HSM – Meshed Hypersphere Model

A geometric and solitonic construction of fields, particles, and effective constants

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General Introduction

This document is an English translation of the French reference version of HSM. The French version remains the canonical scientific text. In case of ambiguity, the French formulation shall be considered authoritative.

Genesis of the construction. The starting point of this work does not come from an abstract attempt to unify existing physical theories, but from a simpler observation drawn from nonlinear mechanical systems.

Some nonlinear classical systems already show that a vibrational excitation is not necessarily bound to disperse. The sine–Gordon equation, an emblematic nonlinear differential equation of classical soliton theory [18, 19, 20], provides such an example: it admits localized soliton-type solutions, interactions between internal orientations, and bound states of breather type.

These systems also show that an interaction between solutions can emerge naturally without being postulated as an external force: attraction or repulsion depending on the internal orientation, low-energy bound states (*breathers*), and non-massive propagating modes whose energy can appear in the form of localized packets.

These observations raise a natural question: if a classical mechanical system can already produce localized solutions carrying energy, effective inertia, interactions and bound states, how far can this idea be pushed when the support is no longer an open chain, but a closed geometry such as a sphere or a hypersphere?

The founding mechanical idea. The framework constructed in this manuscript starts from a hypersphere $\mathbb{S}^3$ endowed with an internal meshed mechanical structure capable of carrying families of solutions more structured than those encountered in one-dimensional solitonic systems. This structure may be pictured as a network of inertial elements coupled by mechanical links: these elements can vibrate tangentially to the support, locally describe rotational motions in tangent planes and, because of the curvature of the hypersphere, induce a radial deformation of it.

This deformation locally modifies the mechanical tension state and introduces a geometric nonlinearity. Two consequences then appear naturally. On the one hand, the internal sense of rotation can generate oriented interactions between solutions. On the other hand, a localized solution that radially deforms the support modifies the mechanical tension state in its neighborhood. Two nearby solutions can then interact through this common deformation of the support, which provides a natural attraction mechanism linked to the amplitude of the radial response.

The support is then no longer a mere geometric background: it becomes an actor in the dynamics.

Why a compact hypersphere? A second idea plays a central role: the compactness of the support.

On an open space, a localized excitation can often be studied as an isolated object, without a global closure constraint. On a compact support, this freedom disappears. Fields and solutions

must reconnect to themselves, satisfy global return conditions and fit into a spectral structure imposed by the geometry. Even when the solutions are nonlinear, their existence remains constrained by the eigenmodes of the support, the matching conditions and the admissible levels associated with relooping.

But compactness produces more than a mere energy discretization. It modifies the very nature of the admissible solutions. Configurations with several localized centers, spatially distinct but linked by the relooping conditions of the support, then become admissible solutions of the system.

The initial study of these mechanisms on spherical geometries naturally leads to the hypersphere $\mathbb{S}^3$. Whereas an ordinary sphere essentially constrains tangential rotations to a two-degree-of-freedom structure, $\mathbb{S}^3$ opens additional classes of solutions with two or three degrees of freedom. Geometry therefore does not merely set the background; it directly enriches the dynamic catalogue of possible solutions.

The HSM proposal. HSM (*Meshed Hypersphere*) is first the study of a compact mechanical system defined on the hypersphere $\mathbb{S}^3$. The starting point is not the immediate identification of this system with the physical Universe, but the analysis of the solutions that such a mechanics can carry when the support has tangential degrees of freedom, radial deformations, global returns, localized matchings and an internal mesh structure.

Within this framework, the primary question is purely mechanical: which families of solutions emerge from a compact support where tangential motions can induce a radial response, where compactness imposes return conditions, and where internal rotations provide additional orientation degrees of freedom?

The radial–tangential coupling specific to the hyperspherical support locally leads to a nonlinear differential equation of cubic Klein–Gordon type, whose localized solutions provide the first families of HSM solutions.

This dynamics admits several structured families: bump-type solutions, NSL solutions, propagating perturbative modes, purely radial solutions, configurations with two or three degrees of freedom, multi-center bound solutions and composite structures.

The massive families follow an energy law of the form

$$E = E_0\,\gamma, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c_t^2}},$$

where v is the effective propagation speed of the solution and c_t is the maximum tangential propagation speed in the mesh.

More generally, the constructed families share the same effective action constant h_{eff} in a common law

$$E = h_{\text{eff}}\,\omega.$$

Internal rotation also makes it possible to define a geometric spin number, which naturally classifies the solutions into spin-1/2, spin-1, or spin-0 families for certain purely radial excitations.

Only after this mechanical catalogue of solutions has been built does the physical question arise: can the properties that emerge from HSM be put into correspondence with certain structures observed in known physics?

What HSM brings that is new. HSM is a deterministic theory at the fundamental level: it starts from a classical mechanical dynamics on $\mathbb{S}^3$, and not from a primary probabilistic law. In the regimes studied, certain structures usually associated with quantum mechanics appear as effective readings of the underlying dynamics, and not as starting postulates.

The novelty of HSM is not to add an interaction or an additional phenomenological sector to known physics. It is to propose a single mechanical framework in which several structures usually postulated separately appear as properties of solutions on $\mathbb{S}^3$: relativistic-type effective inertia, common action constant, geometric spin, oriented charges, interactions between localized solutions, bound states, non-massive modes and composite structures.

Even before numerical calibration, the analysis of HSM solutions reveals a discrete architecture that is not imposed as initial data: reduced field-charge coefficients for the charged families, the motifs 1, 2/3 and 1/3 transmitted by the source–field pairing of the channel associated with the electromagnetic field, a dimensionless fine-structure constant close to 1/137 arising from the closure of the same channel, three admissible rest energy levels for the families associated with leptons and quarks, and a single level for the solutions associated with the W and Z^0 bosons.

The same analysis also reveals four identifiable interaction channels, in the appropriate regimes, with the gravitational, electromagnetic, weak and strong sectors. The photon-like sector appears in particular as a non-massive transverse mode, with two physical polarizations, compatible at leading order with a Maxwell-type reading.

The compactness of $\mathbb{S}^3$ also allows global composite solutions, some of which realize several localized centers within a single state. In the sector associated with non-massive modes, this property supports the analysis of non-factorizable correlations.

In the usual formulations of the Standard Model, particle masses are largely introduced through parameters or couplings determined experimentally, rather than deduced from a single common mechanics. HSM proposes another possibility: a small number of calibration inputs fix the mechanical scales and certain structural thresholds, then the masses of the identifiable solutions are calculated from the internal dynamics of the model.

The mechanical catalogue of HSM is broader than the current inventory of the identified states of the Standard Model. Some families find a natural correspondence with observed structures; others remain without direct experimental identification and constitute candidates for new predictions of the framework.

Calibration, outputs and falsifiability. The numerical comparison of HSM relies on a limited number of calibration inputs, all explicitly declared. The observables declared as calibration inputs are:

- Planck's constant, to fix the effective action scale,
- the speed of light, to fix the maximum propagation speed,
- the mass of the W boson, to fix the weak scale,
- the vacuum impedance, to fix the internal electromagnetic scale,
- Newton's constant, to fix the global gravitational scale,
- the hierarchy of charged-lepton masses, to fix the spectral triplet.

Once these inputs have been declared, the closed chains of the manuscript produce quantitative outputs comparable to observed reference values:

- the mass of the HSM solution associated with the electron,
- the mass of the neutral HSM solution associated with the Z^0,
- the mass of the radial sector associated with the Higgs boson,
- the masses of HSM composites identifiable with observed baryons,
- the mass splittings of the sector associated with neutrinos,
- the internal angles of the tri–breather composite, subsequently connected to the PMNS structure,
- the elementary quantum of charge deduced from electromagnetic closure,
- the properties of the solution associated with the photon: zero mass at leading order and two transverse

In each numerical chain, the status of a value is explicitly distinguished. A quantity consumed as a calibration input is not presented as an independent prediction. Conversely, a quantity displayed as a prediction or numerical output must be traceable to an equation, a closure, a hypothesis or a clearly announced normalization convention.

This discipline also applies to approximation levels. The manuscript distinguishes results obtained at leading order, higher-order refinements, independent predictions, calculated theoretical outputs, branch-selection diagnostics and the associated domains of validity.

Organization of the manuscript. The document is organized into three main parts.

The first part constructs the theoretical framework. It introduces HSM mechanics on $\mathbb{S}^3$, the equations of motion, the elementary solutions with two and three degrees of freedom, the associated effective fields, the field–solution couplings, and then the composites with two and three degrees of freedom.

The second part gathers the active observational calibrations, the closed products, the separated constants and the numerical predictions. It distinguishes the inputs used to fix a scale or a structural product, the outputs obtained without recourse to the compared observable, the calculated theoretical outputs and the branch-selection diagnostics. The synthesis tables should be read as a traceability map of the closure chains, and not as a homogeneous list of numbers with the same status.

The constructive realization of the mesh takes place after the separation of the HSM constants. It is not used to calibrate the observational predictions; it establishes the compatibility of a discrete structure with the separated constants and the retained spectral constraints.

The third part establishes the link with effective physical theories. It relies on the elements constructed in the theoretical part and shows how certain regimes of HSM can be read as effective domains: reduced quantum mechanics, local effective field theory, relativistic kinematics, the gravitational limit and low-energy readings. Each link is presented with its domain of validity and its logical status.

Scope. The manuscript presents the closure chains that produce calculated outputs, predictions comparable to observables and branch-selection diagnostics. Each result is associated with its approximation rank, its domain of validity and the equations that define it.

The objective is to provide a theory that can be checked line by line: every result must be connectable to an equation, a closure, a normalization, a calibration input or an explicitly identified hypothesis.

In this perspective, the value of the document does not rest on a vocabulary analogy with existing theories, but on the logical continuity between the internal equations, the solutions, the fields, the closures, the traceable numerical outputs and the observational comparisons.

Part I

HSM theory: deductive construction

Chapter 1

General framework and assumptions of the HSM model

1.1 Guiding principle: classical mechanics on $\mathbb{S}^3$

The starting point of HSM is the postulate of a fundamental geometric support: a 3-sphere $\mathbb{S}^3$ of reference radius R_0, that is, the hypersphere embedded in $\mathbb{R}^4$. The ambient space $\mathbb{R}^4$ is used here only as a tool for geometric representation; the fundamental dynamics is formulated on the manifold $\mathbb{S}^3$ itself.

This structure is not introduced as a reformulation of the usual frameworks of relativistic or quantum physics, but as the fundamental carrier manifold of the model.

On this manifold, the fundamental dynamics is postulated to fall exclusively within classical continuum mechanics. The time parameter t is an absolute time in the sense of the fundamental model: no primitive Lorentzian structure is postulated at this stage.

The fundamental equations are assumed to be invariant under time translation and under the action of the isometry group of $\mathbb{S}^3(R_0)$.

The quantity c_t introduced later denotes a characteristic propagation speed of elastic propagation in the medium, determined by its mechanical parameters. It is not postulated as a universal kinematic invariant.

Description frames. We shall distinguish an observation frame attached to the reference configuration on $\mathbb{S}^3(R_0)$, and comoving descriptions associated with localized solutions in motion. These descriptions correspond to choices of variables and frames on one and the same support manifold; they introduce no relativistic kinematic structure at the fundamental level.

Standard mathematical results. The developments below use classical mathematical objects: the canonical metric of $\mathbb{S}^3$, intrinsic operators, the spectrum of the Laplacian, hyperspherical harmonics, Gegenbauer polynomials, and Green functions associated with elliptic operators of Helmholtz or Yukawa type. When these results are used in their standard form without complete rederivation, they fall under the usual mathematical references [13, 14, 15]. The constants, normalizations, and kernels specific to HSM, however, are defined and closed by the equations and internal labels of the manuscript.

1.2 Reference geometry, metric, and basic operators

The reference sphere $\mathbb{S}^3(R_0)$ is endowed with its canonical Riemannian metric. In the standard hyperspherical coordinates (χ, θ, ϕ), with

$$(\chi, \theta, \phi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi],$$

the metric is

$$ds^2 = R_0^2\Big(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi\, \sin^2\theta\, d\phi^2\Big).$$

The associated Riemannian volume measure is

$$dV = \sqrt{\det g}\, d\chi\, d\theta\, d\phi = R_0^3 \sin^2\chi\, \sin\theta\, d\chi\, d\theta\, d\phi.$$

Unless explicitly stated otherwise, integrals are understood over the reference sphere:

$$\int_{\mathbb{S}^3(R_0)} (\cdot)\, dV.$$

Unless stated otherwise, all differential operators are understood intrinsically on $\mathbb{S}^3(R_0)$. The basic operators are the intrinsic gradient ∇ and the Beltrami Laplacian $\Delta_{\mathbb{S}^3}$ associated with this metric. For a scalar field u, in coordinates one has

$$\Delta_{\mathbb{S}^3} u = \frac{1}{R_0^2}\left(\partial_{\chi\chi} u + 2\cot\chi\, \partial_\chi u + \frac{1}{\sin^2\chi}\Big[\partial_{\theta\theta} u + \cot\theta\, \partial_\theta u + \frac{1}{\sin^2\theta}\partial_{\phi\phi} u\Big]\right).$$

These expressions form the geometric basis for the subsequent developments. The detailed displacement fields, the specialized mechanical operators, and the effective energies will be introduced later, after setting up the discrete construction and the passage to the continuum.

1.3 Classical kinematics, velocities, and material derivative

We shall distinguish two levels of kinematics:

- a *global kinematics* of large-scale transport on $\mathbb{S}^3$, characterized by a typical rate Ω. When a pattern is advected with a characteristic tangential velocity v on the geometric scale R_0, one introduces the kinematic order of magnitude

$$v = O(\Omega R_0).$$

- a *fast local kinematics* of oscillation, described by a characteristic angular frequency ω_0 associated with the internal dynamics of localized structures.

These two levels of description generally correspond to distinct scales: slow transport on the geometric scale of the support, and fast dynamics on the local scale of the solutions.

The material derivative acting on a scalar field u carried by the medium is defined by

$$D_t u = \partial_t u + \vec{v} \cdot \nabla u,$$

where $\vec{v}(\chi, \theta, \phi, t)$ denotes the material velocity field of the medium, understood as a tangent field on $\mathbb{S}^3$. In this notation, $\vec{v} \cdot \nabla u$ denotes the intrinsic directional derivative of u along the velocity field. For vector or tensor fields, the corresponding covariant material derivative will be introduced when needed.

This purely classical structure will be used to describe localized solutions in motion, as well as the comoving descriptions associated with them.

1.4 Scale and regularity assumptions

The model is developed in a scale-separation regime

$$\ell_0 \ll \lambda \ll R_0,$$

where ℓ_0 denotes the microscopic scale of the discrete network, λ the typical scale of the localized structures under study, and R_0 the global curvature scale of the reference configuration.

In this regime, the passage from the discrete network to the continuum medium is understood through a Cauchy–Born-type procedure adapted to the geometry of $\mathbb{S}^3$. Unless stated otherwise, the continuous fields introduced later will be assumed sufficiently regular for the integrations by parts, projections, and asymptotic expansions used below to be well defined.

When leading-order (LO) linearizations are invoked, they will concern weak local deformations at the scale of the structures considered. In the regime $\lambda \ll R_0$, the global curvature of the support then enters through the intrinsic operators of the background geometry, while local curvature corrections are controlled by the small parameter λ/R_0.

1.5 Description of the HSM structure

The HSM model (mass–spring meshed hypersphere) describes an elastic medium built on the reference sphere $\mathbb{S}^3$ of radius R_0. The medium is represented by a network of nodes, connected by *central bonds* (axial springs) carried by geodesic edges. We denote

$\ell_0 > 0$ (network spacing), k (bond stiffness), $z \in \mathbb{N}$ (coordination: number of bonds per node).

Each node is associated with a reference volume cell $\mathcal{V}_{\ell_0}$ of order ℓ_0^3.[1]

For two neighboring nodes (i, j), the edge length at time t is denoted $L_{ij}(t)$; the axial extension

$$\delta\ell_{ij} := L_{ij}(t) - \ell_0$$

carries the local deformation of the bond.

1.5.1 Local reference mesh and discrete–continuum transfer

Geometric framework. We consider a local reference mesh on the 3-sphere $\mathbb{S}^3(R_0)$. At this stage, its precise geometry is not fixed: the theory uses only the discrete–continuum transfer quantities needed for the local passage to the effective medium.

A particular regular realization belongs to mesh analysis. It is not postulated in the local closures of the solution families.

In the present section, only the generic characteristics of a local mesh are used: the spacing ℓ_0, the effective coordination z, the elementary volume $\mathcal{V}_{\ell_0}$, as well as the geometric transfer factors that appear in the discrete–continuum passage.

Neighborhood notation. For a cell C of the tiling, $\mathcal{N}(C)$ denotes the set of its rank-1 adjacent cells, that is, its immediate neighborhood. The cardinality of $\mathcal{N}(C)$ is denoted z when the context imposes a uniform coordination.

The geometry of this tiling provides the discrete scale ℓ_0, taken below as the characteristic length of the discrete reference support.

[1] At this stage of the local construction, $\mathcal{V}_{\ell_0}$ denotes the elementary volume associated with the discrete–continuum passage. Its fine geometry is not required to define the local equations: only the effective quantities that enter the mechanical closures are explicitly used.

Cauchy–Born approximation (quasi-flat cap). In the quasi-flat cap[2] and for small displacements, we consider on the reference sphere $\mathbb{S}^3(R_0)$ a radial scalar field $r(x)$, representing the amplitude of the normal displacement to the reference sphere, and a tangential displacement field $\vec{u}_{\tan}(x)$.

For an edge (i,j) of geodesic rest length ℓ_0, with unit tangent $\hat{t}_{ij}$ oriented from x_i to x_j, the first-order axial extension receives two contributions: a tangential variation of the endpoints on the reference sphere, and an arc-length variation induced by the radial displacement.

At fixed metric on $\mathbb{S}^3(R_0)$, the first variation of the geodesic distance between two tangentially displaced points gives

$$\delta\ell_{ij}^{\tan} = \hat{t}_{ij}\cdot\left(\Pi_{j\to i}\vec{u}_{\tan}(x_j) - \vec{u}_{\tan}(x_i)\right),$$

where $\Pi_{j\to i}$ denotes parallel transport along the geodesic edge from x_j to x_i.

Moreover, if the edge is carried by a sphere of local radius $R_0 + \bar{r}_{ij}$, with

$$\bar{r}_{ij} := \frac{r(x_i) + r(x_j)}{2},$$

then, at fixed geodesic angle, its arc length is multiplied at first order by the factor $1 + \bar{r}_{ij}/R_0$. Since $\ell_0 = R_0\alpha_{ij}$, one obtains

$$\delta\ell_{ij}^{\mathrm{rad}} = \alpha_{ij}\,\bar{r}_{ij} = \frac{\ell_0}{R_0}\,\bar{r}_{ij}.$$

Thus,

$$\delta\ell_{ij} = \hat{t}_{ij}\cdot\left(\Pi_{j\to i}\vec{u}_{\tan}(x_j) - \vec{u}_{\tan}(x_i)\right) + \frac{\ell_0}{R_0}\,\frac{r(x_i)+r(x_j)}{2} + O\left(\|\vec{u}_{\tan}\|^2, \frac{|r|^2}{R_0}, \ell_0\,|\nabla r|\,\frac{\ell_0}{R_0}\right). \quad (1.1)$$

In the quasi-flat cap, parallel transport between neighbors is defined by the Levi–Civita connection of $\mathbb{S}^3(R_0)$. For a tangent field sufficiently regular at the scale ℓ_0, one then has the first-order expansion

$$\Pi_{j\to i}\vec{u}_{\tan}(x_j) - \vec{u}_{\tan}(x_i) = \ell_0\,\nabla^{\mathbb{S}^3}_{\hat{t}_{ij}}\vec{u}_{\tan}(x_i) + O(\ell_0^2), \quad (1.2)$$

where $\nabla^{\mathbb{S}^3}$ denotes the covariant derivative associated with the metric of the reference sphere.

Radial channel of discrete coherence. In the discrete closure based only on central geodesic bonds, the energy obtained from (1.1) produces, at first order, the tangent channel as well as a local quadratic contribution in r. In contrast, it does not generate at this level a neighborhood term of the type $|\nabla_{\mathbb{S}^3} r|^2$, because the radial contribution to $\delta\ell_{ij}$ depends on the average

$$\bar{r}_{ij} := \frac{r(x_i) + r(x_j)}{2},$$

and not on the difference $r(x_j) - r(x_i)$.

The radial gradient term must therefore come from a distinct discrete channel: a radial coherence channel between neighboring cells. This assumption does not fix a particular mesh. It states only that the local datum $\mathcal{M}$ contains, in addition to the central bonds, a radial sub-channel capable of comparing the radial amplitudes of two adjacent cells.

