does not replace the effective coordinate $A_2^{\rm stab}$, which is separately constrained in the calibration.

This quantity must not be confused with a free phenomenological parameter Ω_k fitted in an effective FLRW metric. Here it measures the global geometric curvature associated with the hyperspherical radius R_0 inherited from the HSM/HSMD construction. Its very small value implies that late-time distances can remain quasi-flat in the usual tests, even if the underlying global geometry is hyperspherical.

Thus, the main prediction of this channel is not a strong deviation of BAO or SNe distances, but the existence of a global curvature scale traced by

$$\varepsilon_{\rm curv}^{\rm HS} = \left(\frac{c_{\rm lum}}{R_0^{\rm HSM} H_{\rm ref}^{\rm cal}}\right)^2 .$$

5.1 Traceability table for prospective and structural outputs

Quantity	Status	Calculation chain
Effective trajectory $R(t)$, possible future turnaround and constant residue $A_0^{\rm stab}$	Structural output dependent on the A_0 branch	(5.1)–(5.6), 3.1
$\rho_{a^{-3},0}^{\rm HSMD} \simeq 3.03 \times 10^{-27}\ \mathrm{kg\,m^{-3}}$	Equivalent mass density, self-consistent branch	(5.9)–(5.12)
$u_{a^{-3},0}^{\rm HSMD} \simeq 2.72 \times 10^{-10}\ \mathrm{J\,m^{-3}}$	Equivalent energy density, self-consistent branch	(5.11)–(5.14)
Sandage–Loeb drift $\Delta v(z)$	Prospective target; central LO value at the point $A_0^\star$ and profiled scan envelope	(5.16)–(5.20), 5.1
Global geometric curvature $\varepsilon_{\rm curv}^{\rm HS} \simeq 8.16 \times 10^{-7}$	Central HSM/HSMD structural output, self-consistent branch	(5.21)–(5.25)

These outputs do not modify the HSM constants or the calibrated coordinates of the stabilized kernel. They specify what the calibrated branch implies beyond direct observational confrontations: effective geometric evolution, dimensional translation of the a^{-3} sector, future spectroscopic drift and global curvature scale.

The quantities in this table are central outputs of the stabilized kernel. The uncertainties from the global calibration are not propagated within this numerical scope. They must therefore not be read as results equipped with a full covariance.

The trajectory $R(t)$ describes the effective evolution in the stabilized regime. The position of the current radius relative to the natural radius is reconstructed from the elastic coordinate A_1, via

$$a_{\rm nat} = \frac{3}{A_1}.$$

In the self-consistent branch, this reading gives

$$a_{\rm nat}^\star \simeq 4.34.$$

The sign and value of the constant term (5.6) indicate how the balance between solitonic radial closure and homogeneous elastic restoring contributes to the asymptotic evolution within the scope of this branch.

The Sandage–Loeb drift is a prospective prediction: it provides a kinematic target to compare with future redshift-drift measurements. The global geometric curvature is a structural HSM/HSMD output; it is not presented as a current detection of cosmological curvature.

Part IV

Comparison with ΛCDM

Chapter 6

Comparative reading with the ΛCDM framework

6.1 Purpose of the comparison

The ΛCDM model and HSMD both describe cosmological background observables, but they do not decompose the dynamics in the same variables. The comparison must therefore be made at two distinct levels.

The first level is observational. It concerns the quantities directly reconstructed or predicted:

$$H(z), \qquad E(z) = \frac{H(z)}{H_0}, \qquad D_M(z), \qquad D_H(z), \qquad D_V(z), \qquad \mu(z), \qquad q_0, \qquad z_t.$$

These quantities can be compared directly between ΛCDM and HSMD.

The second level is interpretive. It concerns the coefficients that appear in the effective expansion law. In flat ΛCDM, one schematically writes

$$E^2_{\Lambda\mathrm{CDM}}(a) = \Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}.$$

In HSMD, the stabilized branch is written

$$E^2_{\mathrm{HSMD}}(a) = A_0 + A_1 a^{-1} + A_2 a^{-2} + A_{3,\mathrm{sol}} a^{-3} + A_4 a^{-4}.$$

Coefficients of the same rank in a may therefore be compared at the kinematic level, but they must not be identified term by term as equivalent physical components.