For each bond (i,j), we introduce a positive reduced stiffness

$$\alpha_r := \frac{k_{\mathrm{coh}}}{k} > 0,$$

[2]Regime $\lambda/R_0 \ll 1$ for the characteristic wavelengths λ of the fields considered.

where k is the central stiffness appearing in (1.8) and k_{coh} the stiffness of the radial coherence channel. The associated minimal discrete energy is then

$$E_{r,\mathrm{disc}} = \frac{k_{\mathrm{coh}}}{2} \sum_{(i,j)} \big(r(x_j) - r(x_i)\big)^2 = \frac{k\alpha_r}{2} \sum_{(i,j)} \big(r(x_j) - r(x_i)\big)^2. \tag{1.3}$$

The positivity of α_r ensures that this channel increases the energy when two neighboring cells carry different radial amplitudes.

For scalar fields smooth at the scale ℓ_0, one has

$$r(x_j) - r(x_i) = \ell_0 \, \nabla^{\mathbb{S}^3}_{\hat{t}_{ij}} r(x_i) + O(\ell_0^2). \tag{1.4}$$

After orientation averaging and passage to the continuum, this term provides the radial channel

$$E_{r,\mathrm{cont}} = \int_{\mathbb{S}^3(R_0)} \frac{T_r}{2} \, |\nabla_{\mathbb{S}^3} r|^2 \, dV, \qquad T_r = \frac{k\alpha_r \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_r, \tag{1.5}$$

where Ξ_r denotes the geometric factor resulting from the orientation average of the neighborhood.

We then introduce the dimensionless constant κ_r by

$$T_r = \kappa_r \, T. \tag{1.6}$$

Using

$$T = \frac{k\ell_0^2}{\mathcal{V}_{\ell_0}} \Xi_{\tan}, \qquad T_r = \frac{k\alpha_r \ell_0^2}{\mathcal{V}_{\ell_0}} \Xi_r,$$

one obtains the structural closure

$$\boxed{\kappa_r = \alpha_r \, \frac{\Xi_r}{\Xi_{\tan}}.} \tag{1.7}$$

Thus, κ_r is not an independent constant added at the continuum level: it is the effective output of the reduced radial stiffness ratio α_r and of the geometric transfer factors contained in $\mathcal{M}$.

1.5.2 Discrete energy and orientation averages

Effective discrete energy. Counting each link only once, we consider the following effective discrete parametrization of the linearized sector:

$$E_{\mathrm{disc}} = \sum_i \frac{m}{2} \Big(\dot{r}_i^{\,2} + |\dot{\vec{u}}_{\tan,i}|^2 \Big) + \sum_{(i,j)} \frac{k}{2} \, \delta\ell_{ij}^2 + \sum_{(i,j)} \frac{k\alpha_r}{2} \big(r(x_j) - r(x_i)\big)^2, \tag{1.8}$$

where m is the mass associated with the node, k the stiffness of the central links, and $\alpha_r > 0$ the reduced stiffness ratio of the radial coherence channel.

The third term does not result from the geodesic central links alone. It represents the radial coherence channel introduced in (1.3). This channel belongs to the enriched local datum $\mathcal{M}$: it does not impose a particular geometric realization of the mesh, but only the existence of a positive radial sub-channel between neighboring cells. The effective value of α_r, and therefore that of κ_r, will depend on the local realization chosen for $\mathcal{M}$.

The sum over links is written as

$$\sum_{(i,j)} (\cdot) = \frac{1}{2} \sum_i \sum_{j \in \mathcal{N}(i)} (\cdot),$$

the factor $1/2$ avoiding double counting. When one passes to a local average over neighboring directions, one then uses

$$\sum_{j \in \mathcal{N}(i)} (\cdot) = z \, \langle \cdot \rangle_{\mathrm{neighbors}},$$

where z denotes the number of rank-1 neighbors of the node considered.

Orientation moments. In the quasi-flat cap, the neighborhood of a node is identified at first order with a local three-dimensional Euclidean space, decomposed into a radial direction $\hat{e}_r$ and a local tangent plane. The distribution of link directions around a node is then described by the average tensor

$$\mathsf{M} := \langle \hat{t} \otimes \hat{t} \rangle_{\text{links}}.$$

We denote by $P_{\tan} := I_3 - \hat{e}_{\text{rad}} \otimes \hat{e}_{\text{rad}}$ the orthogonal projector onto the plane transverse to the local radial geodesic direction. The tangential trace of a tensor A is then defined by

$$\text{tr}_{\tan} A := \text{tr}(P_{\tan} A).$$

Under the assumption of local isotropy, we introduce the orientation moments

$$\eta_{\tan} := \text{tr}_{\tan} \mathsf{M}, \qquad \eta_r := \langle (\hat{t} \cdot \hat{e}_{\text{rad}})^2 \rangle_{\text{links}}, \qquad \eta_0 := 1. \tag{1.9}$$

Values under ideal local isotropy. In the ideal 3D isotropic approximation,

$$\langle \hat{t} \otimes \hat{t} \rangle = \frac{1}{3} I_3.$$

One then obtains

$$\eta_r = \hat{e}_r^{\top} \Big\langle \hat{t} \otimes \hat{t} \Big\rangle \hat{e}_r = \frac{1}{3}, \qquad \eta_{\tan} = \text{tr}_{\tan} \langle \hat{t} \otimes \hat{t} \rangle = 1 - \eta_r = \frac{2}{3}.$$

These quantities are therefore pure numbers of order 1; in the local isotropic approximation,

$$\eta_{\tan} = \frac{2}{3}, \qquad \eta_r = \frac{1}{3}.$$

Geometric return factors. We group the angular dependence and the coordination into

$$\Xi_{\tan} := \Big(\frac{z}{2}\Big) \eta_{\tan}, \qquad \Xi_r := \Big(\frac{z}{2}\Big) \eta_r, \qquad \Xi_0 := \Big(\frac{z}{2}\Big) \eta_0 = \frac{z}{2}. \tag{1.10}$$

For the radial scalar channels, Ξ_r and Ξ_0 correspond to the isotropic return of the second moment. More precisely, Ξ_r here denotes the directional second moment associated with the gradient term of the radial scalar field r, whereas Ξ_0 denotes the counting factor of the local quadratic term in r^2. The two factors therefore do not play the same role: Ξ_r weights a gradient invariant, whereas Ξ_0 weights a local invariant. By contrast, for the vector tangential sector, $\Xi_{\tan}$ must be understood here as the effective factor associated with the reduced tangential closure retained below, and not as the reconstruction of the most general tensorial isotropic elasticity.

Continuum passage. By dividing the discrete energy by the elementary volume $\mathcal{V}_{\ell_0}$ and then passing to the continuum limit for fields smooth at the scale ℓ_0, we adopt a minimal effective closure of the linearized sector. For the scalar radial sector, the second-order quadratic contributions are grouped into the invariants $|\nabla_{\mathbb{S}^3} r|^2$ and r^2.

For the vector tangential sector, we do not seek here to reconstruct the most general tensorial isotropic elasticity with two Lamé constants. Instead, we retain the reduced tangential closure used below, compatible with the transverse branch $\nabla \cdot_{\tan} \vec{u}_{\tan} = 0$, and summarized by a single effective coefficient T. Here, $\nabla_{\mathbb{S}^3} \vec{u}_{\tan}$ denotes the intrinsic covariant derivative of the tangent field on $\mathbb{S}^3(R_0)$.

We then obtain:

- for the effective tangential channel retained in the transverse branch,

$$E_{\tan,\text{cont}} = \int_{\mathbb{S}^3(R_0)} \frac{T}{2} \, |\nabla_{\mathbb{S}^3} \vec{u}_{\tan}|^2 \, dV, \qquad T = \frac{k \, \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_{\tan};$$

- for the radial gradient channel arising from the effective inter-site radial coupling,

$$E_{r,\mathrm{cont}} = \int_{\mathbb{S}^3(R_0)} \frac{T_r}{2} \, |\nabla_{\mathbb{S}^3} r|^2 \, dV, \qquad T_r = \frac{k \, \alpha_r \, \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_r$$

- for the local term in r^2 arising from the arc dilation of the central links,

$$E_{\mathrm{loc},r} = \int_{\mathbb{S}^3(R_0)} \frac{\mu_2^{(\mathrm{geom})}}{2} \, r^2 \, dV, \qquad \mu_2^{(\mathrm{geom})} = \frac{k}{\mathcal{V}_{\ell_0}} \Big(\frac{\ell_0}{R_0} \Big)^2 \Xi_0.$$

Combining this expression with

$$T = \frac{k \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_{\tan},$$

one recovers the structural closure

$$\kappa_r = \frac{T_r}{T} = \alpha_r \, \frac{\Xi_r}{\Xi_{\tan}},$$

consistent with (1.7).

The coefficient $\mu_2^{(\mathrm{geom})}$ here represents the minimal geometric contribution arising from the arc dilation of the central links. Below, μ_2 will denote the effective local radial coefficient appearing in the continuum energy.

We first treat the potential part of the continuum passage. It is written as

$$E_{\mathrm{pot,cont}} = \int_{\mathbb{S}^3(R_0)} \Big(\frac{T}{2} \, |\nabla_{\mathbb{S}^3} \vec{u}_{\tan}|^2 + \frac{T \kappa_r}{2} \, |\nabla_{\mathbb{S}^3} r|^2 + \frac{\mu_2}{2} \, r^2 \Big) \, dV. \tag{1.11}$$

with the structuring parameters of the continuum passage

$$\boxed{\rho = \frac{m}{\mathcal{V}_{\ell_0}} \quad [\mathrm{kg\,m^{-3}}], \qquad T = \frac{k \, \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_{\tan} \quad [\mathrm{Pa}].} \tag{1.12}$$

The associated continuum kinetic part is obtained directly by volumetric return of the nodal mass:

$$K_{\mathrm{cont}} = \int_{\mathbb{S}^3(R_0)} \frac{\rho}{2} \Big(\dot{r}^{\,2} + |\partial_t \vec{u}_{\tan}|^2 \Big) \, dV, \qquad \rho = \frac{m}{\mathcal{V}_{\ell_0}}. \tag{1.13}$$

The corresponding linearized continuum Lagrangian will be introduced below from $K_{\mathrm{cont}} - E_{\mathrm{pot,cont}}$.

The minimal geometric contribution of the radial sector arising from the arc dilation of the central links is given by

$$\mu_2^{(\mathrm{geom})} = \frac{k}{\mathcal{V}_{\ell_0}} \Big(\frac{\ell_0}{R_0} \Big)^2 \Xi_0, \qquad [\mu_2^{(\mathrm{geom})}] = \mathrm{N\,m^{-4}}.$$

It generally constitutes only one part of the effective local radial coefficient μ_2.

We then introduce the dimensionless radial weight

$$\boxed{\kappa_r := \frac{T_r}{T}, \qquad [\kappa_r] = 1,} \tag{1.14}$$

as well as the local radial coefficient

$$\boxed{\kappa_\mu := \frac{\mu_2}{T}, \qquad [\kappa_\mu] = \mathrm{m^{-2}}.} \tag{1.15}$$

Thus, κ_r and κ_μ play *a priori* distinct roles: κ_r weights the radial gradient term, whereas κ_μ measures the local stiffness of the quadratic term r^2.

At this stage, no general identity is postulated between these two coefficients. In particular, we do not assume $\mu_2 = T\kappa_r$. The relation between κ_r and κ_μ must be derived from the structure of the retained discrete mesh; it will therefore be a *mesh output*, and not a starting assumption.

The kinematic parameters nevertheless satisfy

$$c_t^2 = \frac{T}{\rho}, \qquad \omega_{r,0}^2 = \frac{\mu_2}{\rho} = \kappa_\mu\, c_t^2,$$

which is dimensionally consistent since $[\kappa_\mu] = \mathrm{m}^{-2}$.

We emphasize that ρ is a *volume density* ($\mathrm{kg\,m^{-3}}$), in accordance with the three-dimensional character of $\mathbb{S}^3$.

Functional assumptions. In this chapter, unless stated otherwise, we assume that $r, \vec{u}_{\tan} \in H^1(\mathbb{S}^3)$, in the scale regime $\ell_0 \ll \lambda \ll R_0$. The leading-order linearizations apply to amplitudes satisfying

$$\|r\|_\infty, \|\vec{u}_{\tan}\|_\infty \ll R_0.$$

1.6 Fundamental constants: definition and role

Principle. At the structural level at leading order, only the data needed to fix the continuum mechanical framework, the global geometric scales, and the local structure of the discrete mesh are retained as *fundamental*. The effective coefficients of the continuum radial sector are therefore not postulated as fundamental when they must be derived from this mesh.

Fundamental constants at leading order. We therefore retain as fundamental data

ρ [$\mathrm{kg\,m^{-3}}$], T [Pa], ℓ_0 (explicit discrete scale), R_0 (global radius), $\mathcal{M}$ (enriched local structure: neigh

Here, $\mathcal{M}$ denotes the enriched local datum in the broad sense: neighborhood, coordination, elementary volume, geometric return factors, active local channels, and reduced stiffness ratios such as α_r. As long as no specialized realization is chosen, ℓ_0 is kept as an explicit independent scale. For a given regular realization, its value then becomes a deterministic function of $(\mathcal{M}, R_0)$.

Structural outputs associated with the radial sector. From the enriched local datum $\mathcal{M}$, one derives the effective coefficients that control the continuum radial sector:

$\kappa_r = \alpha_r \dfrac{\Xi_r}{\Xi_{\tan}}$ (dimensionless, radial-gradient weight), $\qquad \kappa_\mu := \dfrac{\mu_2}{T}$ (dimension $\mathrm{m^{-2}}$, local radial scale).

At this stage, no general identity is imposed between κ_r and κ_μ. The first quantity belongs to the radial coherence channel, whereas the second belongs to the local quadratic term in r^2.

Scope of the list. The specialized constants associated with a particular realization of the mesh, as well as the effective coefficients specific to certain channels or reduced sectors, will be introduced when they become necessary. They are not part of the present structural list.

Methodological remark. The presentation above deliberately separates: (i) the fundamental structural data $\rho, T, \ell_0, R_0, \mathcal{M}$, and (ii) the effective coefficients of the continuum radial sector. This separation avoids prematurely introducing an arbitrary closure of the type $\mu_2 = T\kappa_r$ before the explicit analysis of the mesh.

1.7 Linear Lagrangian and equations of motion

Transverse branch and tangential operators. Any tangent field $\vec{u}_{\tan}$ on $\mathbb{S}^3(R_0)$ decomposes, in the Helmholtz–Hodge sense, as

$$\vec{u}_{\tan} = \vec{u}_{\perp} + \nabla_{\tan}\phi, \qquad \nabla\cdot_{\tan} \vec{u}_{\perp} = 0.$$

In the present section, we restrict to the transverse branch

$$\vec{u}_{\tan} = \vec{u}_{\perp}, \qquad \nabla\cdot_{\tan} \vec{u}_{\tan} = 0.$$

For a tangent field $\vec{v}$, we denote by $v^\flat$ the 1-form associated by the metric. The tangent curl and the tangent Laplacian are then defined by

$$(\operatorname{curl}_{\tan} \vec{v})^\flat := \star\, d(v^\flat), \qquad \Delta_{\tan}\vec{v} := \left(\Delta_H v^\flat\right)^\sharp,$$

where $\star$ is the Hodge star on $\mathbb{S}^3(R_0)$ and $\Delta_H = d\delta + \delta d$ is the Hodge Laplacian on 1-forms.

Local transverse Lagrangian at leading order. According to (1.11), the tangential part of the continuum energy is of the type

$$E_{\tan} = \frac{T}{2}\int_{\mathbb{S}^3(R_0)} |\nabla_{\mathbb{S}^3}\vec{u}_{\tan}|^2\, dV.$$

Since $\mathbb{S}^3(R_0)$ is compact and without boundary, integrations by parts produce no boundary term. In the transverse branch $\nabla\cdot_{\tan}\vec{u}_{\tan} = 0$, the Hodge–Weitzenböck identity shows that the difference between

$$\int_{\mathbb{S}^3(R_0)} |\nabla_{\mathbb{S}^3}\vec{u}_{\tan}|^2\, dV \quad \text{and} \quad \int_{\mathbb{S}^3(R_0)} |\operatorname{curl}_{\tan}\vec{u}_{\tan}|^2\, dV$$

is a curvature term controlled by the Ricci tensor, hence of order $R_0^{-2}\int |\vec{u}_{\tan}|^2\, dV$. For a mode with characteristic wavelength $\lambda \ll R_0$, this term is smaller than the gradient energy by a relative factor $O(\lambda^2/R_0^2)$. Thus, at leading order, one obtains

$$E_{\tan} = \frac{T}{2}\int_{\mathbb{S}^3(R_0)} |\operatorname{curl}_{\tan}\vec{u}_{\tan}|^2\, dV \;+\; O\!\left(\frac{\lambda^2}{R_0^2}\right).$$

Combining this expression with the continuum kinetic part (1.13), one obtains the local transverse Lagrangian

$$\mathcal{L}_{\tan} = \tfrac{1}{2}\,\rho\,|\partial_t\vec{u}_{\tan}|^2 - \tfrac{1}{2}\,T\,|\operatorname{curl}_{\tan}\vec{u}_{\tan}|^2. \tag{1.16}$$

Transverse waves and speed c_t. The variation of (1.16) first gives the equation

$$\rho\,\partial_{tt}\vec{u}_{\tan} \;+\; T\,\operatorname{curl}_{\tan}(\operatorname{curl}_{\tan}\vec{u}_{\tan}) = 0.$$

In the transverse branch, this writing is equivalent to

$$\rho\,\partial_{tt}\vec{u}_{\tan} \;-\; T\,\Delta_{\tan}\vec{u}_{\tan} \;=\; 0 \qquad \text{up to } O(R_0^{-2}). \tag{1.17}$$

In a quasi-Euclidean local chart of size $\lambda \ll R_0$, the tangent Laplacian reduces to the flat Laplacian at first order. For a local plane wave

$$\vec{u}_{\tan}(x,t) \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)},$$

one then has

$$\Delta_{\tan}\vec{u}_{\tan} \simeq -|\vec{k}|^2\,\vec{u}_{\tan},$$

which gives the dispersion relation

$$\rho\,\omega^2 = T\,|\vec{k}|^2.$$

It follows that the local phase speed is

$$\boxed{c_t^2 = \frac{T}{\rho}}. \tag{1.18}$$

Global curvature effects ($R_0 < \infty$) therefore appear as relative corrections of order $O(\lambda^2/R_0^2)$, without modifying at first order the local identification $c_t^2 = T/\rho$.

Chapter 2

Equations of motion (continuum on $\mathbb{S}^3$)

2.1 Continuous fields, reference configuration, and radial/tangential decomposition

The continuum passage established in the previous chapter leads to describing the HSM medium by fields defined on the reference configuration $\mathbb{S}^3(R_0)$. Throughout this chapter, we keep the geometric conventions and asymptotic assumptions introduced in the previous chapter, in particular those of Sections 1.2 and 1.4.

Unless explicitly stated otherwise, the geometric notation and intrinsic operators used in this chapter are those fixed in § 1.2. We systematically denote by $\nabla\cdot_{\tan}$ the intrinsic divergence of tangent fields on $\mathbb{S}^3(R_0)$, and by $\Delta_{\mathbb{S}^3}$ the Beltrami Laplacian on scalars.

Reference configuration and deformed configuration. In hyperspherical coordinates (χ, θ, ϕ), the reference position of a material point is written

$$\vec{X}(\chi, \theta, \phi) = R_0\, \vec{e}_r(\chi, \theta, \phi),$$

where $\vec{e}_r$ denotes the unit radial vector of $\mathbb{R}^4$, outward normal to $\mathbb{S}^3(R_0)$.

Under the effect of deformations, the material position becomes

$$\vec{x}(\chi, \theta, \phi, t) = \vec{X}(\chi, \theta, \phi) + r(\chi, \theta, \phi, t)\, \vec{e}_r(\chi, \theta, \phi) + \vec{u}_{\tan}(\chi, \theta, \phi, t),$$

with

$$\vec{u}_{\tan}(\chi, \theta, \phi, t) \in T_{\vec{X}(\chi,\theta,\phi)}\mathbb{S}^3.$$

The three terms of this decomposition are viewed as vectors of $\mathbb{R}^4$: $\vec{X}$ is the reference point, $r\,\vec{e}_r$ the radial contribution, and $\vec{u}_{\tan}$ the tangential contribution.