6.2 Effective dictionary of coefficients

The coefficient $A_{3,\mathrm{sol}}$ is the HSMD coefficient of the comoving component of rank a^{-3}. In the background law, it therefore plays a kinematic role analogous to that of Ω_m in ΛCDM. This analogy is not an identity of definition:

$$A_{3,\mathrm{sol}} \not\equiv \Omega_m^{\Lambda\mathrm{CDM}}.$$

In HSMD, $A_{3,\mathrm{sol}}$ denotes the stabilized solitonic component that contributes to the homogeneous background as a comoving density. Relativistic excitations, which dilute like radiation, are not included in this coefficient when they belong to rank a^{-4}.

Likewise, the coefficient A_0 occupies the constant rank in $E^2(a)$, but it must not be identified with a fundamental cosmological constant. In HSMD, the constant term results from the balance between the radial closure associated with the comoving solitonic component and the homogeneous elastic restoring term:

$$A_0 = A_{3,\mathrm{sol}} - \frac{A_1^2}{6}.$$

Thus,

$$A_0 \not\equiv \Omega_\Lambda.$$

It plays an analogous kinematic role in the effective expansion law, but its physical origin is different.

The coefficient A_1 has no equivalent in the minimal ΛCDM model. It carries the trace of the global elastic restoring term and makes it possible to reconstruct the position of the current radius relative to the natural radius:

$$a_{\mathrm{nat}} = \frac{3}{A_1}.$$

The self-consistent branch gives

$$a_{\mathrm{nat}}^{\star} \simeq 4.34.$$

This reading is specific to HSMD: it relates the observed expansion to a geometrical position in the radial dynamics of the stabilized hypersphere.

6.3 Reading of the H_0 discrepancy

In ΛCDM, the discrepancy on H_0 is formulated as a tension between a global value inferred from background observables and a local value reconstructed from distance ladders [20]. In HSMD, these two values are not interpreted as two strictly identical measurements of the same homogeneous scalar.

The calibrated background branch fixes

$$H_{\mathrm{ref}}^{\mathrm{cal}} \simeq 67.91\ \mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{6.1}$$

This value describes the stabilized background rate. It is comparable to the global values reconstructed by background channels.

In HSMD, the apparent local value of H_0 is interpreted as an output of the local quadratic projection of the global rotation. The PMNS channel fixes the effective latitude of the observer,

$$\chi_{\mathrm{obs}} \simeq 66.4^\circ. \tag{6.2}$$

At this latitude, the radius projected into the rotation plane is

$$R_\perp = R_0 \cos\chi_{\mathrm{obs}}, \tag{6.3}$$

and the local quadratic projection factor is

$$M_{\mathrm{loc}} = \left(\frac{R_\perp}{R_0}\right)^2 = \cos^2\chi_{\mathrm{obs}}. \tag{6.4}$$

The primary correction acts on the squared rate:

$$\frac{\left(H_0^{\mathrm{loc,pred}}\right)^2}{\left(H_{\mathrm{ref}}^{\mathrm{cal}}\right)^2} = 1 + D_{\mathrm{rot}} M_{\mathrm{loc}}. \tag{6.5}$$

It follows that

$$H_0^{\mathrm{loc,pred}} = H_{\mathrm{ref}}^{\mathrm{cal}} \sqrt{1 + D_{\mathrm{rot}} \cos^2\chi_{\mathrm{obs}}}. \tag{6.6}$$

The linearized form is

$$\frac{H_0^{\rm loc,pred}}{H_{\rm ref}^{\rm cal}} - 1 \simeq \frac{D_{\rm rot}}{2}\cos^2\chi_{\rm obs}. \tag{6.7}$$

With

$$H_{\rm ref}^{\rm cal} = 67.91\ {\rm km\,s^{-1}\,Mpc^{-1}}, \qquad D_{\rm rot} = 0.933, \qquad \chi_{\rm obs} \simeq 66.4^\circ,$$

one obtains

$$H_0^{\rm loc,pred} \simeq 72.8\ {\rm km\,s^{-1}\,Mpc^{-1}}. \tag{6.8}$$

This value is to be compared with the local value published by SH0ES/Pantheon+ [11], used here as an external observational reference:

$$H_0^{\rm loc,obs} = 73.04 \pm 1.04\ {\rm km\,s^{-1}\,Mpc^{-1}}.$$

This writing relies on two steps internal to the HSMD framework. The first is geometrical: $\chi_{\rm obs}$ is a latitude relative to the rotation plane, so that $R_\perp = R_0\cos\chi_{\rm obs}$ and the projected quadratic rotational term is $M_{\rm loc} = \cos^2\chi_{\rm obs}$. The second connects the sectors: the Dirac phase observed in the PMNS channel is interpreted as the local projection of the global rotation,

$$\delta_{\rm CP}^{\rm obs} = 2\pi A_{\rm rot}^{\rm glob}\cos\chi_{\rm obs}.$$

This closure does not mean that PMNS directly measures a cosmological coordinate in the standard observational sense; it provides an angular projection whose local consequence is tested by $H_0^{\rm loc,pred}$. The local SH0ES/Pantheon+ value is not consumed to fix $\chi_{\rm obs}$; it serves only as a downstream comparison.

HSMD thus provides a geometrical reading of the H_0 discrepancy. The global background value and the apparent local value do not play the same role: the first describes the stabilized homogeneous branch; the second is reproduced as an output of the local projection of the global rotation. This reading consumes neither an HSC amplitude nor the Pantheon+SH0ES mask kernel.

6.4 Statistical background comparison with ΛCDM

The quantitative comparison with ΛCDM is made here at the cosmological background level used in the calibration. It does not constitute a complete comparison of perturbations, growth, or lensing.

We write

$$\Delta\chi^2 = \chi^2_{\Lambda{\rm CDM}} - \chi^2_{\rm HSMD}.$$

The background fit gives

$$\Delta\chi^2 \simeq 4.5, \tag{6.9}$$

in favor of HSMD on the considered dataset. The penalized comparison must, however, account for the fact that the HSMD branch consumes one additional effective parameter within this perimeter:

$$\Delta k = k_{\rm HSMD} - k_{\Lambda{\rm CDM}} = 1. \tag{6.10}$$

With the convention

$$\Delta{\rm IC} = {\rm IC}_{\rm HSMD} - {\rm IC}_{\Lambda{\rm CDM}},$$

one obtains

$$\Delta{\rm AIC} = -\Delta\chi^2 + 2\Delta k, \tag{6.11}$$

$$\Delta{\rm BIC} = -\Delta\chi^2 + \Delta k \ln N, \tag{6.12}$$

Criterion	Expression	Value / reading
Raw gain in χ^2	$\Delta\chi^2$	4.5
Additional effective parameter	Δk	1
AIC	$-4.5 + 2$	-2.5 : slightly favors HSMD
Standard BIC	$-4.5 + \ln(1624)$	$+2.9$: slightly favors ΛCDM

Table 6.1: Penalized comparison of the HSMD background branch with ΛCDM. The signs are given for $\mathrm{IC}_{\mathrm{HSMD}} - \mathrm{IC}_{\Lambda\mathrm{CDM}}$: a negative value favors HSMD, whereas a positive value favors ΛCDM.

where N denotes the number of data points in the background dataset used in the comparison. For $N \simeq 1624$, one has $\ln N \simeq 7.39$.

This table does not show a decisive statistical superiority of HSMD over ΛCDM. The AIC slightly favors HSMD, whereas the standard BIC slightly favors ΛCDM. The conclusion is therefore a statistically overall neutral match at the background level. The main contribution of HSMD is not a massive statistical gain on the background, but the geometrical interpretation of the effective coefficients and the linking of the H_0, PMNS, HSC, and CMB channels.