Local radius at first order. The deformed configuration is in general no longer contained in the reference sphere. Since $\vec{u}_{\tan} \perp \vec{e}_r$, one has

$$\|\vec{x}\|^2 = (R_0 + r)^2 + |\vec{u}_{\tan}|^2,$$

hence

$$\|\vec{x}\| = R_0 + r + O\left(\frac{r^2}{R_0}, \frac{|\vec{u}_{\tan}|^2}{R_0}\right).$$

It is therefore convenient to introduce

$$R(\chi, \theta, \phi, t) = R_0 + r(\chi, \theta, \phi, t),$$

which denotes the local radius at first order in the deformation.

Operational conventions. Unless explicitly stated otherwise, the differential operators ∇ and $\Delta_{\mathbb{S}^3}$ used in this chapter are those of the reference sphere. Corrections due to the locally deformed geometry will be introduced only when they become necessary at the considered approximation order.

Scope of the chapter. This radial/tangential decomposition provides the framework for the developments that follow: we will use the continuum energies obtained in the previous chapter to write the equations of motion, then to analyze the geometric couplings specific to the HSM medium on $\mathbb{S}^3(R_0)$.

2.2 Central links and geodesic elongation

The Cauchy–Born approximation of the geodesic elongation of central links was established in the previous chapter; here we recall only the operational formula used in what follows.

For two neighboring nodes (i, j), if $\hat{t}_{ij}$ denotes the unit tangent to the geodesic $i \to j$ at x_i, if

$$\bar{r}_{ij} = \tfrac{1}{2}\big(r(x_i) + r(x_j)\big),$$

and if $\Pi_{j\to i}$ is parallel transport along the geodesic edge from x_j to x_i, then

$$\delta\ell_{ij} \;\simeq\; \hat{t}_{ij}\cdot\Big(\Pi_{j\to i}\vec{u}_{\tan}(x_j) - \vec{u}_{\tan}(x_i)\Big) + \frac{\ell_0}{R_0}\,\bar{r}_{ij}. \tag{2.1}$$

In the quasi-flat cap $\lambda/R_0 \ll 1$, and for fields smooth at the scale ℓ_0, one also has

$$\Pi_{j\to i}\vec{u}_{\tan}(x_j) - \vec{u}_{\tan}(x_i) = \ell_0\,\nabla_{\hat{t}_{ij}}\vec{u}_{\tan}(x_i) + O(\ell_0^2).$$

The geometric derivation of (2.1) belongs to the previous chapter; the present section retains only the form useful for writing the continuous equations.

2.3 Discrete–continuum transfer of radial coefficients

Purpose. Here we reformulate, under a general discrete parametrization, the radial sector already introduced in the previous chapter, in order to separate cleanly: (i) the weight of the continuous *radial gradient*, governed by κ_r, and (ii) the *local stiffness* of the quadratic radial term, governed by $\kappa_\mu = \mu_2/T$. This writing must not be read as a consequence of geodesic central links alone. It extends the radial channel of coherence introduced in (1.3), which belongs to the enriched local data $\mathcal{M}$ without imposing a particular geometric realization of the mesh.

Counting convention and discrete radial variable. We denote by r_i the local radial displacement at site i, with dimension of length. Throughout this section, the notation

$$\sum_{\langle i,j\rangle}$$

denotes the sum over unordered pairs of first-neighbor sites, that is, each link counted only once. With this convention, the coefficient B_r introduced below coincides with the coefficient of the radial coherence channel in (1.3):

$$\boxed{B_r = k\,\alpha_r\,.}$$

General discrete form of the radial sector. We then consider the quadratic parametrization

$$E_r^{\mathrm{disc}} = \frac{A_r}{2} \sum_i r_i^2 + \frac{B_r}{2} \sum_{\langle i,j \rangle} (r_j - r_i)^2, \tag{2.2}$$

where $A_r > 0$ measures the local penalty and $B_r > 0$ the inter-site coupling.

Continuum transfer. The associated continuous radial sector is written

$$E_{r,\mathrm{cont}} = \int_{\mathbb{S}^3(R_0)} \frac{T_r}{2} \, |\nabla_{\mathbb{S}^3} r|^2 \, dV, \qquad T_r = \frac{k \, \alpha_r \, \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_r. \tag{2.3}$$

Combining this expression with

$$T = \frac{k \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_{\tan},$$

one recovers the structural closure

$$\kappa_r = \frac{T_r}{T} = \alpha_r \, \frac{\Xi_r}{\Xi_{\tan}},$$

consistent with (1.7).

The retained discrete–continuum transfer is

$$\boxed{\mu_2 = \frac{A_r}{\mathcal{V}_{\ell_0}} \, ,} \tag{2.4}$$

and

$$\boxed{T \kappa_r = \frac{B_r \, \ell_0^2}{\mathcal{V}_{\ell_0}} \, \Xi_r \, ,} \tag{2.5}$$

where $\mathcal{V}_{\ell_0}$ is the elementary volume associated with the local mesh and Ξ_r the geometric transfer factor of the discrete radial Laplacian.

General mesh relation. Dividing the two preceding relations by T, then using

$$\kappa_\mu = \frac{\mu_2}{T},$$

one obtains the general structural relation

$$\boxed{\frac{\kappa_r}{\kappa_\mu} = \frac{B_r \, \ell_0^2 \, \Xi_r}{A_r} \, .} \tag{2.6}$$

Physical reading. Relation (2.6) shows that κ_r and κ_μ encode two distinct aspects of the radial sector:

- κ_μ comes from the local term $A_r \sum_i r_i^2$;
- κ_r comes from the inter-site coupling $B_r \sum_{\langle i,j \rangle} (r_j - r_i)^2$, via the geometric transfer Ξ_r.

Their ratio therefore depends on the effective mesh structure, and not on a universal identity imposed *a priori*.

Methodological consequence. In what follows, any particular relation between κ_r and κ_μ must be derived for a given family of meshes. A possible equality, or a precise numerical value of the ratio κ_r / κ_μ, must be interpreted as a *mesh output*, never as a fundamental assumption.

2.4 Tangential sector: intrinsic reference formulation

Notation. We denote by g the metric of $\mathbb{S}^3(R_0)$, by ∇ its Levi–Civita connection, by $\langle\cdot,\cdot\rangle$ the induced product, and

$$dV = \sqrt{\det g}\, d\chi\, d\theta\, d\phi.$$

Intrinsic formulation. The tangential energy is written intrinsically as

$$E_{\tan} = \frac{T}{2}\int_{\mathbb{S}^3(R_0)} \langle \nabla \vec{u}_{\tan},\, \nabla \vec{u}_{\tan}\rangle\, dV. \tag{2.7}$$

The variation of this energy leads to the intrinsic equation

$$\rho\, \ddot{\vec{u}}_{\tan} - T\, \nabla^*\nabla \vec{u}_{\tan} = \vec{0}, \tag{2.8}$$

where $\nabla^*\nabla$ denotes the Bochner Laplacian acting on tangent fields.

Curvature of the round sphere. On $\mathbb{S}^3(R_0)$, the Ricci tensor is

$$\mathrm{Ric} = \frac{2}{R_0^2}\, g, \qquad \mathrm{Ric}(\vec{u}_{\tan}) = \frac{2}{R_0^2}\, \vec{u}_{\tan}. \tag{2.9}$$

This identity will be used when one wants to compare $\nabla^*\nabla$ to other intrinsic vector Laplacians via the Weitzenböck formulas.

Transverse branch at leading order. In the transverse branch $\nabla \cdot_{\tan} \vec{u}_{\tan} = 0$, and in the local regime $\lambda/R_0 \ll 1$, the intrinsic equation (2.8) reduces, at the operational level, to the transverse wave equation already obtained above, cf. (1.17), with local speed c_t given by (1.18). The curvature corrections associated with the difference between the various intrinsic vector Laplacians are relative corrections of order $O(\lambda^2/R_0^2)$.

Representation remark. If a local component writing becomes necessary later, it must be obtained directly from (2.8) in an explicitly fixed frame. The connection and curvature terms that then appear are entirely determined by the geometry of $\mathbb{S}^3(R_0)$ and do not constitute new free parameters of the model.

2.5 Effective energies and Lagrangian

Radial parameters and characteristic speeds. We introduce

$$\kappa_\mu := \frac{\mu_2}{T}, \qquad [\kappa_\mu] = \mathrm{m}^{-2},$$

so that the local radial term is written $\mu_2 = T\kappa_\mu$. We recall

$$c_t^2 = \frac{T}{\rho}, \qquad c_r^2 := \frac{T\kappa_r}{\rho} = \kappa_r\, c_t^2, \qquad \omega_{r,0}^2 = \frac{\mu_2}{\rho} = \kappa_\mu\, c_t^2.$$

The quantity c_r plays the role of a characteristic radial speed. In general, $c_r \neq c_t$ as soon as $\kappa_r \neq 1$.

Intrinsic measure. We denote by

$$dV = \sqrt{\det g}\, d\chi\, d\theta\, d\phi$$

the intrinsic measure on $\mathbb{S}^3(R_0)$.

Kinetic energy.

$$E_{\mathrm{cin}} = \frac{\rho}{2} \int_{\mathbb{S}^3(R_0)} \left(\dot{r}^{\,2} + |\dot{\vec{u}}_{\tan}|^2 \right) dV.$$

Radial energy.

$$E_{\mathrm{rad}} = \frac{T\kappa_r}{2} \int_{\mathbb{S}^3(R_0)} |\nabla_{\mathbb{S}^3} r|^2 \, dV + \frac{\mu_2}{2} \int_{\mathbb{S}^3(R_0)} r^2 \, dV.$$

Tangential energy.

$$E_{\tan} = \frac{T}{2} \int_{\mathbb{S}^3(R_0)} \langle \nabla \vec{u}_{\tan}, \, \nabla \vec{u}_{\tan} \rangle \, dV.$$

Minimal effective Lagrangian at LO. At the continuum level, to leading order, we adopt the minimal effective Lagrangian

$$\mathcal{L} = \frac{\rho}{2} \int_{\mathbb{S}^3(R_0)} \left(\dot{r}^{\,2} + |\dot{\vec{u}}_{\tan}|^2 \right) dV - \frac{T\kappa_r}{2} \int_{\mathbb{S}^3(R_0)} |\nabla_{\mathbb{S}^3} r|^2 \, dV - \frac{\mu_2}{2} \int_{\mathbb{S}^3(R_0)} r^2 \, dV - \frac{T}{2} \int_{\mathbb{S}^3(R_0)} \langle \nabla \vec{u}_{\tan}, \, \nabla \vec{u}_{\tan} \rangle \, dV. \tag{2.10}$$

Remark on mixed terms. Bilinear couplings of the type

$$r \, \nabla_{\cdot \tan} \, \vec{u}_{\tan}$$

or, after integration by parts on $\mathbb{S}^3(R_0)$,

$$-\nabla_{\mathbb{S}^3} r \cdot \vec{u}_{\tan},$$

are not retained in the minimal closure adopted here at leading order. Their appearance would require a specific discrete derivation or an additional symmetry breaking which is not available at the closure level considered.

Linearized Lagrangian. In this minimal closure, the effective Lagrangian is already quadratic. It therefore coincides with its linearized Lagrangian around the uniform background $(r, \vec{u}_{\tan}) = (0, \vec{0})$:

$$\mathcal{L}_{\mathrm{lin}} = \mathcal{L}. \tag{2.11}$$

Coercivity and conservation (linear). Under $T > 0$, $\kappa_r > 0$, and $\mu_2 \geq 0$, we define the quadratic energy

$$\mathcal{E}(t) = \frac{\rho}{2} \int_{\mathbb{S}^3(R_0)} (\dot{r}^{\,2} + |\dot{\vec{u}}_{\tan}|^2) \, dV + \frac{T\kappa_r}{2} \int_{\mathbb{S}^3(R_0)} |\nabla_{\mathbb{S}^3} r|^2 \, dV + \frac{\mu_2}{2} \int_{\mathbb{S}^3(R_0)} r^2 \, dV + \frac{T}{2} \int_{\mathbb{S}^3(R_0)} \langle \nabla \vec{u}_{\tan}, \, \nabla \vec{u}_{\tan} \rangle \, dV.$$

There exist $c, C > 0$ such that, for every smooth $(r, \vec{u}_{\tan}, \dot{r}, \dot{\vec{u}}_{\tan})$ (and for r of zero mean if $\mu_2 = 0$),

$$c \, \| (r, \vec{u}_{\tan}, \dot{r}, \dot{\vec{u}}_{\tan}) \|^2_{H^1 \times H^1 \times L^2 \times L^2} \leq \mathcal{E}(t) \leq C \, \| (r, \vec{u}_{\tan}, \dot{r}, \dot{\vec{u}}_{\tan}) \|^2_{H^1 \times H^1 \times L^2 \times L^2}.$$

Moreover, $\dot{\mathcal{E}}(t) = 0$ for the associated linear Euler–Lagrange equations.

2.6 Euler–Lagrange equations

Starting point. We start from the Lagrangian (2.10), with dV the fixed measure of the reference sphere $\mathbb{S}^3(R_0)$.

Variation with respect to r. The radial variation gives

$$\delta\mathcal{L}_r = \int_{\mathbb{S}^3(R_0)} \Big(\rho\, \dot{r}\, \delta\dot{r} - T\kappa_r\, \nabla_{\mathbb{S}^3} r \cdot \nabla_{\mathbb{S}^3} \delta r - \mu_2\, r\, \delta r \Big)\, dV.$$

After integrations by parts in time and in space,

$$\delta\mathcal{L}_r = \int_{\mathbb{S}^3(R_0)} \Big(-\rho\, \ddot{r} + T\kappa_r\, \Delta_{\mathbb{S}^3} r - \mu_2\, r \Big)\, \delta r\, dV,$$

from which the radial equation follows:

$$\boxed{\rho\, \ddot{r} - T\kappa_r\, \Delta_{\mathbb{S}^3} r + \mu_2\, r = 0.} \tag{2.12}$$

Variation with respect to $\vec{u}_{\tan}$. The tangential variation is written

$$\delta\mathcal{L}_u = \int_{\mathbb{S}^3(R_0)} \Big(\rho\, \dot{\vec{u}}_{\tan} \cdot \delta\dot{\vec{u}}_{\tan} - T\, \langle \nabla\vec{u}_{\tan},\, \nabla\delta\vec{u}_{\tan} \rangle \Big)\, dV,$$

where $\langle \nabla\vec{u}_{\tan},\, \nabla\delta\vec{u}_{\tan} \rangle$ denotes the double contraction of the tensors $\nabla\vec{u}_{\tan}$ and $\nabla\delta\vec{u}_{\tan}$. Integration by parts leads to

$$\delta\mathcal{L}_u = \int_{\mathbb{S}^3(R_0)} \Big(-\rho\, \ddot{\vec{u}}_{\tan} + T\, \nabla^*\nabla\vec{u}_{\tan} \Big) \cdot \delta\vec{u}_{\tan}\, dV,$$

from which the intrinsic tangential equation follows:

$$\boxed{\rho\, \ddot{\vec{u}}_{\tan} - T\, \nabla^*\nabla\vec{u}_{\tan} = \vec{0}.} \tag{2.13}$$

Linearized system. In the minimal closure retained here, the system is already linear in (r, u). The following equations therefore do not constitute an additional approximation: they rewrite (2.12) and (2.13) in the form used for the modal analysis:

$$\boxed{\rho\, \ddot{r} - T\kappa_r\, \Delta_{\mathbb{S}^3} r + \mu_2\, r = 0,} \tag{2.14}$$

and

$$\boxed{\rho\, \ddot{\vec{u}}_{\tan} - T\, \nabla^*\nabla\vec{u}_{\tan} = \vec{0}.} \tag{2.15}$$

Local transverse branch. In the transverse branch $\nabla\cdot_{\tan} \vec{u}_{\tan} = 0$ and in the local regime $\lambda/R_0 \ll 1$, the intrinsic tangential equation is rewritten, at the operational level, as the transverse wave equation (1.17), with speed c_t given by (1.18).

2.7 Linear wave equations on $\mathbb{S}^3$ (uniform background)

At the level linearized around the uniform background $r = 0$, the radial sector and the transverse tangential branch admit linear wave equations on $\mathbb{S}^3(R_0)$.

Radial mode. Dividing (2.14) by ρ, one obtains the Klein–Gordon-type equation

$$\partial_t^2 r \;-\; c_r^2\, \Delta_{\mathbb{S}^3} r \;+\; \omega_{r,0}^2\, r \;=\; 0, \tag{2.16}$$

with

$$c_r^2 = \frac{T\kappa_r}{\rho} = \kappa_r\, c_t^2, \qquad \omega_{r,0}^2 = \frac{\mu_2}{\rho} = \kappa_\mu c_t^2.$$

Transverse tangential branch. In the transverse branch $\nabla \cdot_{\tan} \vec{u}_{\tan} = 0$, and in the local regime $\lambda/R_0 \ll 1$, the intrinsic tangential equation reduces operationally to the wave equation

$$\partial_t^2 \vec{u}_{\tan} \; - \; c_t^2 \, \Delta_{\tan} \vec{u}_{\tan} \; = \; \vec{0}, \qquad c_t^2 = \frac{T}{\rho}. \tag{2.17}$$

In a quasi-Euclidean local chart, $\Delta_{\tan}$ reduces to the flat Laplacian, so that for a local plane wave

$$\vec{u}_{\tan}(x,t) \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)},$$

one obtains the dispersion relation

$$\omega^2 = c_t^2 \, |\vec{k}|^2.$$

High-frequency limit. In the radial sector, the local term becomes subdominant when

$$\omega_{r,0}^2 \ll c_r^2 \, k^2,$$

that is, when the local wavelength is small compared with $c_r/\omega_{r,0}$. In this limit, the radial equation also reduces to a wave equation at first order.

Radial spectrum on $\mathbb{S}^3$. Since $\mathbb{S}^3(R_0)$ is compact, the spectrum of the scalar Laplacian is discrete:

$$-\Delta_{\mathbb{S}^3} Y_\ell = \frac{\ell(\ell+2)}{R_0^2} \, Y_\ell, \qquad \ell = 0, 1, 2, \ldots$$

Consequently, the radial eigenfrequencies satisfy

$$\omega_\ell^2 = \frac{\ell(\ell+2)}{R_0^2} \, c_r^2 + \omega_{r,0}^2.$$

This spectral discretization will play an important role below.

Chapter 3

Axisymmetric reduction and comoving frame on $\mathbb{S}^3$

3.1 Axisymmetric reduction: 1D Laplacian along a geodesic

Axisymmetric field and reduced measure. We consider a time-dependent axisymmetric field

$$u = u(s,t), \qquad s = R_0\chi,$$

with no dependence on (θ,ϕ). The geometric conventions on $\mathbb{S}^3(R_0)$ are those fixed in § 1.2. The volume measure then reduces to

$$dV = J_{S^3}(s)\,ds\,d\Omega, \qquad J_{S^3}(s) = R_0^2\sin^2\Big(\frac{s}{R_0}\Big), \qquad \int_{\mathbb{S}^2} d\Omega = 4\pi.$$

Under this axisymmetry assumption, one has

$$|\nabla_{\mathbb{S}^3}u|^2 = u_s^2.$$

1D Laplacian along a geodesic. The axisymmetric restriction of the scalar Laplacian is

$$\boxed{\Delta_{\mathbb{S}^3}u = \frac{1}{J_{S^3}}\partial_s(J_{S^3}u_s) = u_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\,u_s.} \tag{3.1}$$

General 1D equation and high-frequency limit. From the linearized Euler–Lagrange equations, the 1D PDE for an axisymmetric mode $u(s,t)$ is

$$\partial_t^2 u \;-\; c^2\Big[u_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)u_s\Big] \;+\; \omega_0^2\,u \;=\; 0, \tag{3.2}$$

where (c,ω_0) denote the characteristic speed and the local frequency of the mode considered. For the scalar radial sector, one has

$$(c,\omega_0) = (c_r,\omega_{r,0}), \qquad \omega_{r,0}^2 = \frac{\mu_2}{\rho}.$$

The same form applies to any axisymmetric scalar mode, or to any projected amplitude for which a pair (c,ω_0) has been fixed.