6.5 Constant term and reading of dark energy

In ΛCDM, cosmological acceleration is carried by a constant term interpreted as a cosmological constant or effective dark energy, in the continuation of the supernova observations that established the late-time acceleration of the expansion [21, 22]. The model therefore introduces a component Ω_Λ whose density remains constant as the universe expands.

HSMD does not remove the need for a constant term in the effective expansion law. However, this term is not introduced as an independent fundamental component. It appears as the residue of a balance:

$$A_0 = A_{3,\mathrm{sol}} - \frac{A_1^2}{6}.$$

The HSMD constant term therefore expresses the difference between a comoving solitonic closure contribution and a homogeneous elastic restoring contribution.

This difference is one of the main contributions of HSMD relative to ΛCDM. The effective acceleration is not merely parameterized by a fluid added to the cosmological background; it is related to the geometrical and elastic structure of the stabilized hypersphere.

6.6 Specific contributions of HSMD

The comparison highlights several specific contributions of HSMD.

First, HSMD explicitly distinguishes the background value $H_{\mathrm{ref}}^{\mathrm{cal}}$ and the oriented local branch fixed by H_0^{loc}. This distinction provides an internal reading of the H_0 discrepancy, without identifying the local distance channel with a simple redundant measurement of the homogeneous background.

Second, HSMD gives a structural origin to the constant term in the expansion law. The term A_0 is not postulated as an independent dark energy; it is related to $A_{3,\mathrm{sol}}$ and A_1 by a consistency condition.

Third, HSMD introduces an elastic component of rank a^{-1}, absent from the minimal ΛCDM model. This component makes it possible to reconstruct a position relative to the natural radius a_{nat}, thereby giving a geometrical interpretation of the current state of the universe.

Fourth, the comoving density of rank a^{-3} is read as a stabilized solitonic component. It plays the role of a gravitational background density within the homogeneous perimeter. Growth and lensing observables belong to the perturbative sector and are not used as constraints in this chapter.

6.7 Scope of the comparison

The comparison presented here concerns the stabilized cosmological background and the effective coefficients of the same rank in $E^2(a)$. It establishes a background dictionary between HSMD and ΛCDM, without identifying the HSMD coefficients with fundamental ΛCDM fluids.

Growth and lensing observables,

$$\sigma_8, \qquad S_8, \qquad f\sigma_8(z), \qquad P(k), \qquad C_\ell^{\text{lensing}},$$

are not used here as independent constraints on $A_{3,\text{sol}}$ or on A_0. They belong to the perturbative sector of HSMD, distinct from the homogeneous dictionary treated in this chapter.

The conclusion of this chapter is therefore limited but clear: HSMD reproduces the level of description of the cosmological background while proposing a different interpretation of the cosmological terms. The penalized comparison (6.9)–6.1 shows a statistically competitive HSMD branch, without decisive superiority: the AIC slightly favors HSMD, whereas the standard BIC slightly favors ΛCDM. The background quantities are comparable to those of ΛCDM, but the HSMD coefficients are not simple renamings of Ω_m and Ω_Λ.

Chapter 7

General conclusion of HSMD

7.1 General assessment

HSMD extends HSM by introducing a global cosmological dynamics of the meshed hypersphere. The central point is not to replace the structural constants of HSM, but to describe the collective evolution of the global geometry from a calibrated dynamical core. The HSM constants remain the structural constants; HSMD adds global branch, rotation and collective-content conditions that make it possible to produce an effective cosmological trajectory.

In this construction, late-time expansion is not postulated as an independent FLRW dynamics. It is read as the effective response of a compact, meshed, elastic and globally rotating hypersphere. The current radius R_0 and the current mesh ℓ_0 are those of the present state; they must not be confused with a natural radius or a rest state of the structure. HSMD dynamics therefore distinguishes the present state, the stabilized branch and the position of the radius relative to its natural radius.

The late-time calibration leads to a stabilized core capable of reproducing the BAO observables used in the calibration, and then of generating post-calibration outputs. These outputs do not all have the same status: some are current observational confrontations, others are structural outputs, and others still are prospective predictions. This distinction is essential in order not to confuse calibration, confrontation and future falsifiability.