In the high-frequency regime

$$\omega_0^2 \ll c^2k^2 \qquad (\text{or equivalently } k \gg \omega_0/c),$$

the zero-order term becomes subdominant compared with the dispersive term, and one obtains the local massless approximation

$$\partial_t^2 u \;-\; c^2\Big[u_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)u_s\Big] \simeq 0.$$

3.1.1 Proof of the 1D form of the Laplacian by weighted IBP

Reduced axisymmetric measure. For an axisymmetric function $u = u(s)$ on $\mathbb{S}^3(R_0)$, the volume measure is

$$dV = J_{S^3}(s)\, ds\, d\Omega, \qquad J_{S^3}(s) = R_0^2\, \sin^2\Big(\frac{s}{R_0}\Big),$$

where $d\Omega$ denotes the standard measure on $\mathbb{S}^2$, with

$$\int_{\mathbb{S}^2} d\Omega = 4\pi.$$

Derivation by weighted integration by parts. We restrict the variation to the axisymmetric class, that is $\delta u = \delta u(s)$. Then

$$\delta \int_{\mathbb{S}^3} \tfrac{1}{2}|\nabla_{\mathbb{S}^3} u|^2\, dV = \delta \int_{\mathbb{S}^3} \tfrac{1}{2} u_s^2\, dV = \int J_{S^3}(s)\, u_s\, (\delta u)_s\, ds\, d\Omega.$$

An integration by parts gives

$$\delta \int_{\mathbb{S}^3} \tfrac{1}{2} u_s^2\, dV = \Big[J_{S^3} u_s \delta u\Big]_0^{\pi R_0} \int_{\mathbb{S}^2} d\Omega - \int (J_{S^3} u_s)_s\, \delta u\, ds\, d\Omega.$$

Since

$$J_{S^3}(0) = J_{S^3}(\pi R_0) = 0,$$

and since the C^∞ regularity of an axisymmetric field imposes in particular

$$u_s(0) = u_s(\pi R_0) = 0,$$

one deduces

$$J_{S^3}(s)\, u_s(s) \to 0 \qquad \text{at the poles.}$$

Consequently,

$$\delta \int_{\mathbb{S}^3} \tfrac{1}{2} u_s^2\, dV = - \int_{\mathbb{S}^3} \frac{1}{J_{S^3}} \partial_s (J_{S^3} u_s)\, \delta u\, dV,$$

which leads to (3.1).

Regularity conditions at the poles. The C^∞ regularity of an axisymmetric field imposes in particular

$$u_s(0) = u_s(\pi R_0) = 0.$$

This condition implies

$$J_{S^3}(s)\, u_s(s) \to 0 \qquad \text{at the poles,}$$

which suffices to cancel the boundary terms in the weighted integration by parts.

1D axisymmetric energy and conservation. For the general 1D equation

$$u_{tt} - c^2 \Big(u_{ss} + \frac{2}{R_0} \cot\Big(\frac{s}{R_0}\Big)\, u_s \Big) + \omega_0^2 u = 0,$$

we introduce the energy density

$$e = \frac{\rho}{2}\, u_t^2 + \frac{\rho c^2}{2}\, u_s^2 + \frac{\rho \omega_0^2}{2}\, u^2,$$

and the flux

$$f = -\,\rho c^2\, u_t\, u_s.$$

They satisfy the local law

$$\partial_t e + \partial_s f + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\, f = 0.$$

By integrating with the reduced axisymmetric measure

$$dV = J_{S^3}(s)\, ds\, d\Omega, \qquad J_{S^3}(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big),$$

one obtains, for sufficiently regular axisymmetric solutions (for example $u \in C^\infty$ or $u \in H^2$ with the compatibilities at the poles), the global conservation law

$$\frac{d}{dt}\int_{\mathbb{S}^3} e\, dV = 0.$$

3.2 Moving frames: uniform rotation and local translation along a geodesic

Choice of a reference geodesic. By invariance of $\mathbb{S}^3(R_0)$ under its isometry group, one may choose without loss of generality a large reference geodesic and parametrize it by its arc length

$$s = R_0\, \chi.$$

The axisymmetric profiles studied in this chapter will be described along this variable s.

Rotating frame. To describe a pattern in uniform rotation, one introduces the coordinate change

$$\chi' = \chi, \qquad \theta' = \theta, \qquad \phi' = \phi - \Omega t, \qquad t' = t.$$

The chain rule gives

$$\partial_{t'} = \partial_t + \Omega\, \partial_\phi, \qquad \partial_t = \partial_{t'} - \Omega\, \partial_{\phi'},$$

and

$$\partial_t^2 = \partial_{t'}^2 - 2\Omega\, \partial_{t'}\partial_{\phi'} + \Omega^2\, \partial_{\phi'}^2.$$

Since the rotation $\phi \mapsto \phi + \Omega t$ is an isometry of $\mathbb{S}^3(R_0)$, the intrinsic geometric operators keep the same form in the coordinates (χ', θ', ϕ'). Under axisymmetry in the rotating frame ($\partial_{\phi'} \equiv 0$), the Coriolis and centrifugal terms vanish, and one recovers

$$\partial_t^2 = \partial_{t'}^2.$$

In this case, the wave equation in the rotating frame coincides with that of the inertial frame.

3.2.1 Local comoving translation along a geodesic

Local comoving approximation. For a pattern moving at constant speed v along the chosen geodesic, one locally introduces the comoving variable

$$s' = s - vt, \qquad t' = t.$$

This transformation must be understood as a local approximation valid in a quasi-flat regime ($\lambda \ll R_0$). Globally, on $\mathbb{S}^3(R_0)$, motion along a large geodesic corresponds to an isometry of the sphere, that is, to a rotation in the appropriate plane of $\mathbb{R}^4$. The writing $s' = s - vt$ is its local form at first order.

The chain rule gives

$$\partial_t = \partial_{t'} - v\, \partial_{s'}, \qquad \partial_s = \partial_{s'},$$

and therefore

$$\partial_t^2 = \partial_{t'}^2 - 2v\, \partial_{t'}\partial_{s'} + v^2\, \partial_{s'}^2.$$

Stationary comoving ansatz. A stationary comoving profile is an ansatz of the form

$$u(s,t) = \widetilde{u}(s - vt) = \widetilde{u}(s'),$$

that is, satisfying

$$\partial_{t'} u \equiv 0.$$

In this framework, the evolution equations reduce locally to differential equations in the single variable s'.

Geometric scope. This reduction is strictly stationary in s' only in the local regime where the geometric coefficients vary little at the scale of the structure. When explicit curvature terms, such as $\cot(s/R_0)$, must be retained globally, one must return to the full geometry on $\mathbb{S}^3(R_0)$, or to the global isometric description of the motion. Moreover, since the large geodesic is closed, any moving pattern on it is globally periodic, with period

$$T_{\text{orb}} = \frac{2\pi R_0}{v}.$$

3.3 Translating solutions: local comoving ansatz and moving branches

3.3.1 Local comoving ansatz

In the quasi-flat regime along a reference geodesic, one introduces the comoving variable

$$\xi = s - vt.$$

One then looks for profiles of the form

$$U(s,t) = \mathcal{U}(\xi),$$

that is, stationary in this local moving frame. One then has

$$\partial_t U = -v\,\mathcal{U}'(\xi), \qquad \partial_t^2 U = v^2\,\mathcal{U}''(\xi), \qquad \partial_s U = \mathcal{U}'(\xi), \qquad \partial_s^2 U = \mathcal{U}''(\xi).$$

Thus, stationarity in the comoving frame does not imply the absence of temporal inertia in the initial reference frame.

In the local 1D case

$$\partial_t^2 U - c^2 \partial_s^2 U + \omega_0^2 U = 0,$$

the ansatz $U(s,t) = \mathcal{U}(s - vt)$ leads to

$$(v^2 - c^2)\,\mathcal{U}''(\xi) + \omega_0^2\,\mathcal{U}(\xi) = 0. \tag{3.3}$$

This reduction is local: if the complete geometric coefficients of $\mathbb{S}^3$ are retained, the equation is no longer strictly autonomous in ξ alone.

Globally, on $\mathbb{S}^3(R_0)$, a translation along a large geodesic corresponds to a closed isometry of the support; a moving pattern on it is therefore periodic, with orbital period

$$T_{\text{orb}} = \frac{2\pi R_0}{v}.$$

3.3.2 Local moving branch

Proposition 3.1 (Local existence of a moving branch)**.** *Let*

$$\Psi_0 = (r_0, \vec{u}_{\tan,0})$$

be a static solution that is non-degenerate modulo translation. Then, for $|v|$ sufficiently small, there exists a local branch of moving solutions

$$v \longmapsto \Psi_v = (r_v, \vec{u}_{\tan,v})$$

such that

$$\Psi_{v=0} = \Psi_0.$$

Here, non-degenerate modulo translation *means that the linearized operator around Ψ_0 has no neutral mode other than the mode generated by translation along the chosen geodesic.*

Proof. After quotienting by translation along the chosen geodesic, the non-degeneracy condition makes the linearized operator invertible on the complement of the translation mode. The stationary equation written in the moving frame depends regularly on the parameter v and coincides, for $v = 0$, with the static equation satisfied by Ψ_0. The implicit-function theorem therefore provides a unique local branch

$$v \longmapsto \Psi_v$$

in a neighborhood of $v = 0$, with $\Psi_{v=0} = \Psi_0$. □

3.3.3 Limits of a strictly comoving profile in multisector kinematics

In a system coupled to several dynamic sectors, the linear characteristic speeds of the internal sectors may differ. In HSM, the radial sector and the tangential sector thus generally carry distinct characteristic speeds (c_r and c_t).

A coherent moving structure nevertheless remains associated with a single global propagation speed v. However, this does not imply that all internal sectors are transported in the form of a single strictly comoving profile $\mathcal{U}(s - vt)$. One must therefore distinguish:

existence of a moving branch and existence of a single strictly comoving profile.

This distinction will play an important role in the detailed analysis of the 2DOF solutions.

3.4 Axisymmetric 2DOF ansatz with circular polarization

We look for an axisymmetric solution on $\mathbb{S}^3(R_0)$, described in the local comoving frame introduced above along a large geodesic. The tangential sector performs a uniform rotation of angular frequency ω in a local tangential 2-plane, whereas the radial background is described by a slow deformation $r(s) \ll R_0$.

Under the axisymmetric/comoving assumption, the 2DOF system is parameterized by two scalar profiles

$$A(s) \qquad \text{and} \qquad r(s),$$

where $A(s)$ denotes the tangential amplitude and $r(s)$ the radial deformation of the background.

Circular-polarization ansatz. One chooses a local orthonormal basis $(\vec{e}_1(s), \vec{e}_2(s))$ of the tangential 2-plane orthogonal to the reference geodesic, parallel transported at the level of approximation retained. One then sets

$$\boxed{\vec{u}_{\tan}(s,t') = A(s)\,\big[\cos(\omega t')\,\vec{e}_1(s) + \sin(\omega t')\,\vec{e}_2(s)\big].} \tag{3.4}$$

Writing

$$\mathbf{q}(s,t') := \cos(\omega t')\,\vec{e}_1(s) + \sin(\omega t')\,\vec{e}_2(s),$$

one has $|\mathbf{q}| = 1$ and

$$\partial_{t'}\mathbf{q} = \omega J\mathbf{q}, \qquad \partial_{t'}^2\mathbf{q} = -\omega^2\mathbf{q},$$

where J denotes the rotation by $+\pi/2$ in the plane $(\vec{e}_1, \vec{e}_2)$.

Effective tangential closure at LO. In the regime $\lambda \ll R_0$, gradient terms dominate curvature corrections. In the transverse axisymmetric branch retained here, one adopts an effective closure at LO in which the tangential operator takes the form

$$\Delta_{\mathbb{S}^3}^{(t)}\vec{u}_{\tan} = \vec{u}_{\tan,ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\,\vec{u}_{\tan,s} - 2\,\kappa_{\mathrm{curv}}(s)\,\vec{u}_{\tan}. \tag{3.5}$$

On the ideal round sphere ($r \equiv 0$), this writing is compatible with the Ricci contribution

$$\mathrm{Ric} = \frac{2}{R_0^2}g,$$

which then fixes

$$\beta_0 = \frac{2T}{R_0^2}.$$

Effective local curvature. The correction induced by the radial background at dominant order is modeled by

$$\kappa_{\mathrm{curv}}(s) \simeq \frac{1}{R(s)^2}, \qquad R(s) = R_0 + r(s),$$

that is, at first order,

$$\kappa_{\mathrm{curv}}(s) = \frac{1}{R_0^2}\Big(1 - \frac{2r(s)}{R_0} + \cdots\Big).$$

This parameterization must be read as an effective geometric closure at LO. The dominant tangential coupling with the radial background is then of type $r(s)\,A(s)$.

Amplitude equation at fixed r. Projecting the equation

$$-\omega^2\,\vec{u}_{\tan} = c_t^2\,\Delta_{\mathbb{S}^3}^{(t)}\vec{u}_{\tan}$$

on $\mathbf{q}$, one obtains

$$A''(s) + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)A'(s) + \Big(\frac{\omega^2}{c_t^2} - 2\,\kappa_{\mathrm{curv}}(s)\Big)A(s) = 0. \tag{3.6}$$

This is a linear equation in A for fixed r. When $r \equiv 0$, it reduces to

$$A''(s) + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)A'(s) + \Big(\frac{\omega^2}{c_t^2} - \frac{2}{R_0^2}\Big)A(s) = 0.$$

Regularity at the poles. Since $\vec{u}_{\tan}$ is a regular tangential field on $\mathbb{S}^3(R_0)$, the amplitude must vanish at the poles:

$$A(0) = A(\pi R_0) = 0.$$

Additional conditions on the derivatives may then be imposed by the fine regularity of the solution.

3.5 Geometry–amplitude link: $\kappa_{\mathrm{curv}}(s)$ as a function of $r(s)$

We write

$$R(s) = R_0 + r(s), \qquad r(s) \ll R_0, \qquad |\partial_s r| \ll 1.$$

The local curvature contribution to the tangential sector comes from the Weitzenböck reduction of the covariant tangential operator. For a transverse tangential field $u_{\tan}$, the tangential Hodge operator decomposes locally as

$$\Delta_H u_{\tan} = \nabla^\dagger_{\tan}\nabla_{\tan} u_{\tan} + \mathrm{Ric}^\sharp(u_{\tan}),$$

where $\mathrm{Ric}^\sharp$ denotes the action of the Ricci tensor on tangential fields. On the round sphere $\mathbb{S}^3(R)$,

$$\mathrm{Ric}^\sharp = \frac{2}{R^2}\,\mathrm{Id}_{T\mathbb{S}^3}.$$

The projected tangential quadratic part therefore contains

$$\frac{1}{2}\int T\,\frac{2}{R^2}\,|u_{\tan}|^2\,dV.$$

The tangential geometric coefficient is thus defined by

$$\beta_{\mathrm{geom}}(R) := 2T\,\kappa_{\mathrm{curv}}(R). \tag{3.7}$$

On the round sphere, one obtains

$$\beta_{\mathrm{geom}}(R) = \frac{2T}{R^2}, \tag{3.8}$$

and in particular

$$\beta_0 = \frac{2T}{R_0^2}.$$

The factor 2 is therefore that of the Ricci tensor of $\mathbb{S}^3(R)$,

$$\mathrm{Ric} = \frac{2}{R^2}\,g,$$

after projection on the transverse tangential branch.

At leading order (weakly deformed background),

$$\kappa_{\mathrm{curv}}(s) = \frac{1}{R(s)^2} = \frac{1}{(R_0+r)^2} = \frac{1}{R_0^2}\Big(1 - \frac{2\,r(s)}{R_0}\Big) + \mathcal{O}\Big(\frac{r^2}{R_0^4}\Big),$$

and therefore

$$\boxed{\beta_{\mathrm{geom}}(R_0+r) = \beta_0 + \beta_1\,r + \mathcal{O}(r^2), \qquad \beta_0 = \frac{2T}{R_0^2}, \qquad \beta_1 = -\frac{4T}{R_0^3}.} \tag{3.9}$$

Injecting this expansion into (3.6), one obtains

$$A'' + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)A' + \Big[\frac{\omega^2}{c_t^2} - \frac{2}{R_0^2} + \frac{4}{R_0^3}\,r(s)\Big]A = 0 \quad + \ \mathcal{O}\Big(\frac{r^2}{R_0^4}\Big)A. \tag{3.10}$$

The term $\frac{4}{R_0^3}r(s)\,A$ represents the curvature correction induced by the radial deformation $r(s)$; at first order, it determines the coefficient β_1.

3.6 Stationary radial equation and coupled 2DOF system

We start from the radial Euler–Lagrange equation

$$\rho\,\ddot{r} \;-\; T\kappa_r\,\Delta_{\mathbb{S}^3} r \;+\; \mu_2\, r \;=\; -\tfrac{1}{2}\,\beta'_{\mathrm{geom}}(R_0+r)\,|\vec{u}_{\tan}|^2, \qquad \beta_{\mathrm{geom}}(R_0+r) = \beta_0 + \beta_1 r + \mathcal{O}(r^2).$$

With the tangential closure adopted in (3.6) and the definition (3.7), the tangential geometric factor is that of the Ricci tensor on $\mathbb{S}^3(R_0)$. The expansion (3.9) therefore gives

$$\beta_0 = \frac{2T}{R_0^2}, \qquad \beta_1 = -\,\frac{4T}{R_0^3}.$$

For a stationary radial profile in the comoving frame, one indeed has

$$\partial_{t'} r = 0.$$

However, this does not generally imply

$$\ddot{r} = 0$$

in the initial reference frame. Indeed, for a moving branch of the form

$$r(s,t) = \mathcal{R}(s - vt),$$

one obtains

$$\partial_t r = -v\,\mathcal{R}'(s-vt), \qquad \partial_t^2 r = v^2\,\mathcal{R}''(s-vt).$$

The complete treatment of the moving radial sector must therefore retain this convective term. The present section is restricted to the *rest case*, for which

$$\ddot{r} = 0.$$

In axisymmetry, this gives

$$-\,T\kappa_r\Big(r'' + \frac{2}{R_0}\cot\frac{s}{R_0}\,r'\Big) + \mu_2 r = -\frac{1}{2}\,\beta_1\,|\vec{u}_{\tan}|^2.$$

Under the ansatz $|\vec{u}_{\tan}|^2 = A^2(s)$, one obtains

$$r'' + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)r' - \Big(\frac{\mu_2}{T\kappa_r}\Big)\,r = \Big(\frac{\beta_1}{2T\kappa_r}\Big)\,A^2(s). \tag{3.11}$$

We set

$$\boxed{\lambda_r^2 := \frac{\mu_2}{T\kappa_r} = \frac{\omega_{r,0}^2}{c_r^2}}, \qquad \boxed{\gamma := -\frac{\beta_1}{2T\kappa_r} = \frac{2}{\kappa_r R_0^3} \;>\; 0}.$$

Since $\beta_1 < 0$, one indeed has

$$\frac{\beta_1}{2T\kappa_r} = -\,\gamma.$$

The radial equation therefore reads

$$\boxed{r'' + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)r' - \lambda_r^2\, r = -\,\gamma\,A^2(s).} \tag{3.12}$$

Interpretation of the sign. With the geometric convention

$$R(s) = R_0 + r(s), \qquad r > 0 \iff R_0 + r > R_0,$$

a radial bump corresponds to $r > 0$. In the strong-screening regime ($\lambda_r \lambda \gg 1$), the local equation

$$-\lambda_r^2 r \simeq -\gamma A^2$$

does give

$$r \simeq \frac{\gamma}{\lambda_r^2} A^2 > 0,$$

in agreement with this convention.

Coupled 2DOF system (summary).

$$A'' + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)A' + \Big[\frac{\omega^2}{c_t^2} - \frac{2}{R_0^2} + \frac{4}{R_0^3}\, r(s)\Big]A = 0, \tag{3.13}$$

$$r'' + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)r' - \lambda_r^2\, r = -\,\gamma\, A^2(s), \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}, \qquad \gamma = \frac{2}{\kappa_r R_0^3}. \tag{3.14}$$

Conditions at the poles.

$$A(0) = A(\pi R_0) = 0, \qquad r'(0) = r'(\pi R_0) = 0.$$

Quasi-flat core limit ($s \ll R_0$). Since

$$\frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big) = \frac{2}{s} + \mathcal{O}\Big(\frac{s}{R_0^2}\Big),$$

one obtains at first order

$$A'' + \frac{2}{s}A' + \Big(\frac{\omega^2}{c_t^2} - \frac{2}{R_0^2} + \frac{4}{R_0^3} r\Big)A \simeq 0, \qquad r'' + \frac{2}{s}r' - \lambda_r^2 r \simeq -\,\gamma\, A^2.$$

Rewriting with a local geometric coefficient. It is convenient to introduce the local geometric coefficient

$$V_{\text{eff}}(s) := T\,\kappa_{\text{curv}}(s) = \frac{1}{2}\,\beta_{\text{geom}}(R(s)), \qquad c_t^2 = \frac{T}{\rho}.$$

The amplitude equation (3.6) then reads

$$A_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)A_s + \Big[\frac{\omega^2}{c_t^2} - \frac{2\,V_{\text{eff}}(s)}{T}\Big]\,A = 0. \tag{3.15}$$

At leading order,

$$V_{\text{eff}}(s) = \frac{T}{R_0^2}\left(1 - \frac{2\,r(s)}{R_0} + \mathcal{O}\Big(\frac{r^2}{R_0^2}\Big)\right),$$

which indeed recovers (3.13).

Chapter 4

Reduction of the 2DOF amplitude equation to a cubic Klein–Gordon equation

4.1 Local effective reduction in the core

We introduce a central zone $\{|s| \le s_c\}$, called the *core*, of characteristic width $s_c = O(\lambda)$, in the regime

$$\lambda \ll R_0, \qquad |r(s)| \ll R_0, \qquad |r'(s)| \ll 1.$$

In this zone, the exact hyperspherical description is replaced by a *local effective reduction* adapted to the deformed geometry of the core.

Local effective radius. We write

$$R(s) = R_0 + r(s),$$

where R_0 is the radius of the reference sphere and $r(s)$ is the background radial deformation. In the adopted local closure, the angular part of the volume element is then parameterized by the effective radius $R(s)$.

Reduced measure in the core. In the zone $|s| \le s_c$, we use the effective measure

$$dV_{\mathrm{c}} = J_{\mathrm{c}}(s)\, ds\, d\Omega, \qquad J_{\mathrm{c}}(s) = R(s)^2. \tag{4.1}$$

This writing is not an exact reformulation of the global intrinsic measure on $\mathbb{S}^3(R_0)$, but a local asymptotic reduction valid to first order in the quasi-flat core.

Exterior measure. Outside the core, we keep the exact hyperspherical measure of the reference sphere:

$$dV_{\mathrm{ext}} = J_0(s)\, ds\, d\Omega, \qquad J_0(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big). \tag{4.2}$$

Status of the reduction. The reduction (4.1)–(4.2) defines a *local effective core model*. It is used only to extract the dominant contributions of the coupled dynamics in the zone $|s| \le s_c$, while the reference geometry on $\mathbb{S}^3(R_0)$ remains the global description of the exterior.

Core/exterior matching. At the boundary $s = \pm s_c$, the fields are matched by continuity of the considered variable and of the reduced flux:

$$[u]_{s=\pm s_c} = 0, \qquad [J\, u_s]_{s=\pm s_c} = 0, \tag{4.3}$$

where J denotes J_{c} in the core and J_0 outside it.

Use below. All local variations performed in the core will be taken with the reduced measure dV_{c}. Exterior variations and matching conditions will remain formulated with the exact measure dV_{ext}.

4.2 Variation of the radial energy in axisymmetry: operator L_J and matching conditions

Reminder. In axisymmetry on $\mathbb{S}^3$, one has

$$\Delta_{\mathbb{S}^3} u = \frac{1}{J_{S^3}} \partial_s (J_{S^3} u_s), \qquad J_{S^3}(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big),$$

see the proof by weighted IBP in §3.1.1.

Variation of the radial energy. We consider the radial energy

$$E_{\mathrm{rad}} = \frac{T\kappa_r}{2} \int_{\mathbb{S}^3} (r_s)^2 \, dV.$$

In the piecewise axisymmetric reduction used in this chapter, we write

$$dV = J(s)\, ds\, d\Omega,$$

with

$$J(s) = \begin{cases} J_{\mathrm{c}}(s), & |s| \le s_c \qquad \text{(reduced core)}, \\ R_0^2 \sin^2(s/R_0), & |s| > s_c \qquad \text{(spherical exterior)}. \end{cases}$$

Here $J_{\mathrm{c}}(s)$ denotes the reduced Jacobian used in the core. It is not identified with the exact spherical Jacobian; it represents the reduced measure adapted to the quasi-flat local closure of the core.

The same weighted integration by parts then gives

$$\delta E_{\mathrm{rad}} = T\kappa_r \int J\, r_s\, (\delta r)_s \, ds\, d\Omega = \Big[T\kappa_r\, J\, r_s\, \delta r\Big]_{\partial(\text{zones})} \int d\Omega - T\kappa_r \int \partial_s (J r_s)\, \delta r \, ds\, d\Omega.$$

Under matching conditions canceling the boundary terms (continuity of r, continuity of the flux Jr_s, zero flux at the poles), we obtain

$$\boxed{\delta E_{\mathrm{rad}} = -\,T\kappa_r \int_{\mathbb{S}^3} \underbrace{\frac{1}{J} \partial_s (J\, r_s)}_{:= L_J r} \, \delta r \, dV, \qquad L_J r = r'' + \frac{J'}{J}\, r'.}$$

Thus explicitly,

$$L_J r = \begin{cases} r'' + \dfrac{J_{\mathrm{c}}'}{J_{\mathrm{c}}}\, r', & |s| \le s_c, \\ r'' + \dfrac{2}{R_0} \cot(s/R_0)\, r', & |s| > s_c. \end{cases}$$

Remark (quasi-uniform core). In the local closure used in the core, we will take below

$$J_{\mathrm{c}}(s) \simeq R_0^2 \qquad \text{at leading order,}$$

so that

$$\frac{J_{\mathrm{c}}'}{J_{\mathrm{c}}} = O\Big(\frac{r'}{R_0}\Big) \ll 1$$

when the radial background varies slowly. This gives the approximation

$$L_J r \simeq r''.$$

4.3 Variation of E_{cin} and $E_{\tan}$ in a quasi-flat core

4.3.1 Quasi-flat core assumptions and order rules

We work in a central zone (the *core*) of characteristic width λ such that

$$\lambda \ll R_0,$$

and where the slope of the radial deformation remains small. We introduce the two small dimensionless parameters

$$\varepsilon_{\mathrm{geo}} := \frac{\lambda}{R_0} \ll 1, \qquad \varepsilon_{\mathrm{slope}} := \max_{|s| \lesssim \lambda} |r'(s)| \ll 1.$$

In the geometric reduction adopted for the core, we use a reduced measure

$$dV_{\mathrm{core}} = J_{\mathrm{c}}(s)\, ds\, d\Omega,$$

where $J_{\mathrm{c}}(s)$ denotes the reduced core Jacobian. In the weakly deformed quasi-flat regime, we will take at leading order

$$J_{\mathrm{c}}(s) = R_0^2(1 + O(r/R_0)), \qquad \frac{J_{\mathrm{c}}'(s)}{J_{\mathrm{c}}(s)} = O\Big(\frac{\varepsilon_{\mathrm{slope}}}{R_0}\Big).$$

Outside the core, we keep the exact hyperspherical measure

$$dV_{\mathrm{ext}} = J_0(s)\, ds\, d\Omega, \qquad J_0(s) = R_0^2 \sin^2(s/R_0).$$

Consequences in the core.

- **Geometric drift term.** In the local core closure,

  $$\frac{J_{\mathrm{c}}'(s)}{J_{\mathrm{c}}(s)} = O\Big(\frac{\varepsilon_{\mathrm{slope}}}{R_0}\Big).$$

  Using

  $$\frac{|A'|}{|A|} \sim \frac{1}{\lambda}, \qquad \frac{|A''|}{|A|} \sim \frac{1}{\lambda^2},$$

  one obtains

  $$\frac{|(J_{\mathrm{c}}'/J_{\mathrm{c}})A'|}{|A''|} = O(\varepsilon_{\mathrm{slope}}\varepsilon_{\mathrm{geo}}) \ll 1.$$

  The term $(J_{\mathrm{c}}'/J_{\mathrm{c}})A'$ can therefore be neglected in the local core ODE to first order.

- **Background hyperspherical curvature.** Linear contributions of type A/R_0^2 are of relative order

  $$\frac{1/R_0^2}{1/\lambda^2} = \varepsilon_{\mathrm{geo}}^2 \ll 1,$$

  and are therefore neglected to first order in the quasi-flat core.

Approximation control rule. To target a relative error $< \eta$ on the local core ODE, it is sufficient to impose

$$\varepsilon_{\text{slope}}\varepsilon_{\text{geo}} \le \eta, \qquad \varepsilon_{\text{geo}}^2 \le \eta.$$

The parameter η is a numerical control tolerance of the local approximation; it is not a structural constant of the model and enters no downstream closure. For the numerical figures in this chapter, the displayed tolerance is

$$\eta = 10^{-2}.$$

4.3.2 Outside the core

Outside the core, the exact hyperspherical measure is restored,

$$J_0(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big),$$

as well as the associated geometric terms. In particular, on $\mathbb{S}^3$, the 1D reduction gives

$$\frac{J_0'}{J_0} = \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big), \qquad \Delta_{\mathbb{S}^3} u = \frac{1}{J_0}\partial_s(J_0 u_s) = u_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)u_s.$$

The core/exterior matching is performed by continuity of A and of the reduced flux $J\,A_s$ (or, equivalently, $T\,J\,A_s$ up to a constant factor).

4.3.3 Useful variations

Throughout what follows, we use $T = \rho c_t^2$ and the harmonic ansatz of circular polarization. The energy expressions are understood after averaging over one period of the fast time $2\pi/\omega$.

We then have

$$\delta_A E_{\text{cin}} = \rho\,\omega^2 \int_{\mathbb{S}^3} A\,\delta A\,dV,$$

and

$$\delta_A E_{\text{tan}} = T\int_{\mathbb{S}^3} \langle \nabla A, \nabla \delta A\rangle\,dV = -\,T\int_{\mathbb{S}^3} \Delta_{\mathbb{S}^3} A\;\delta A\,dV,$$

where the integration by parts is understood with the reduced measure

$$dV = J(s)\,ds\,d\Omega,$$

the boundary terms being zero by regularity at the poles and/or by matching conditions.

In the quasi-flat core, to first order,

$$\Delta_{\mathbb{S}^3} A \;\simeq\; A_{ss}, \qquad \text{car } \Big(\frac{J_{\text{c}}'}{J_{\text{c}}}\Big)A' = O(\varepsilon_{\text{slope}}\,\varepsilon_{\text{geo}})\,A''.$$

4.3.4 Status of $\delta_r E_{\text{eff}}$

The coherent variation of E_{eff} in the core is treated in §4.5. Under the reduced measure $dV = R^2\,ds\,d\Omega$, with $\beta_{\text{geom}}(R) = 2T/R^2$, the product $\beta_{\text{geom}}(R)dV$ is independent of R. Hence

$$\delta_r E_{\text{eff}} = 0$$

in the full local variation, and no separate local source from E_{eff} is added to the reduced radial equation.

Outside the core, we place ourselves in the zone where the radial background has returned to its reference value, so that

$$R(s) = R_0, \qquad V_{\text{eff}} = \frac{T}{R_0^2},$$

and consequently

$$\delta_r E_{\text{eff}} = 0 \qquad \text{outside the core.}$$

4.4 Local variations in the core: $E_{\rm cin}$ and $E_{\rm tan}$

Throughout this section, for brevity we write

$$dV = dV_{\rm c} = J(s)\,ds\,d\Omega, \qquad J(s) = J_{\rm c}(s) = R(s)^2, \qquad R(s) = R_0 + r(s),$$

in the sense of the local effective reduction introduced in §4.1. Moreover, in the harmonic tangential sector, $E_{\rm cin}$ denotes the energy averaged over one period of the fast rotation.

4.4.1 Variation of the kinetic energy in the core

With the temporal ansatz of circular polarization, we have

$$|\partial_t \mathbf{u}|^2 = \omega^2 A^2,$$

and, in the local core reduction,

$$E_{\rm cin} = \frac{\rho\omega^2}{2}\int A^2\,dV, \qquad dV = J(s)\,ds\,d\Omega, \qquad J(s) = R(s)^2.$$

Step 1 — Variation with respect to r at fixed A. Since $dV = J\,ds\,d\Omega$, one has $\delta(dV) = \delta J\,ds\,d\Omega$ and

$$\delta J = \delta(R^2) = 2R\,\delta r.$$

Thus

$$\delta E_{\rm cin} = \frac{\rho\omega^2}{2}\int A^2\,\delta(dV) = \rho\omega^2\int A^2\,R\,\delta r\,ds\,d\Omega.$$

Step 2 — Rewriting under the measure dV. Since $dV = R^2\,ds\,d\Omega$, one has $ds\,d\Omega = dV/R^2$, hence

$$\delta E_{\rm cin} = \rho\omega^2\int A^2\,\frac{\delta r}{R}\,dV.$$

Variational reading. By definition,

$$\delta E_{\rm cin} = \int \frac{\delta E_{\rm cin}}{\delta r}\,\delta r\,dV,$$

therefore

$$\boxed{\frac{\delta E_{\rm cin}}{\delta r} = \rho\omega^2\,\frac{A^2}{R}}.$$

In the quasi-flat core, $R = R_0 + r \simeq R_0$, hence

$$\boxed{\frac{\delta E_{\rm cin}}{\delta r} \;\simeq\; \frac{\rho\omega^2}{R_0}\,A^2}.$$

Consequence for the local radial EL equation. In the local core reduction, where to first order one keeps only the term r'' in the radial operator, the kinetic contribution appears as a positive inertial source:

$$-\,T\kappa_r\,r''(s) \;+\; \frac{\rho\omega^2}{R_0}\,A^2(s) \;+\; \cdots \;=\; 0.$$

Remark (exact exterior). In the exterior zone, one returns to the exact hyperspherical measure

$$dV_{\rm ext} = J_0(s)\,ds\,d\Omega, \qquad J_0(s) = R_0^2\sin^2(s/R_0),$$

which does not depend on r. Thus

$$\delta_r E_{\rm cin} = 0 \qquad \text{for } |s| > s_c.$$

Synthesis (global variation). We thus obtain

$$\boxed{\frac{\delta E_{\text{cin}}}{\delta r}(s) = \begin{cases} \rho\,\omega^2\,\dfrac{A^2(s)}{R(s)}, & |s| \le s_c, \\ 0, & |s| > s_c, \end{cases}}$$

or equivalently

$$\frac{\delta E_{\text{cin}}}{\delta r}(s) = \chi_{\text{c}}(s)\,\rho\,\omega^2\,\frac{A^2(s)}{R(s)}, \qquad \chi_{\text{c}}(s) = \mathbf{1}_{\{|s|\le s_c\}}.$$

In the quasi-flat core,

$$\boxed{\frac{\delta E_{\text{cin}}}{\delta r}(s) \;\simeq\; \chi_{\text{c}}(s)\,\frac{\rho\,\omega^2}{R_0}\,A^2(s)}.$$

4.4.2 Variation of the tangential energy (gradient effect)

In the local core reduction, we have

$$|\nabla A|^2 \simeq (A_s)^2,$$

and

$$E_{\text{tan}} = \frac{T}{2}\int (A_s)^2\,dV, \qquad dV = J(s)\,ds\,d\Omega.$$

The only dependence on r then passes through the measure, whence

$$\delta E_{\text{tan}} = \frac{T}{2}\int (A_s)^2\,\delta(dV) = \frac{T}{2}\int (A_s)^2\,(2R\,\delta r)\,ds\,d\Omega = T\int (A_s)^2\,\frac{\delta r}{R}\,dV.$$

Thus

$$\boxed{\frac{\delta E_{\text{tan}}}{\delta r} = T\,\frac{(A_s)^2}{R}} \qquad \Longrightarrow \qquad \boxed{\frac{\delta E_{\text{tan}}}{\delta r} \;\simeq\; \frac{T}{R_0}\,(A_s)^2} \text{ (in the core).}$$

Remark (exact exterior). In the exterior zone, one returns to the exact hyperspherical measure

$$dV_{\text{ext}} = J_0(s)\,ds\,d\Omega, \qquad J_0(s) = R_0^2\sin^2(s/R_0),$$

which does not depend on r. Thus

$$\delta_r E_{\text{tan}} = 0 \qquad \text{for } |s| > s_c.$$

Synthesis (global variation). We obtain

$$\boxed{\frac{\delta E_{\text{tan}}}{\delta r}(s) = \begin{cases} T\,\dfrac{(A_s(s))^2}{R(s)}, & |s| \le s_c, \\ 0, & |s| > s_c, \end{cases}}$$

or equivalently

$$\frac{\delta E_{\text{tan}}}{\delta r}(s) = \chi_{\text{c}}(s)\,T\,\frac{(A_s(s))^2}{R(s)}.$$

In the quasi-flat core,

$$\boxed{\frac{\delta E_{\text{tan}}}{\delta r}(s) \;\simeq\; \chi_{\text{c}}(s)\,\frac{T}{R_0}\,(A_s(s))^2}.$$

4.5 Scope of the local core reduction

The local core reduction is used only to control the signs and orders of magnitude of the kinetic and tangential contributions in a quasi-flat cap. It is not an independent closure of the radial channel. The operative closure used in the rest of the manuscript is the global closure by the elliptic radial operator and its reduced Green function.

Coherent variation of the local geometric term. In the 2DOF closure, the tangential geometric coefficient is

$$\beta_{\mathrm{geom}}(R) = \frac{2T}{R^2}.$$

If one locally varies the reduced core measure at the same time as this coefficient, one must use

$$dV = R^2\, ds\, d\Omega.$$

The corresponding effective term is then written

$$E_{\mathrm{eff}} = \frac{1}{2}\int \beta_{\mathrm{geom}}(R)\, A^2\, dV = \frac{1}{2}\int \frac{2T}{R^2} A^2\, R^2\, ds\, d\Omega = T\int A^2\, ds\, d\Omega.$$

Thus, in this complete local variation,

$$\boxed{\frac{\delta E_{\mathrm{eff}}}{\delta r} = 0.} \tag{4.4}$$

The effective geometric term must therefore not be counted as a separate local source when the variation of the measure $R^2\, ds\, d\Omega$ is treated simultaneously.

Status of the local kinetic and tangential contributions. The local variations of the kinetic and tangential contributions remain meaningful as sign checks in the core cap:

$$\frac{\delta E_{\mathrm{cin}}}{\delta r} \simeq \frac{\rho\omega^2}{R_0} A^2, \qquad \frac{\delta E_{\mathrm{tan}}}{\delta r} \simeq \frac{T}{R_0}(A_s)^2,$$

at quasi-flat order. These expressions do not, however, define by themselves a universal radial closure. They only provide local sign and sensitivity checks of the quasi-flat core. They must not be identified with the global radial source used by the downstream closures.

Retained radial closure. The closure used afterwards is the global closure of the radial channel. We introduce the reduced operator

$$\mathcal{L}_{\lambda_r} r := -\frac{1}{J(s)}\partial_s\big(J(s)\partial_s r\big) + \lambda_r^2 r, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}.$$

The radial response is given by

$$\mathcal{L}_{\lambda_r} r = S_{\mathrm{glob}}[A], \qquad r = \mathcal{G}_{\lambda_r} S_{\mathrm{glob}}[A],$$

where $\mathcal{G}_{\lambda_r}$ is the Green function of the radial operator with the boundary and core matching conditions. In the local strong-screening limit, this global closure can be read in the algebraic form

$$r \simeq \frac{\gamma}{\lambda_r^2} A^2,$$

but this expression is a limit of the Green response, not an independent local closure.

Consequence for the cubic reduction. A local cubic reduction of the amplitude equation is legitimate only as a projected limit of the preceding global closure. At the LO order used in the operative closures of the manuscript, the radial source used by the downstream chains is therefore the one obtained from

$$r = \mathcal{G}_{\lambda_r} S_{\mathrm{glob}}[A],$$

then, when the strong-screening regime is verified, from its local approximation

$$r \simeq \frac{\gamma}{\lambda_r^2} A^2.$$

No additional local source from E_{eff} is added to this closure.

In particular, the reduced kinetic source obtained by local variation of the measure,

$$S_{\mathrm{loc}}(s) \simeq \frac{\rho\omega^2}{T\kappa_r R_0}\, A^2(s),$$

is not a second operative radial source. Compared with the retained global source

$$S_{\mathrm{glob}}(s) = \gamma A^2(s) = \frac{2}{\kappa_r R_0^3} A^2(s),$$

it formally satisfies

$$\frac{S_{\mathrm{loc}}}{S_{\mathrm{glob}}} \simeq \frac{\rho\omega^2 R_0^2}{2T} = \frac{\omega^2 R_0^2}{2c_t^2} \simeq \frac{1}{2}\,\varepsilon_{\mathrm{geo}}^{-2} \gg 1, \qquad \varepsilon_{\mathrm{geo}} := \frac{c_t}{\omega R_0}.$$

This magnitude difference confirms that S_{loc} cannot be added to S_{glob} without double counting or changing the closure. Its use is diagnostic; the operative closure remains $r = \mathcal{G}_{\lambda_r} S_{\mathrm{glob}}[A]$.