7.2 Stabilized results

The stabilized core of HSMD provides a coherent late-time branch in which the main cosmographic quantities are computed from the BAO calibration. The reference Hubble constant, the coefficients of the dynamical core and the effective constant term together define the branch used in the post-calibration predictions.

The selected branch organizes in particular the present deceleration, the transition redshift, the cosmological age and the equivalent densities associated with the matter-like dilutive sector. These quantities are outputs of the calibrated branch, not independent constants.

The structure of the effective constant term is also clarified. In HSMD, this residue measures the balance between radial solitonic closure and homogeneous elastic restoring. It is therefore not merely a fit parameter: it carries information on the global equilibrium of the dynamical branch. The minimal compensated branch describes the case in which this balance is close to an effective compensation, while the profiles of the constant term are used to explore the sensitivity of the late-time outputs.

7.3 Rotational closure and multi-channel tests

The rotational closure is one of the most structuring results of the document. In HSM, the internal geometry of neutrino-like solutions provides the real PMNS mixing angles. At the order where no global dynamical orientation is introduced, the effective PMNS matrix can be taken to be real: the Jarlskog invariant is then zero and the Dirac phase is reduced to $\delta_{\mathrm{CP}} = 0$ or π.

By contrast, for the Dirac phase to depart from the trivial real values 0 and π, an additional dynamical orientation is required. The experimental uncertainties are still broad: the neighbourhood of π is not excluded at the current order. The rotational closure must therefore be read as a testable dynamical consequence of the HSMD framework, conditional on the future stabilization of δ_{CP}. In HSMD, the dynamical orientation able to carry a non-trivial phase is supplied by the global rotation of the hypersphere. The proposed closure then identifies the observed Dirac phase with a rotational phase projected onto the effective angular position of our observation domain.

This relation has the status of an HSMD closure of the complex phase left open by the real PMNS structure. Its downstream consequences are computed without readjusting the angular position, which directly fixes its falsification conditions.

Once the angular position is fixed by the PMNS channel, three downstream confrontations are obtained without readjusting this angle: the apparent local Hubble value, the CMB birefringence and the HSC/spin channel.

These three confrontations are not three independent angular levers. Two geometrical dependences enter: a quadratic branch, common to the apparent local Hubble value and to the HSC/spin channel, and a transverse branch, carried by CMB birefringence. The HSC channel therefore does not provide a new angular dependence relative to the local Hubble value, but it provides an independent directional data set on the same geometrical branch.

The status of the PMNS–rotation closure is that of a multi-channel compatibility. A single projection derived from PMNS feeds two angular branches and three distinct data sets. The result is therefore not an isolated coincidence on the local Hubble value, but a confrontable geometrical closure. It is falsifiable by an incompatible stabilization of the measurements of the Dirac phase, of CMB birefringence, of the local Hubble value or of galaxy-spin catalogues.

7.4 What is resolved, what is explained, what remains prospective

HSMD should not be read as a collection of numerical coincidences. The document distinguishes three levels of results.

The first level is that of calibrated quantities. The BAO data used in the calibration fix the late-time core and must not be counted again as independent predictions. They define the working branch.

The second level is that of confrontable post-calibration outputs. This is the case of the apparent local Hubble value, CMB birefringence and the HSC/spin channel in its active window. These outputs consume the calibrated branch and the PMNS–rotation closure, and are then compared with distinct observables. Their status is that of a multi-channel compatibility, not that of a complete statistical proof.

The third level is that of structural or prospective outputs. The Sandage–Loeb drift, the effective trajectory $R(t)$, the position of the current radius relative to the natural radius and the global curvature scale are consequences of the stabilized branch. They provide future targets or internal diagnostics, but they are not presented as current observational detections.

This separation of statuses is an important methodological result of the document. It prevents the concordances from being overinterpreted, but it also makes the predictions falsifiable:

improved data may reinforce the selected branch, shift it or invalidate it.

7.5 Validity conditions and confrontation points

The validity conditions of the selected branch are explicit.