4.5.1 Exact radial energy

We introduce the physical radial operator

$$\mathcal{L}_{\lambda_r} r := -\frac{1}{J(s)}\partial_s\big(J(s)\,\partial_s r\big) + \lambda_r^2\, r, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}.$$

The total radial energy is then written

$$\boxed{E_{\mathrm{rad}} = \frac{1}{2} T\kappa_r \int \Big((\partial_s r)^2 + \lambda_r^2 r^2\Big)\, J(s)\, ds = \frac{1}{2} T\kappa_r \,\langle r, \mathcal{L}_{\lambda_r} r\rangle_J = \frac{1}{2} T\kappa_r \,\langle S, \mathcal{L}_{\lambda_r}^{-1} S\rangle_J,}$$

where $\langle f, g\rangle_J := \int f(s) g(s) J(s)\, ds$, and where $\mathcal{L}_{\lambda_r}^{-1}$ is the inverse of the operator defined with the same boundary/matching conditions.

4.5.2 Nonlocal Green form

Let $G_{\lambda_r}(s, x)$ be the Green function of the elliptic operator $\mathcal{L}_{\lambda_r}$ on the considered reduced geometry, with regularity conditions at the poles and matching conditions compatible with the weight J.

Case without mass term. For the pure operator

$$\mathcal{L}_0 := -\,J(s)^{-1}\partial_s\big(J(s)\partial_s\,\cdot\,\big),$$

the kernel contains the constants. We then understand G_0 as the pseudo-inverse Green function on $\{1\}^{\perp}$, that is, after orthogonal projection of the source onto the zero-mean subspace.

Physical radial case. When the radial screening term is kept, the relevant operator is

$$\mathcal{L}_{\lambda_r} = \mathcal{L}_0 + \lambda_r^2,$$

which is invertible on regular functions and does not require a pseudo-inverse. Throughout what follows, the reduction to $\mathcal{L}_0$ alone must therefore be understood as a local leading-order approximation when the term $\lambda_r^2 r$ is neglected in the zone considered.

4.5.3 Local limit as a rank-1 Galerkin projection

We seek a local approximation of the form

$$r_{\mathrm{loc}}(s) = \alpha\, S(s).$$

A natural variational estimator is the best $L^2(J)$ approximant of the solution

$$r = \mathcal{L}_{\lambda_r}^{-1} S$$

in the line $\mathrm{Span}\{S\}$:

$$\min_{\alpha} \; \| \, r - \alpha S \, \|^2_{L^2(J)}.$$

The orthogonality condition

$$\langle r - \alpha S, S \rangle_J = 0$$

gives

$$\alpha = \frac{\langle r, S \rangle_J}{\langle S, S \rangle_J} = \frac{\langle \mathcal{L}_{\lambda_r}^{-1} S, S \rangle_J}{\langle S, S \rangle_J}.$$

We therefore introduce the effective Green coefficient

$$\boxed{C_G \;:=\; \frac{\langle \mathcal{L}_{\lambda_r}^{-1} S, S \rangle_J}{\langle S, S \rangle_J} \;>\; 0, \qquad \mathcal{L}_{\lambda_r}^{-1} S \;\approx\; C_G\, S.}$$

In kernel form,

$$\boxed{\int G_{\lambda_r}(s,x)\, S(x)\, J(x)\, dx \;\approx\; C_G\, S(s),}$$

which provides precisely the nonlocal $\to$ local rank-1 closure.

Remarks. (i) C_G has the dimension of $\mathcal{L}_{\lambda_r}^{-1}$, hence $[C_G] = \mathrm{length}^2$, and depends on the geometry as well as on the shape of S through the weighted scalar products. For a source concentrated over a width $\lambda \ll R_0$, one typically has $C_G = \mathcal{O}(\lambda^2)$, up to numerical factors.

(ii) In the particular case without radial screening term ($\lambda_r = 0$), the pure operator $\mathcal{L}_0$ has the constants in its kernel. One then replaces S by its projection $S_\perp$ onto $\{1\}^\perp$, and uses the corresponding pseudo-inverse.

4.5.4 Practical expression of C_G in the quasi-flat cap

In the quasi-flat core, one has $J \simeq$ const., and in practice

$$J \simeq R_0^2.$$

If the radial screening term is neglected at leading order, the reduced operator becomes

$$\mathcal{L}_0 \simeq -\,\partial_{ss}.$$

For any smooth function u satisfying the homogeneous conditions compatible with the local reduction, one then has

$$\langle \mathcal{L}_0 u, u\rangle_J = \int J\, u_s^2\, ds \simeq R_0^2 \int (u')^2\, ds.$$

In particular, if $r = \mathcal{L}_0^{-1} S$, then

$$\langle \mathcal{L}_0^{-1} S, S\rangle_J = \langle r, S\rangle_J = \langle \mathcal{L}_0 r, r\rangle_J.$$

The rank-1 Galerkin approximation then consists in seeking

$$\mathcal{L}_0^{-1} S \approx C_G\, S,$$

with

$$C_G \approx \frac{\langle S, S\rangle_J}{\langle \mathcal{L}_0 S, S\rangle_J} \simeq \frac{\int S^2\, ds}{\int (S')^2\, ds}.$$

One obtains

$$\boxed{C_G \;\simeq\; \frac{\int S^2\, ds}{\int (S')^2\, ds} = \mathcal{O}(\lambda^2), \qquad C_G > 0,}$$

for a localized source of characteristic width λ.

In other words, in this local approximation, the integral operator behaves like a multiplier of size $\sim \lambda^2$ on the class of narrow, weakly oscillatory sources.

4.5.5 Error and validity domain

The quality of the approximation

$$\mathcal{L}_0^{-1} S \approx C_G\, S$$

degrades when the geometric coefficients vary significantly inside the core, or when the source feels the boundaries/matchings. In the regime

$$\lambda \ll R_0,$$

and for a localized unimodal source, one heuristically expects a relative error of size

$$\|\mathcal{L}_0^{-1} S - C_G S\|_{L^2(J)} = \mathcal{O}\Big(\frac{\lambda}{R_0}\Big)\, \|S\|_{L^2(J)}.$$

This estimate must be understood as an asymptotic validity criterion of the local reduction, not as a universal rigorous bound.

Diagnostic local form. In the quasi-flat core, the local variation of the reduced measure leads to the sign check

$$r''(s) \simeq -\frac{\rho\omega^2}{T\kappa_r R_0}\, A^2(s).$$

If one introduces the associated local source

$$\mathcal{L}_0 r = S_{\mathrm{loc}}, \qquad S_{\mathrm{loc}}(s) \simeq \frac{\rho\omega^2}{T\kappa_r R_0}\, A^2(s),$$

one obtains a useful form for diagnosing the sign and local scale of the radial coupling. This source S_{loc} is not the operative source S_{glob} used by the global closure $r = \mathcal{G}_{\lambda_r} S_{\mathrm{glob}}[A]$. It must therefore not be used as an independent input of the mass or coupling chains.

It follows, for this local diagnostic only, that the reduced radial energy is

$$E_{\mathrm{rad}} \simeq \frac{1}{2}\, T\kappa_r\, C_G \int S(s)^2\, J(s)\, ds,$$

or equivalently

$$E_{\mathrm{rad}} \simeq \frac{1}{2}\, T\kappa_r\, C_G \left(\frac{\rho\omega^2}{T\kappa_r R_0}\right)^2 \int A^4(s)\, J(s)\, ds.$$

This expression does not replace the radial energy obtained from the global Green function; it is only a local control form. The operative results of the manuscript use the global closure S_{glob} defined above.

4.5.6 Practical conclusion

For energy calculations, it is preferable to keep the global nonlocal form in terms of the Green function $G_{\lambda_r}(s, x)$, which takes into account the contributions of all zones, including near the matchings. By contrast, for the tangential dynamics in the quasi-flat core, one may use the local reduction leading to a cubic Klein–Gordon-type equation, provided that its validity assumptions are checked a posteriori.

Chapter 5

Nonlocal energy representation and weighted Green function

5.1 Hyperspherical framework, operator, and weighted Green function

Retained global framework. In this chapter, we return to the global radial operator posed on the reference sphere $\mathbb{S}^3(R_0)$, equipped with its exact intrinsic measure. The local core reductions introduced above enter here only a posteriori, as asymptotic approximations of the global nonlocal representation.

Measure conventions (axisymmetry). We use the reduced measure

$$dV = J(s)\, ds\, d\Omega, \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big),$$

where $d\Omega$ denotes the standard measure on $\mathbb{S}^2$, with

$$\int_{\mathbb{S}^2} d\Omega = 4\pi.$$

In purely one-dimensional writings along the meridian, this angular factor may be absorbed into the global normalization.

Weighted scalar product. We work with

$$\langle f, g\rangle_J = \int_0^{\pi R_0} f(s)\, g(s)\, J(s)\, ds.$$

Physical radial operator. The radial operator relevant to the physical sector with screening is

$$\boxed{\mathcal{L}_{\lambda_r} = -\frac{1}{J}\partial_s(J\,\partial_s\,\cdot\,) + \lambda_r^2, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}.}$$

In hyperspherical axisymmetry,

$$\frac{J'}{J} = \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big),$$

so that

$$\mathcal{L}_{\lambda_r} = -\Big(\partial_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\partial_s\Big) + \lambda_r^2.$$

Regularity conditions are imposed at the poles, equivalent to zero flux $J\,u_s \to 0$.

Weighted Green function: physical case. The Green function $G_{\lambda_r}(s,x)$ of the operator $\mathcal{L}_{\lambda_r}$ is defined by

$$\boxed{\mathcal{L}_{\lambda_r,s}\, G_{\lambda_r}(s,x) = \frac{\delta(s-x)}{J(s)},}$$

with the same boundary conditions. This weighted normalization makes $\mathcal{L}_{\lambda_r}^{-1}$ the inverse for the scalar product $\langle\cdot,\cdot\rangle_J$.

For any source S, one then has

$$\boxed{r(s) = (\mathcal{L}_{\lambda_r}^{-1}S)(s) = \int_0^{\pi R_0} G_{\lambda_r}(s,x)\, S(x)\, J(x)\, dx.}$$

Case without mass term. If the radial screening is neglected, one obtains the pure operator

$$\mathcal{L}_0 := -\frac{1}{J}\partial_s(J\,\partial_s\ \cdot\ ).$$

In this case, the kernel contains the constants. The corresponding Green function G_0 must then be understood as a pseudo-inverse on $\{1\}^\perp$, that is, after projection of the source onto the zero-mean subspace.

Energy representation. The physical radial energy is written

$$E_{\rm rad} = \frac{1}{2}\,T\kappa_r \int_0^{\pi R_0} \Big((r_s)^2 + \lambda_r^2 r^2\Big)\, J(s)\, ds = \frac{1}{2}\,T\kappa_r\, \langle r, \mathcal{L}_{\lambda_r} r\rangle_J.$$

If $\mathcal{L}_{\lambda_r} r = S$, then

$$\boxed{E_{\rm rad} = \frac{1}{2}\,T\kappa_r\, \langle \mathcal{L}_{\lambda_r}^{-1}S,\ S\rangle_J.}$$

In kernel form,

$$\boxed{E_{\rm rad} = \frac{1}{2}\,T\kappa_r \iint_{[0,\pi R_0]^2} S(s)\, G_{\lambda_r}(s,x)\, S(x)\, J(s)\, J(x)\, ds\, dx.}$$

Link with the local core reduction. When the source S is concentrated in a region of width $\lambda \ll R_0$, the global representation by $G_{\lambda_r}(s,x)$ reduces asymptotically to the local closure obtained in the quasi-flat core. The description by the effective coefficient C_G introduced in §4.5.4 must then be understood as the local rank-1 projection of the global inverse $\mathcal{L}_{\lambda_r}^{-1}$ onto the class of narrow sources.

5.2 Tangential energy identity on $\mathbb{S}^3$

In this section, we work in the tangential branch with circular polarization already introduced, and we simply denote by $A(s)$ the associated scalar amplitude. The stationary amplitude equation is written

$$-\rho\,\omega^2 A = T\,\Delta_{\mathbb{S}^3} A - 2\,V_{\rm eff}(s)\,A, \qquad c_t^2 = \frac{T}{\rho}, \qquad V_{\rm eff}(s) = T\,\kappa_{\rm curv}(s).$$

Equivalently,

$$-\omega^2 A = c_t^2\,\Delta_{\mathbb{S}^3} A - \frac{2V_{\rm eff}(s)}{\rho}\,A.$$

The tangential energies averaged over one period are then

$$\overline{E_{\rm cin}^{\rm tan}} = \frac{\rho\omega^2}{2}\int_{\mathbb{S}^3} A^2\,dV, \qquad \overline{E_{\rm elas}^{\rm tan}} = \frac{T}{2}\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} A|^2\,dV.$$

Multiplying the stationary equation by A and integrating over $\mathbb{S}^3$, then using integration by parts on the compact boundaryless sphere, gives

$$\rho\omega^2\int_{\mathbb{S}^3} A^2\,dV = T\int_{\mathbb{S}^3} |\nabla_{\mathbb{S}^3} A|^2\,dV + 2\int_{\mathbb{S}^3} V_{\rm eff}(s)\,A^2\,dV.$$

We therefore introduce the geometric coupling energy

$$E_{\rm cpl} := \int_{\mathbb{S}^3} V_{\rm eff}(s)\,A^2\,dV,$$

and the tangential energy identity takes the form

$$\boxed{\overline{E_{\rm cin}^{\rm tan}} = \overline{E_{\rm elas}^{\rm tan}} + E_{\rm cpl}.}$$

This identity is the natural form of the energy balance in the tangential sector. If one then chooses to regroup $E_{\rm cpl}$ into the radial/coupled sector, one obtains the purely tangential decomposition

$$E_c := \overline{E_{\rm elas}^{\rm tan}} = \overline{E_{\rm cin}^{\rm tan}} - E_{\rm cpl}.$$

In other words, equality between the kinetic and elastic parts of the tangential sector is not here a primitive identity, but a writing obtained after reclassification of the coupling term.

In this convention,

$$E_{\rm tot} = \overline{E_{\rm cin}^{\rm tan}} + \overline{E_{\rm elas}^{\rm tan}} + E_{\rm rad},$$

where $E_{\rm rad}$ must be understood with the counting convention adopted for the geometric coupling part.

5.3 First-order energy consistency in the strong-screening regime

Framework. We work in the axisymmetric 2DOF branch, in the quasi-flat core

$$\lambda \ll R_0, \qquad J(s) \simeq J_{\rm c} \simeq R_0^2,$$

and in the strong-screening regime

$$\lambda_r\,\lambda \gg 1.$$

In this regime, the local radial closure is algebraic:

$$r(s) \simeq \frac{\gamma}{\lambda_r^2}\,A(s)^2, \qquad \gamma = \frac{2}{\kappa_r R_0^3}, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}. \tag{5.1}$$

Local amplitude equation. In the same approximation, the amplitude equation becomes

$$A'' - \Omega_0^2 A + \beta_{\rm alg} A^3 \simeq 0, \qquad \beta_{\rm alg} = \frac{4\gamma}{\lambda_r^2 R_0^3} = \frac{8}{\kappa_\mu R_0^6}, \tag{5.2}$$

where

$$\kappa_\mu = \frac{\mu_2}{T}, \qquad [\kappa_\mu] = \mathrm{m}^{-2}.$$

Localized profile. Equation (5.2) admits the standard localized solution

$$A(s) = A_0\,\mathrm{sech}(\Omega_0 s), \qquad A_0^2 = \frac{2\Omega_0^2}{\beta_{\rm alg}}. \tag{5.3}$$

Pure tangential energy. After reclassification of the geometric coupling into the radial/coupled sector, we define

$$E_c := \overline{E^{\tan}_{\text{elas}}} = \overline{E^{\tan}_{\text{cin}}} - E_{\text{cpl}}.$$

In the local core, one then has

$$E_c = \frac{TJ_{\text{c}}}{2} \int_{\mathbb{R}} (A')^2 \, ds.$$

Now, for the profile (5.3),

$$A'(s) = -A_0\Omega_0 \, \operatorname{sech}(\Omega_0 s) \tanh(\Omega_0 s),$$

and the identity

$$\int_{\mathbb{R}} \operatorname{sech}^2 u \, \tanh^2 u \, du = \frac{2}{3}$$

gives

$$E_c = \frac{TJ_{\text{c}}}{2} \, A_0^2\Omega_0 \cdot \frac{2}{3} = \frac{TJ_{\text{c}}A_0^2\Omega_0}{3} = \frac{2TJ_{\text{c}}\Omega_0^3}{3\beta_{\text{alg}}}. \tag{5.4}$$

Radial energy. In the regime $\lambda_r\lambda \gg 1$, the dominant contribution to the radial energy comes from the local screening term:

$$E_{\text{rad}} \simeq \frac{T\kappa_r\lambda_r^2 J_{\text{c}}}{2} \int_{\mathbb{R}} r(s)^2 \, ds.$$

Using (5.1),

$$r(s)^2 \simeq \frac{\gamma^2}{\lambda_r^4} A(s)^4,$$

and, for the profile (5.3),

$$\int_{\mathbb{R}} A^4 \, ds = A_0^4 \int_{\mathbb{R}} \operatorname{sech}^4(\Omega_0 s) \, ds = A_0^4 \, \frac{4}{3\Omega_0},$$

since

$$\int_{\mathbb{R}} \operatorname{sech}^4 u \, du = \frac{4}{3}.$$

Thus one obtains

$$E_{\text{rad}} = \frac{T\kappa_r\lambda_r^2 J_{\text{c}}}{2} \cdot \frac{\gamma^2}{\lambda_r^4} \cdot \frac{4A_0^4}{3\Omega_0}. \tag{5.5}$$

Replacing

$$A_0^4 = \frac{4\Omega_0^4}{\beta_{\text{alg}}^2}, \qquad \beta_{\text{alg}} = \frac{4\gamma}{\lambda_r^2 R_0^3},$$

simplifies (5.5) into

$$E_{\text{rad}} = \frac{T\kappa_r\lambda_r^2 J_{\text{c}} R_0^6 \Omega_0^3}{6}. \tag{5.6}$$

Comparison. From (5.4) and

$$\beta_{\text{alg}} = \frac{8}{\kappa_\mu R_0^6}, \qquad T\kappa_\mu = \mu_2 = T\kappa_r\lambda_r^2,$$

one obtains

$$E_c = \frac{T\kappa_r\lambda_r^2 J_{\text{c}} R_0^6 \Omega_0^3}{12}. \tag{5.7}$$

Comparing (5.6) and (5.7), one concludes

$$\boxed{E_{\text{rad}} \simeq 2E_c.} \tag{5.8}$$

Total energy. In the adopted reclassification convention, the pure tangential part is

$$E_{\tan}^{\mathrm{pur}} = 2E_c.$$

Consequently,

$$\boxed{E_{\mathrm{tot}} = E_{\tan}^{\mathrm{pur}} + E_{\mathrm{rad}} \simeq 2E_c + 2E_c = 4E_c.} \tag{5.9}$$

Scope. Relations (5.8)–(5.9) are demonstrated in the regime

$$\lambda_r \lambda \gg 1$$

with local algebraic radial closure and sech profile derived from (5.2). In this LO reduction, the numerical factor 2 between E_{rad} and E_c is exact. The relation $E_{\mathrm{rad}} \simeq 2E_c$ therefore holds in the strong-screening regime with a quasi-flat core. In the soft radial regime,

$$\lambda_r \lambda \ll 1,$$

the algebraic closure is replaced by the differential closure, which is not required in the present chapter.

5.4 Geometric spin index: definition, counting, and tests

Complex tangential rotator and effective number of polarizations. The tangential field is written in a local orthonormal basis $(\vec{e}_1, \vec{e}_2)$ in complex form at the carrier ω_0:

$$\mathbf{u}_{\parallel}(t,s) = \Re\left[\sum_{j=1}^{2} a_j(s)\, e^{i\omega_0 t}\, \vec{e}_j\right].$$

The *effective number of circular polarizations* is defined as

$$N_{\mathrm{pol}} := \mathrm{rank}_{\mathbb{C}}\, \mathrm{Span}\{a_1(s), a_2(s)\}$$

the number of independent circular polarizations remaining after accounting for physical constraints ($L^2(\mathbb{S}^3)$ orthogonality, C^1 matching, internal symmetries, *etc.*). Thus,

$$N_{\mathrm{pol}} = 1$$

means that a single independent circular polarization carries the dynamics, whereas

$$N_{\mathrm{pol}} = 2$$

corresponds to two independent polarizations.

Winding and monodromy. Let $w \in \mathbb{N}$ be the winding number of the normalized tangential rotator over one period $2\pi/\omega_0$. For the fundamental branches without internal node, the C^1 matching at LO rank selects the fundamental winding

$$w = 1,$$

cf. (6.106).