First, the PMNS–rotation closure depends on the current value of the Dirac phase, whose measurement is still broad. A more precise future measurement may shift the effective angular position, and therefore simultaneously modify the apparent local Hubble value, CMB birefringence and the HSC/spin channel.

Second, the CMB channel is compatible with the HSMD prediction in the current state of birefringence measurements, but it remains close to the geometrical ceiling of the model. A future value stabilized clearly above this ceiling would put this branch of the rotational closure under pressure.

Third, the HSC/spin channel remains a delicate directional observable. The confrontation concerns the raw slope measured in the active window where the spin axis is freely detected. This output tests the directional coherence of the PMNS–rotation closure in a galaxy catalogue, while remaining sensitive to selection effects, morphological classification and the redshift heterogeneity of the signal.

Fourth, the profiles of the constant term show that some late-time outputs depend on how the balance between elastic restoring and radial closure is closed. The document selects a self-coherent branch, but does not claim that all dynamical variants are already excluded.

7.6 Falsifiability and natural continuations

The falsifiability of HSMD rests on several independent paths.

A first path is the future measurement of the Dirac phase. If this phase stabilizes far from the value consumed here, the effective angular position will be shifted, and the three downstream outputs will have to change simultaneously. The PMNS–rotation closure is therefore directly testable by future neutrino experiments.

A second path is CMB birefringence. The transverse branch of the PMNS–rotation closure imposes an amplitude and a geometrical ceiling. A future measurement that is robust, stable and incompatible with this ceiling would constitute a severe test of the rotational closure.

A third path is the HSC/spin channel, or any equivalent catalogue. The relevant test is not a global average over a heterogeneous catalogue, but the search for an active window in which a free axis is recovered, followed by the measurement of the raw slope on the fixed axis. New catalogues of galaxy morphologies may confirm, shift or invalidate this signature.

A fourth, prospective path is the Sandage–Loeb drift. It does not directly test the PMNS–rotation closure, but it tests the late-time kinematic branch and the stabilized core. It therefore constitutes a future target for distinguishing HSMD from neighbouring effective cosmological scenarios.

7.7 Conclusion

HSMD proposes a dynamical reading of cosmology as a continuation of HSM: the observable Universe is described as the late-time state of a compact, elastic, meshed and globally rotating hypersphere. The late-time calibration fixes an effective branch; the post-calibration predictions then organize the cosmographic, directional and prospective outputs.

The central result of the document is a structured coherence: the real PMNS geometry provided by HSM leaves open the complex Dirac phase, while the global HSMD rotation provides the dynamical closure of this phase consumed in the branch studied. The effective angular position obtained through this closure feeds several downstream confrontations, without readjusting the angle. The current outputs are compatible with this angular position, while remaining sufficiently flexible not to constitute a definitive closure.

The status of HSMD at this stage is therefore that of a dynamic, calibrated, coherent and falsifiable cosmological framework. It provides an exploitable late-time branch, non-circular directional predictions and future observational targets. The confirmation or invalidation of the PMNS–rotation closure, of CMB birefringence, of the HSC/spin channel and of the future spectroscopic drift will make it possible to decide whether this branch describes only a coherent effective parametrization, or a deeper geometrical structure of cosmological dynamics.

Bibliography

[1] HSM Research. HSM – Hypersphere Lattice Model: A Geometric and Solitonic Construction of Fields, Particles and Effective Constants. Zenodo, version 1.0. https://doi.org/10.5281/zenodo.20490570, 2026.

[2] NIST. CODATA Value: speed of light in vacuum. https://physics.nist.gov/cgi-bin/cuu/Value?c=, 2024. CODATA recommended values of the fundamental physical constants: 2022.

[3] NIST. CODATA Value: Newtonian constant of gravitation. https://physics.nist.gov/cgi-bin/cuu/Value?bg=, 2024. CODATA recommended values of the fundamental physical constants: 2022.

[4] DESI Collaboration. DESI DR2 Results II: Measurements of Baryon Acoustic Oscillations and Cosmological Constraints. *Physical Review D*, 112:083515, 2025.

[5] DESI Collaboration. DESI DR2 Results I: Baryon Acoustic Oscillations from the Lyman Alpha Forest. 2025.