The retained counting principle is the following: each independent circular polarization contributes a monodromy factor $(-1)^w$ over a full cycle of the tangential rotator. This monodromy reading is consistent with the geometric reading by Berry connection presented in §8.3, where the geometric phase of the tangential rotator is related to the geometric spin number. For N_{pol} independent polarizations, the total monodromy is therefore

$$\mathcal{R}(2\pi): \quad \mathbf{u}_{\parallel} \mapsto (-1)^{wN_{\mathrm{pol}}}\, \mathbf{u}_{\parallel}.$$

Thus,

$$\mathcal{R}(2\pi) = -\mathrm{Id} \quad \Longleftrightarrow \quad wN_{\mathrm{pol}} \text{ odd},$$

which corresponds to a half-integer class, whereas even wN_{pol} corresponds to an integer class.

Definition of the geometric spin index. The geometric index is introduced as

$$\boxed{s_g := \frac{wN_{\mathrm{pol}}}{2}.}$$

This index must be understood as a geometric counter adapted to the classification of branches at LO. It immediately satisfies:

- if wN_{pol} is odd, then $s_g \in \frac{1}{2} + \mathbb{Z}$;
- if wN_{pol} is even, then $s_g \in \mathbb{Z}$.

Immediate consequences.

1. If N_{pol} is odd within a fundamental branch with $w = 1$, a half-integer class is obtained.
2. If N_{pol} is even within a fundamental branch with $w = 1$, an integer class is obtained.
3. More generally, the parity of the product wN_{pol} fixes the spin class.

Operational tests (determination of N_{pol} and w at LO).

1. L^2 **orthogonality and support:** separate the independent tangential contributions.
2. **Common carrier:** check that they share the same frequency ω_0.
3. **Internal symmetries:** some constraints may reduce the effective number of independent polarizations.
4. C^1 **matching and fundamental structure:** for the fundamental branches without internal node, the LO matching selects $w = 1$, cf. (6.106).

Scope of the geometric counting. The counting by

$$s_g = \frac{wN_{\mathrm{pol}}}{2}$$

classifies the branches at integer or half-integer level and fixes the geometric class of spin at LO. In particular, when wN_{pol} is odd, the branch belongs to a spinorial class with return up to sign on 2π and strict return on 4π. The physical norm of spin and the structure of its projections must then be established family by family; at this stage, they do not fall under the common geometric counting alone.

Chapter 6

Unit 2DOF solutions on $\mathbb{S}^3$: bump and NSL

6.1 Common framework and validity criteria (without a priori closure)

Local reduction in the quasi-flat core. In the quasi-flat cap, with geodesic coordinate s and width $\lambda \ll R_0$, the 2DOF tangential dynamics admits, at leading rank, a local reduction of the form

$$A''(s) - \Omega_0^2 A(s) + \frac{4}{R_0^3}\, r(s)\, A(s) \simeq 0,$$

where $r(s)$ denotes the local radial background. A further reduction to a cubic Klein–Gordon equation

$$A''(s) - \Omega^2 A(s) + \beta A^3(s) = 0$$

is introduced only when an explicit radial regime allows r to be closed locally as a function of A^2.

Decomposition into zones. We choose a threshold $s_c = k\lambda$, with $k \geq 3$. In the central zone $|s| \leq s_c$, the quasi-flat regime is used. In the intermediate zone $s_c < |s| < s_e$, residual contributions remain small; for bump-type profiles, they are typically of order $\mathcal{O}(e^{-k})$. Finally, the exterior zone $|s| \geq s_e$ is governed by the hyperspherical linear equation, and is matched by continuity of A and of the flux JA_s.

Local gauge of the core (equator). We center the core around a latitude $\chi_0 \simeq \pi/2$ (equator) and translate the curvilinear abscissa so that $s = 0$ corresponds to χ_0. In this gauge, for $|s| \lesssim \lambda \ll R_0$,

$$J(s) = R_0^2 \sin^2\chi\big((s + s_0)/R_0\big) \simeq R_0^2.$$

Explicit geometric contributions of type $1/R_0^2$ are then of order

$$\mathcal{O}\big((\lambda/R_0)^2\big),$$

and are neglected at first order in the core.

6.2 Common canonical form of the 2DOF sector and identification of h_{eff}

Purpose. We collect here the elements that are truly common to the 2DOF sector, independently of the detailed identity of the family (bump, NSL, *etc.*). The central point is the

existence of a reduced tangential phase variable and of a common canonical structure at leading rank.

Canonical reduction (reduced tangential phase). At LO, a neutral 2DOF solution admits a reduction to a tangential phase $\phi(t)$ describing the uniform rotation of the vector

$$q(t) = (\cos\phi, \sin\phi)$$

in a local plane. After LO radial projection and spatial integration, the reduced dynamics is that of a periodic mode of frequency ω, characterized by a common phase variable of period 2π.

Canonical action. For a harmonic periodic motion of frequency ω, one can write

$$\phi = A\cos\tau, \qquad \dot{\phi} = -A\omega\sin\tau, \qquad p = I_{\text{eff}}\dot{\phi},$$

where $I_{\text{eff}} > 0$ is an effective shape coefficient. The canonical action is then

$$\mathcal{J} = \frac{1}{2\pi}\oint p\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} I_{\text{eff}}\omega A^2 \sin^2\tau\, d\tau = \frac{I_{\text{eff}}\omega A^2}{2}.$$

Since the reduced energy of the mode is

$$E = \frac{1}{2} I_{\text{eff}}\omega^2 A^2,$$

one obtains the general identity

$$\boxed{\mathcal{J} = \frac{E}{\omega}.}$$

Absorbed EBK convention of the 2DOF sector. The action $\mathcal{J}$ defined above is already the reduced action

$$\mathcal{J} = \frac{1}{2\pi}\oint p\, d\phi.$$

The absorbed EBK convention of the 2DOF sector consists in fixing the fundamental level of this reduced action by

$$\boxed{\mathcal{J} = h_{\text{eff}}.} \tag{6.1}$$

Possible spectral indices are carried by the observed frequency ω, and not by an additional multiplicative factor in $\mathcal{J}$. Combining $\mathcal{J} = E/\omega$ with (6.1), one obtains

$$\boxed{E = h_{\text{eff}}\,\omega.} \tag{6.2}$$

This convention fixes the internal action unit of the 2DOF sector. It does not constitute an observational identification of h_{eff}.

Scope. The previous relations must be understood as common LO results of the 2DOF sector. They concern the reduced canonical structure, the phase periodicity and the common EBK rule of the 2DOF sector, but they do not yet fix, by themselves, the physical norm of spin or the structure of its projections. These points must then be established family by family.

6.3 Bump solution: profile, validity, spin

6.3.1 Local internal solution of the cubic equation

Proposition 6.1 (Local *bump* profile at LO)**.** *In the strong-screening regime where the local radial closure leads to the cubic equation*

$$A'' - \Omega_0^2 A + \beta_{\mathrm{alg}} A^3 = 0, \qquad \Omega_0 > 0, \qquad \beta_{\mathrm{alg}} > 0, \tag{6.3}$$

the local solution centered at $s = 0$, positive, even and bounded, satisfying

$$A(0) = A_0 > 0, \qquad A'(0) = 0, \qquad A(s) \to 0 \quad \textit{when } |s| \to \infty,$$

is

$$\boxed{A_{\mathrm{in}}(s) = A_0 \,\mathrm{sech}(\Omega_0 s) = A_0 \,\mathrm{sech}\Big(\frac{s}{\lambda_{\mathrm{bump}}}\Big), \qquad \lambda_{\mathrm{bump}} := \frac{1}{\Omega_0}, \qquad A_0^2 = \frac{2\Omega_0^2}{\beta_{\mathrm{alg}}}.}$$

Here, the condition $A(s) \to 0$ when $|s| \to \infty$ must be understood in the local quasi-flat core reduction, where the local geodesic coordinate is extended to $\mathbb{R}$ before matching with the exterior solution on $\mathbb{S}^3(R_0)$.

Proof. Multiplying (6.3) by A' and integrating once gives

$$\frac{1}{2}(A')^2 - \frac{\Omega_0^2}{2} A^2 + \frac{\beta_{\mathrm{alg}}}{4} A^4 = C.$$

Since $A(s) \to 0$ and $A'(s) \to 0$ when $|s| \to \infty$ in the local reduction, one obtains $C = 0$. Hence

$$(A')^2 = \Omega_0^2 A^2 - \frac{\beta_{\mathrm{alg}}}{2} A^4 = \Omega_0^2 A^2 \Big(1 - \frac{A^2}{A_0^2}\Big), \qquad A_0^2 = \frac{2\Omega_0^2}{\beta_{\mathrm{alg}}}.$$

For $s \geq 0$, the sought solution is decreasing, therefore

$$A' = -\Omega_0 A \sqrt{1 - \frac{A^2}{A_0^2}}.$$

Separating variables gives

$$\frac{dA}{A\sqrt{1 - A^2/A_0^2}} = -\Omega_0 \, ds.$$

Setting

$$A(s) = A_0 \,\mathrm{sech}\, u(s),$$

one has

$$dA = -A_0 \,\mathrm{sech}\, u \,\tanh u \, du, \qquad \sqrt{1 - \frac{A^2}{A_0^2}} = \tanh u,$$

whence

$$du = \Omega_0 \, ds.$$

The condition $A'(0) = 0$ with $A(0) = A_0$ imposes $u(0) = 0$, hence

$$u(s) = \Omega_0 s \qquad (s \geq 0).$$

Thus,

$$A(s) = A_0 \,\mathrm{sech}(\Omega_0 s) \qquad (s \geq 0).$$

The parity imposed by $A'(0) = 0$ then extends the solution to all $s \in \mathbb{R}$, which gives the announced expression. Uniqueness in this class follows from the Cauchy problem associated with (6.3). □

6.3.2 Kinematic compatibility of the moving bump

The 2DOF bump is built as a localized massive solution in its comoving frame. Its compatibility with uniform motion is treated in Chapter 7. That treatment separately establishes the boost law of the reduced massive core and the localization criterion of the comoving radial sheath.

6.3.3 Validation of the bump by decomposition into three zones

We use again the decomposition introduced in §6.1: central zone $|s| \le s_c$, intermediate zone $s_c < |s| < s_e$, then exterior zone $|s| \ge s_e$. We fix

$$s_c = k\,\lambda_{\mathrm{bump}}, \qquad \lambda_{\mathrm{bump}} := \frac{1}{\Omega_0}, \qquad k > 0.$$

For the internal profile

$$A_{\mathrm{in}}(s) = A_0\ \mathrm{sech}\Big(\frac{s}{\lambda_{\mathrm{bump}}}\Big),$$

one has, for $|s| \ge s_c$,

$$|A_{\mathrm{in}}(s)| \le 2A_0 e^{-k}, \qquad |A'_{\mathrm{in}}(s)| \le \frac{2A_0}{\lambda_{\mathrm{bump}}}\, e^{-k}.$$

The contributions carried by the intermediate zone, as well as the connection terms of the tangential basis in the ODE projected onto the retained mode, are therefore weighted by A_{in} or A'_{in} and remain exponentially small:

$$\text{intermediate error} = \mathcal{O}(e^{-k}).$$

In the central zone, the geometric corrections are controlled by the already established estimates. More precisely,

$$\Big|\frac{J'}{J}A'\Big| \le C\,\varepsilon_{\mathrm{slope}}\varepsilon_{\mathrm{geo}}\,|A''|, \qquad \text{and} \qquad \text{background curvature} = \mathcal{O}\Big(\Big(\frac{\lambda_{\mathrm{bump}}}{R_0}\Big)^2\Big),$$

for a constant $C = O(1)$, with

$$\varepsilon_{\mathrm{geo}} := \frac{\lambda_{\mathrm{bump}}}{R_0}, \qquad \varepsilon_{\mathrm{slope}} := \max_{|s|\le s_c} |r'(s)|.$$

Thus, for the core profile, one obtains the global error estimate

$$\boxed{\text{error} = \mathcal{O}(e^{-k}) + \mathcal{O}(\varepsilon_{\mathrm{slope}}\varepsilon_{\mathrm{geo}}) + \mathcal{O}\Big(\Big(\frac{\lambda_{\mathrm{bump}}}{R_0}\Big)^2\Big).}$$

In particular, the core profile is valid at first order as soon as

$$k \text{ is chosen sufficiently large}, \qquad \frac{\lambda_{\mathrm{bump}}}{R_0} \ll 1, \qquad \max_{|s|\le s_c} |r'(s)| \ll 1.$$

A choice $k \ge 3$ already provides an exponential truncation of size $\mathcal{O}(e^{-3})$.

6.3.4 Geometric spin of the bump

The *bump* mode carries a single independent circular polarization of the tangential field, hence

$$N_{\mathrm{pol}} = 1.$$

For the fundamental branch without internal node, the general criterion recalled in the geometric-counting section fixes the winding

$$w = 1$$

at leading order. One then obtains

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = \frac{1}{2}.$$

For this family, the energy consistency established in §5.3 and the common canonical form of the 2DOF sector (§6.2) give

$$\frac{E}{\omega} = h_{\mathrm{eff}}.$$

The physical modulus of spin is therefore

$$|S| = \frac{1}{2}\,\frac{E}{\omega} = \frac{1}{2}\,h_{\mathrm{eff}}.$$

Thus, for the fundamental bump, one obtains

$$\boxed{s_g = \tfrac{1}{2}, \qquad |S| = \tfrac{1}{2}\,h_{\mathrm{eff}}.}$$

Scope. The previous result is specific to the fundamental bump branch: the geometric counting fixes the half-integer class through

$$wN_{\mathrm{pol}} = 1,$$

whereas the spin modulus is determined here by the energy closure specific to the bump. The corresponding signed projections on the natural cycle will be established in §6.3.5. This result must not be automatically transposed to the other 2DOF families.

6.3.5 Spin projections of the bump on its natural cycle

For the free bump, the natural cycle $\Gamma_{\mathrm{nat}} \simeq S^1$ is the great circle supporting the tangential rotator. It is parametrized by

$$\varphi \in [0, 2\pi).$$

Since the fundamental bump satisfies

$$w = 1, \qquad N_{\mathrm{pol}} = 1,$$

one has

$$wN_{\mathrm{pol}} = 1,$$

so the monodromy on Γ_{nat} is odd. The field lifted onto the natural cycle then satisfies the antiperiodic condition

$$\Psi(\varphi + 2\pi) = -\,\Psi(\varphi), \qquad \Psi(\varphi + 4\pi) = \Psi(\varphi). \tag{6.4}$$

It follows that the spectral expansion is half-integer:

$$\Psi(\varphi) = \sum_{n\in\mathbb{Z}} c_n\, e^{\,i(n+\frac{1}{2})\varphi},$$

for which the generator

$$K := -\,i\,\partial_\varphi$$

satisfies

$$K\,e^{\,i(n+\frac{1}{2})\varphi} = \left(n+\frac{1}{2}\right) e^{\,i(n+\frac{1}{2})\varphi}.$$

The fundamental subspace is therefore

$$\mathcal{H}_{\frac{1}{2}} = \mathrm{Span}_{\mathbb{C}}\left\{e^{+i\varphi/2}, e^{-i\varphi/2}\right\}.$$

Setting

$$|+\rangle_\Gamma := e^{+i\varphi/2}, \qquad |-\rangle_\Gamma := e^{-i\varphi/2}, \qquad \widehat{S}_\Gamma := h_{\mathrm{eff}}\,K\big|_{\mathcal{H}_{1/2}},$$

one immediately obtains

$$\widehat{S}_\Gamma|+\rangle_\Gamma = +\frac{1}{2}\,h_{\mathrm{eff}}\,|+\rangle_\Gamma, \qquad \widehat{S}_\Gamma|-\rangle_\Gamma = -\frac{1}{2}\,h_{\mathrm{eff}}\,|-\rangle_\Gamma.$$

Proposition 6.2 (Projection doublet of the bump)**.** *The fundamental bump has, on its oriented natural cycle, a doublet of spin projections*

$$+\frac{1}{2}\,h_{\mathrm{eff}}, \qquad -\frac{1}{2}\,h_{\mathrm{eff}}.$$

Proof. Condition (6.4) imposes a half-integer spectrum for $K = -i\partial_\varphi$. The restriction of K to the fundamental subspace $\mathcal{H}_{1/2}$ therefore has the eigenvalues $\pm\frac{1}{2}$, hence the eigenvalues $\pm\frac{1}{2}h_{\mathrm{eff}}$ for $\widehat{S}_\Gamma = h_{\mathrm{eff}}K|_{\mathcal{H}_{1/2}}$. □

Degeneracy at leading rank. At leading rank, if the effective problem is invariant under reversal

$$\varphi \mapsto -\varphi,$$

or if the energy depends on K only through K^2, the two states

$$|+\rangle_\Gamma, \qquad |-\rangle_\Gamma$$

are degenerate. This symmetry is natural for the free bump and for the bound problems with central symmetry at first order.

6.4 2DOF NSL solution: construction, validity, spin

6.4.1 Two-leg ansatz and geometric motivation

In what follows, the *2DOF NSL* denotes the two-leg iso-frequency branch described below. We work in a tubular neighborhood of a central geodesic Γ (core axis). The variable

$$s$$

denotes the curvilinear abscissa along Γ, while

$$\vartheta \in [0, 2\pi)$$

denotes the azimuthal angle in the retained transverse tangent 2-plane,

$$\Pi_\perp(s) = \mathrm{Span}\{\vec{e}_1(s), \vec{e}_2(s)\}.$$

The second global transverse angle of $\mathbb{S}^3$ is frozen in this local reduction: the ansatz depends only on the internal azimuthal direction ϑ in $\Pi_\perp(s)$.

We then introduce the fast circular polarization

$$\mathbf{q}(t) := \cos\omega t\,\vec{e}_1 + \sin\omega t\,\vec{e}_2, \qquad J\mathbf{q}(t) := -\sin\omega t\,\vec{e}_1 + \cos\omega t\,\vec{e}_2,$$

where J denotes the rotation by $+\pi/2$ in the plane $(\vec{e}_1, \vec{e}_2)$. The 2DOF NSL ansatz is written as

$$\boxed{\vec{u}_{\mathrm{NSL}}(s,\vartheta,t) = A(s)\Big[\cos\vartheta\,\mathbf{q}(t) + \sin\vartheta\,J\mathbf{q}(t)\Big].}$$

In other words, the solution is constructed as a superposition of two orthogonal tangential legs, iso-frequency, sharing the same temporal carrier ωt, in quadrature in the transverse plane.

Zonality and stationarity of the quadratic observables. Since $|\mathbf{q}| = |J\mathbf{q}| = 1$ and $\mathbf{q}\cdot J\mathbf{q} = 0$, one obtains

$$|\vec{u}_{\mathrm{NSL}}(s,\vartheta,t)|^2 = A(s)^2\big(\cos^2\vartheta + \sin^2\vartheta\big) = A(s)^2.$$

The quantity $|\vec{u}_{\mathrm{NSL}}|^2$ is therefore independent of both ϑ and t. In particular, the quadratic source feeding the radial equation depends only on s, which makes the closure

$$r = r(s).$$

consistent.

Why two legs? With a single leg, for example

$$\vec{u}(s,\vartheta,t) = A(s)\cos\vartheta\,\mathbf{q}(t),$$

one would instead have

$$|\vec{u}|^2 = A(s)^2\cos^2\vartheta = \frac{1}{2}A(s)^2(1+\cos 2\vartheta),$$

so that the quadratic source would contain a non-zonal component. The strictly axisymmetric closure

$$r = r(s)$$

would then be broken.

Status of the ansatz. The two-leg construction is therefore not introduced here as a free artifact: it is the minimal structure that preserves a zonal radial closure while allowing a nontrivial angular dependence of the tangent field. The next subsection will show that, in the core, this construction indeed leads to an effective cubic Klein–Gordon-type reduction for the common envelope $A(s)$.

6.4.2 Tangential reduction in the core: effective equation

In the quasi-flat core, the zonality of the quadratic source

$$|\vec{u}_{\mathrm{NSL}}|^2 = A(s)^2$$

makes the radial closure $r = r(s)$ consistent. After projection of the two-leg NSL ansatz onto the retained mode, the common envelope $A(s)$ satisfies, in the local cubic closure regime, a focal scalar cubic Klein–Gordon-type equation

$$\boxed{A''(s) - \Omega_{\mathrm{NSL}}^2\,A(s) + \beta_{\mathrm{NSL}}\,A^3(s) \;\simeq\; 0, \qquad \Omega_{\mathrm{NSL}}^2 > 0, \qquad \beta_{\mathrm{NSL}} > 0.}$$

The effective coefficients Ω_{NSL} and β_{NSL} collect the local contributions of the core: tangential inertial term, gradients, two-leg angular structure, and retained radial closure. They are therefore not identified *a priori* with those of the bump.