[6] Dan Scolnic et al. The Pantheon+ Analysis: The Full Dataset and Light-Curve Release. *The Astrophysical Journal*, 938(2):113, 2022.

[7] Dillon Brout et al. The Pantheon+ Analysis: Cosmological Constraints. *The Astrophysical Journal*, 938(2):110, 2022.

[8] Michele Moresco, Lucia Pozzetti, Andrea Cimatti, Raul Jimenez, Claudia Maraston, Licia Verde, Daniel Thomas, Annalisa Citro, Rita Tojeiro, and David Wilkinson. A 6% measurement of the Hubble parameter at $z \simeq 0.45$: direct evidence of the epoch of cosmic re-acceleration. *Journal of Cosmology and Astroparticle Physics*, 2016(05):014, 2016.

[9] Michele Moresco. Addressing the Hubble tension with cosmic chronometers. 2023.

[10] Ivan Esteban, M. C. Gonzalez-Garcia, Michele Maltoni, Ivan Martinez-Soler, João Paulo Pinheiro, and Thomas Schwetz. NuFit-6.0: Updated global analysis of three-flavor neutrino oscillations. *Journal of High Energy Physics*, 2024(12):216, 2024.

[11] Adam G. Riess, Wenlong Yuan, Lucas M. Macri, Dan Scolnic, Dillon Brout, Stefano Casertano, David O. Jones, Yukei Murakami, Louise Breuval, Thomas G. Brink, Alexei V. Filippenko, Samantha Hoffmann, Saurabh W. Jha, W. D'Arcy Kenworthy, Gagandeep Anand, John Mackenty, Benjamin E. Stahl, and Weikang Zheng. A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km s^{-1} Mpc^{-1} Uncertainty from the Hubble Space Telescope and the SH0ES Team. *The Astrophysical Journal Letters*, 934:L7, 2022.

[12] Hiroaki Aihara et al. Third data release of the hyper suprime-cam subaru strategic program. *Publications of the Astronomical Society of Japan*, 74:247, 2022.

[13] Lior Shamir. Asymmetry in galaxy spin directions: A fully reproducible experiment using hsc data. *Symmetry*, 16(10):1389, 2024.

[14] Richard Stiskalek and Harry Desmond. Symmetry in hyper suprime-cam galaxy spin directions. *Research Notes of the AAS*, 8:281, 2024.

[15] Planck Collaboration. Planck 2018 results. VI. Cosmological parameters. *Astronomy & Astrophysics*, 641:A6, 2020.

[16] Yuto Minami and Eiichiro Komatsu. New Extraction of the Cosmic Birefringence from the Planck 2018 Polarization Data. *Physical Review Letters*, 125(22):221301, 2020.

[17] P. Diego-Palazuelos, J. R. Eskilt, Y. Minami, M. Tristram, R. M. Sullivan, A. J. Banday, R. B. Barreiro, H. K. Eriksen, K. M. Górski, R. Keskitalo, E. Komatsu, E. Martínez-González, D. Scott, P. Vielva, and I. K. Wehus. Cosmic Birefringence from the Planck Data Release 4. *Physical Review Letters*, 128(9):091302, 2022.

[18] Allan Sandage. The Change of Redshift and Apparent Luminosity of Galaxies due to the Deceleration of Selected Expanding Universes. *The Astrophysical Journal*, 136:319–333, 1962.

[19] Abraham Loeb. Direct Measurement of Cosmological Parameters from the Cosmic Deceleration of Extragalactic Objects. *The Astrophysical Journal Letters*, 499(2):L111–L114, 1998.

[20] Licia Verde, Tommaso Treu, and Adam G. Riess. Tensions between the Early and the Late Universe. *Nature Astronomy*, 3:891–895, 2019.

[21] Adam G. Riess et al. Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. *The Astronomical Journal*, 116:1009–1038, 1998.

[22] S. Perlmutter et al. Measurements of Ω and Λ from 42 High-Redshift Supernovae. *The Astrophysical Journal*, 517:565–586, 1999.