By applying Proposition 6.1 to this scalar ODE, one obtains the localized profile

$$A(s) = A_0 \operatorname{sech}\Big(\frac{s}{\lambda_{\mathrm{NSL}}}\Big), \qquad \lambda_{\mathrm{NSL}} := \frac{1}{\Omega_{\mathrm{NSL}}}, \qquad A_0^2 = \frac{2\Omega_{\mathrm{NSL}}^2}{\beta_{\mathrm{NSL}}}.$$

6.4.3 Hyperspherical exterior and matching

Outside the core, the angular dependence of the NSL is fixed by the two-leg ansatz. The exterior problem therefore reduces to a linear equation for the scalar envelope $A(s)$, associated with the retained angular mode on $\mathbb{S}^3$. Regularity at the poles then selects a discrete family of admissible frequencies.

Matching at the edge $s = s_c$ is imposed on the full field. Since the angular structure is common on both sides of the edge, it reduces, at the envelope level, to the scalar conditions

$$[A]_{s=s_c} = 0, \qquad [JA_s]_{s=s_c} = 0.$$

The zonality of $|\vec{u}_{\mathrm{NSL}}|^2$ again guarantees the consistency of the radial closure.

This section does not spell out the detailed exterior spectral basis, since it is not needed for the energy and spin laws used later.

6.4.4 Validity of the two-zone reduction

Once the core profile is written in the form

$$A(s) = A_0 \operatorname{sech}\Big(\frac{s}{\lambda_{\mathrm{NSL}}}\Big), \qquad s_c = k\lambda_{\mathrm{NSL}},$$

the same decay estimates as for the bump apply *mutatis mutandis* (see §6.3.3):

$$|A(s)| \le 2A_0 e^{-k}, \qquad |A'(s)| \le \frac{2A_0}{\lambda_{\mathrm{NSL}}} e^{-k} \qquad (|s| \ge s_c).$$

The connection terms outside the core therefore contribute $\mathcal{O}(e^{-k})$, while inside the core the geometric corrections remain controlled by

$$\mathcal{O}(\varepsilon_{\mathrm{slope}}\varepsilon_{\mathrm{geo}}) + \mathcal{O}\Big(\Big(\frac{\lambda_{\mathrm{NSL}}}{R_0}\Big)^2\Big), \qquad \varepsilon_{\mathrm{geo}} := \frac{\lambda_{\mathrm{NSL}}}{R_0}, \qquad \varepsilon_{\mathrm{slope}} := \max_{|s|\le s_c} |r'(s)|.$$

Thus, the two-zone NSL reduction is valid at first order under the assumptions

$$k \text{ sufficiently large}, \qquad \frac{\lambda_{\mathrm{NSL}}}{R_0} \ll 1, \qquad \max_{|s|\le s_c} |r'(s)| \ll 1,$$

with a global error of size

$$\mathcal{O}(e^{-k}) + \mathcal{O}(\varepsilon_{\mathrm{slope}}\varepsilon_{\mathrm{geo}}) + \mathcal{O}\Big(\Big(\frac{\lambda_{\mathrm{NSL}}}{R_0}\Big)^2\Big).$$

6.4.5 Energy, action, and integer projections of the 2DOF NSL

2DOF structure of the NSL. The two-leg NSL construction provides two independent tangential contributions sharing the same fast phase. Hence

$$N_{\mathrm{pol}} = 2, \qquad w = 1.$$

The common geometric counting then gives

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = 1.$$

Energy and action. The common canonical structure of the 2DOF sector (§6.2) gives

$$\mathcal{J} = \frac{E_{\mathrm{tot}}}{\omega}, \qquad E_{\mathrm{tot}} = h_{\mathrm{eff}}\,\omega.$$

Monodromy and periodic condition. Since

$$wN_{\mathrm{pol}} = 2$$

is even, the monodromy of the 2DOF NSL is trivial:

$$\boxed{\Psi(\varphi + 2\pi) = \Psi(\varphi).}$$

The 2DOF NSL branch therefore belongs to an integer class.

Integer spectral decomposition. The periodic condition implies the integer Fourier expansion

$$\Psi(\varphi) = \sum_{m\in\mathbb{Z}} c_m\, e^{im\varphi}.$$

Introducing the generator

$$K := -\,i\,\partial_\varphi,$$

one has

$$K\, e^{im\varphi} = m\, e^{im\varphi}, \qquad m \in \mathbb{Z}.$$

Integer fundamental subspace. The mode $m = 0$ corresponds to a constant phase along the natural cycle; it therefore carries no winding and is not compatible with the fundamental branch $w = 1$. The first nontrivial subspace compatible with this branch is then

$$\mathcal{H}_1 = \mathrm{Span}_{\mathbb{C}} \left\{ e^{+i\varphi}, e^{-i\varphi} \right\}.$$

We set

$$|+\rangle_\Gamma := e^{+i\varphi}, \qquad |-\rangle_\Gamma := e^{-i\varphi}.$$

Projected spin operator. In the common spectral normalization of the 2DOF sector, we define

$$\widehat{S}_\Gamma := h_{\mathrm{eff}}\, K\big|_{\mathcal{H}_1}.$$

One obtains

$$\boxed{\widehat{S}_\Gamma|+\rangle_\Gamma = +h_{\mathrm{eff}}\,|+\rangle_\Gamma, \qquad \widehat{S}_\Gamma|-\rangle_\Gamma = -h_{\mathrm{eff}}\,|-\rangle_\Gamma.}$$

Proposition 6.3 (Integer projections of the 2DOF NSL)**.** *The fundamental 2DOF NSL has, on its oriented natural cycle, a spin-projection doublet*

$$+h_{\mathrm{eff}}, \qquad -h_{\mathrm{eff}}.$$

In particular, its spin modulus is

$$\boxed{|S_{\mathrm{NSL}}| = h_{\mathrm{eff}}, \qquad s_g = 1.}$$

Proof. The trivial monodromy imposes the periodicity $\Psi(\varphi + 2\pi) = \Psi(\varphi)$, hence an integer spectrum for $K = -i\partial_\varphi$. The mode $m = 0$ carries no winding and is not suitable for the fundamental branch $w = 1$. The first nontrivial subspace is therefore $\mathcal{H}_1 = \mathrm{Span}_{\mathbb{C}}\{e^{+i\varphi}, e^{-i\varphi}\}$, on which K takes the eigenvalues ± 1. The definition $\widehat{S}_\Gamma = h_{\mathrm{eff}} K|_{\mathcal{H}_1}$ then gives the eigenvalues $\pm h_{\mathrm{eff}}$. The modulus is therefore h_{eff}, and the geometric counting indeed gives $s_g = 1$. □

Consequence for half-integer branches. The half-integer construction of the bump, based on an antiperiodic condition and leading to the doublet

$$\pm\frac{1}{2}\,h_{\mathrm{eff}},$$

does not apply here. The 2DOF NSL instead carries an integer doublet

$$\pm h_{\mathrm{eff}}.$$

Scope. The 2DOF NSL sector thus illustrates the separation between:

- the geometric counting, which fixes the integer class through $wN_{\mathrm{pol}} = 2$;
- and the spectral structure on the natural cycle, which here selects the integer doublet $\pm h_{\mathrm{eff}}$.

6.4.6 Remark on EBK (spectrum placement)

EBK quantization applies to the same reduced phase variable as for the other 2DOF solutions. In this chapter, we write directly

$$\mathcal{J} = \frac{E_{\mathrm{tot}}}{\omega} = h_{\mathrm{eff}},$$

so that any spectrum is carried by the observed frequency ω. The later spectral sections may reintroduce, if necessary, a discrete index q through the spectral structure of ω, without modifying the relations

$$E_{\mathrm{tot}} = h_{\mathrm{eff}}\,\omega, \qquad |S_{\mathrm{NSL}}| = h_{\mathrm{eff}}.$$

6.5 Spectro–geometric filters for 2DOF solutions (common framework)

This section establishes the common framework of the filters applicable to any solution with two degrees of freedom (2DOF) before explicit closure.

We work in a tubular neighborhood of a great circle of $\mathbb{S}^3(R_0)$, locally recentered at the equator. In other words, we introduce a local coordinate

$$s = \frac{\pi R_0}{2} + r, \qquad |r| \lesssim \lambda,$$

so that $r = 0$ corresponds to the point of minimal curvature along the chosen geodesic. The common regime is that of a thin tube and a quasi-flat cap:

$$\varepsilon := \frac{\lambda}{R_0} \ll 1, \qquad A(0) = 1, \qquad A'(0) = 0.$$

For EBK, we use the absorbed convention of the 2DOF sector (6.1)–(6.2). In this section, we recall it in the form

$$\mathcal{J} = \frac{E}{\omega} = h_{\mathrm{eff}}, \tag{6.5}$$

the possible spectral index being carried by the observed frequency ω. When a discrete index $q \in \mathbb{N}$ is reintroduced explicitly, it is introduced through the dependence $\omega = \omega(q)$, without modifying the form $\mathcal{J} = E/\omega$.

We shall finally call *single-ring reduction* the spectral assumption according to which, for fixed core geometry,

$$\varepsilon(q) = \frac{\varepsilon(1)}{q}.$$

This relation will be used below to rewrite the exterior spectral parameter as a function of q.

6.5.1 Quasi-flat $\mathbb{S}^3$ regime, local reduction, and error control

In the preceding local gauge, the hyperspherical Jacobian

$$J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big)$$

is evaluated near the equator as

$$J\Big(\frac{\pi}{2}R_0 + r\Big) = R_0^2 \cos^2\Big(\frac{r}{R_0}\Big) = R_0^2\Big[1 + \mathcal{O}\Big(\frac{r^2}{R_0^2}\Big)\Big] = R_0^2\Big[1 + \mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.6}$$

Thus, at leading order, one may replace J by the constant R_0^2 in the core, with a relative error quadratic in $\varepsilon = \lambda/R_0$.

The local core/exterior reduction, with an exponentially small matching zone, then introduces the corrections

$$\text{LO}: \qquad \mathcal{O}(e^{-k}) \;+\; \mathcal{O}(\varepsilon_{\text{slope}}\varepsilon) \;+\; \mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big), \tag{6.7}$$

where $k \geq 3$ controls the exponential truncation and

$$\varepsilon_{\text{slope}} := \max_{|s|\leq s_c} |r'(s)|.$$

In a rougher notation, these corrections may be condensed as

$$\mathcal{O}(e^{-k}) + \mathcal{O}\Big(\frac{\lambda}{R_0}\Big) + \mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big),$$

but the LO identities of the manuscript must be read in the precise sense of (6.7).

6.5.2 LO matching and DtN map (channel filter)

We call *channel* X the retained exterior-reduction family (bump, NSL, *etc.*), and Δ_X the pure dimensionless shift induced, at leading order, by the Dirichlet–to–Neumann map associated with this channel.

Step 1: interior logarithmic derivative. For a localized core of the type

$$A_{\text{in}}(s) = A_0 \,\operatorname{sech}\Big(\frac{s}{\lambda}\Big), \qquad s_c = u_*\lambda,$$

one obtains at the core boundary

$$\frac{A'_{\text{in}}}{A_{\text{in}}}(s_c) = -\frac{1}{\lambda}\tanh u_*.$$

Step 2: exterior logarithmic derivative regular on $\mathbb{S}^3$. For the regular exterior zonal mode of index $n \in \mathbb{N}$, one has

$$A_{\text{ext}}(\theta) \propto \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad \theta = \frac{s}{R_0},$$

whence

$$\frac{A'_{\text{ext}}}{A_{\text{ext}}}(s_c) = \frac{1}{R_0}\left[(n+1)\cot\big((n+1)\theta_c\big) - \cot\theta_c\right], \qquad \theta_c = \frac{s_c}{R_0} = u_*\varepsilon.$$

Step 3: LO matching. The C^1 matching at the boundary $s = s_c$ imposes equality of the logarithmic derivatives:

$$-\frac{1}{\lambda}\tanh u_* = \frac{1}{R_0}\left[(n+1)\cot\big((n+1)u_*\varepsilon\big) - \cot(u_*\varepsilon)\right].$$

Multiplying by $u_*\lambda$, then introducing

$$y := (n+1)\,u_*\,\varepsilon,$$

one obtains

$$-u_*\tanh u_* = y\cot y - u_*\varepsilon\,\cot(u_*\varepsilon).$$

In other words,

$$y\cot y = u_*\varepsilon\,\cot(u_*\varepsilon) - u_*\tanh u_*.$$

Step 4: layer correction and channel correction. We define the subdominant layer correction by

$$\delta\Lambda(u_*,\varepsilon) := u_*\varepsilon\,\cot(u_*\varepsilon) - 1 = \mathcal{O}\big((u_*\varepsilon)^2\big). \tag{6.8}$$

In channel X, the projected DtN map further adds a pure shift

$$\Delta_X(u_*;\kappa_r),$$

which depends on the channel and the window, but not on the dimensioned constants of the medium. One thus obtains the general LO matching

$$\boxed{h(y) := y\cot y = 1 - u_*\tanh u_* + \Delta_X(u_*;\kappa_r) + \delta\Lambda(u_*,\varepsilon),} \qquad y = (n{+}1)\,u_*\,\varepsilon, \qquad \varepsilon = \frac{\lambda}{R_0} \ll 1. \tag{6.9}$$

To lighten the notation, we introduce the reduced right-hand side

$$G_X(u_*;\kappa_r,\varepsilon) := 1 - u_*\tanh u_* + \Delta_X(u_*;\kappa_r) + \delta\Lambda(u_*,\varepsilon). \tag{6.10}$$

Boundedness and scale independence. After dimensionless reduction at leading order, G_X is a bounded pure number, independent of the dimensioned constants of the medium

$$\rho, \quad T, \quad \mu_2, \quad R_0, \quad \ell_0.$$

Its sign and amplitude depend only on the channel X, on the window u_*, and on the dimensionless parameters of the reduction.

6.5.3 Robust core window and cutoff

The geometric term

$$1 - u\tanh u$$

is decreasing and remains of bounded negative size for $u = \mathcal{O}(1)$. Under the assumption of a bounded boundary shift

$$\Delta_X = \mathcal{O}(1),$$

and of a subdominant layer correction

$$\delta\Lambda = \mathcal{O}\big((u\varepsilon)^2\big),$$

one anticipates, at leading order, a control window for the matching

$$\boxed{u_* \in [1.7,\ 2.1]}. \tag{6.11}$$

This writing is not an autonomous numerical closure of u_*. It is the rounded version of the window derived below from the matching filter

$$-1 \le 1 - u_* \tanh u_* \le -\frac{3}{5},$$

cf. (6.19)–(6.20). The central value consumed by the core closures is then fixed by the canonical matching condition (6.22)–(6.23).

The radial cutoff is then evaluated at the boundary from the canonical core profile

$$A(u) = A(0)\,\operatorname{sech} u.$$

We introduce

$$\Sigma_{\text{eff}} := \sup_{u\in\mathcal{U}_{\text{adm}}} u^2\Big(\tfrac{A(u)}{A(0)}\Big)^2. \tag{6.12}$$

For the canonical profile $A(u) = A(0)\operatorname{sech} u$, and for the complete window

$$\mathcal{U}_{\text{adm}} = [1.7,\, 2.1],$$

one obtains

$$0.2568\ldots \le u^2\operatorname{sech}^2 u \le 0.3613\ldots \qquad (u \in \mathcal{U}_{\text{adm}}).$$

One may therefore use, at LO, the conservative bound

$$u_*^2\operatorname{sech}^2 u_* \le \Sigma_{\text{eff}}^{\text{LO}}, \qquad \Sigma_{\text{eff}}^{\text{LO}} = 0.37.$$

This quantity provides a simple and robust indicator of the boundary cutoff in the admissible window considered.

6.5.4 Absorbed EBK, dispersion, and $y \leftrightarrow q$ map

The 2DOF families share the same phase variable and the same Maslov index. In this section, we use the absorbed EBK convention (6.5): the spectral index q is carried by the dispersion through $\varepsilon(q)$, and is not displayed explicitly in $\mathcal{J}$.

The zonal dispersion on $\mathbb{S}^3$, combined with the single-ring reduction, imposes

$$\varepsilon(q) = \frac{\varepsilon(1)}{q}, \qquad y = (n+1)\,u_*\,\varepsilon(q). \tag{6.13}$$

In the reference single-ring branch, the exterior radial index and the reduced width of the fundamental are related by

$$(n+1)\,\varepsilon(1) = 1 + \mathcal{O}(\varepsilon^2).$$

One therefore obtains, at LO,

$$\boxed{y = \frac{u_*}{q}\Big[1 + \mathcal{O}(\varepsilon^2)\Big].} \tag{6.14}$$

We shall use the sign map of

$$h(y) = y\cot y :$$

$$h(y)\begin{cases} > 0, & 0 < y < \frac{\pi}{2}, \\ < 0, & \frac{\pi}{2} < y < \pi, \end{cases} \qquad h'(y) = \cot y - y\,\csc^2 y \;<\; 0 \quad \text{sur chaque bande ouverte } (k\pi, (k{+}1)\pi). \tag{6.15}$$

Thus, on each open band $(k\pi, (k+1)\pi)$, the function h is strictly decreasing and therefore bijective on its image.

6.5.5 Universal filter: decision by the sign in the window

Proposition 6.4 (Existence/control for $u_* \in [1.7, 2.1]$)**.** *Consider* (6.9)–(6.10) *with the single-ring map* (6.14)*,*

$$y = \frac{u_*}{q}\,[1 + \mathcal{O}(\varepsilon^2)].$$

1. *If $G_X(u_*; \kappa_r, \varepsilon) < 0$ at LO, then for every $q \geq 2$ one has $y \leq u_*/2 < \pi/2$, hence $h(y) > 0$; the equality* (6.9) *is impossible. By contrast, for $q = 1$, one has $y = u_* \in (\pi/2, \pi)$, and since h is continuous and strictly decreasing on this band, there exists a unique solution in this band.*

2. *If $G_X(u_*; \kappa_r, \varepsilon) \in (0, 1)$ at LO, then for each open band where h takes the value G_X, there exists exactly one solution y. Via* (6.13)*, this selects a discrete set of admissible EBK levels.*

Definition 6.5 (Local sign criteria of the LO matching)**.** For an LO matching of the form (6.9), we shall say that the reduced right-hand side is *negative* on the admissible window if

$$G_X(u_*; \kappa_r, 0) < 0 \qquad \text{for all } u_* \in [1.7, 2.1].$$

We shall say that it is *positive bounded* on this window if there exist pure bounds $0 < g_{\min} \leq g_{\max} < 1$ such that

$$G_X(u_*; \kappa_r, 0) \in [g_{\min}, g_{\max}] \qquad \text{for all } u_* \in [1.7, 2.1].$$

These criteria are not family assignments. They are usable only when a sign lemma or an explicit evaluation of the matching map establishes one of the two conditions.

Proof sketch. Point (i) follows directly from (6.15) and from the bound $y \leq u_*/q$. Point (ii) follows from the bijectivity of h on each open band and from the monotonicity $h' < 0$.

LO robustness. The corrections $\delta\Lambda = \mathcal{O}((u\varepsilon)^2)$ and the bounded variations of the DtN (fixed channel) change neither the sign of G_X, nor the monotonicity of h. The decision (uniqueness of $q = 1$ versus discrete set of admissible levels) is therefore robust at leading order.

6.5.6 Selection procedure by bands (generic)

We consider the LO matching (6.9) and the definition (6.10). At leading order, for a given channel X, we bound the right-hand side

$$G_X(u_*; \kappa_r, \varepsilon)$$

by pure bounds $g_{\min}, g_{\max} \in (-\infty, 1)$ on the robust window

$$u_* \in [1.7, 2.1],$$

under the assumptions of bounded DtN and subdominant layer correction. We then use the spectral map (6.13) and the band-by-band monotonicity of $h(y) = y \cot y$ (cf. (6.15)).

Generic algorithm.

1. **Envelope.** Fix an admissible interval

$$[g_{\min}, g_{\max}] \subset (-\infty, 1)$$

for G_X at LO.

2. **Band inversion.** For every $g \in [g_{\min}, g_{\max}]$ and for every open band $(k\pi, (k+1)\pi)$, solve

$$h(y_k) = g.$$

The uniqueness of $y_k(g)$ on each band follows from the strict monotonicity of h.

3. **EBK projection.** Then associate the candidate integer corresponding to the closest value of

$$\frac{u_*}{y_k(g)}, \qquad u_* \in [1.7, 2.1].$$

In other words,

$$q_k(u_*, g) := \operatorname{round}\Big(\frac{u_*}{y_k(g)}\Big),$$

provided it remains uniformly away from half-integers throughout the envelope.

4. **Integer stability (plateaus).** Check that $q_k(u_*, g)$ remains constant when (u_*, g) varies in the envelope, that is, that no half-integer crossing occurs.

Proposition 6.6 (Generic output and LO robustness)**.** *Under the preceding assumptions, the procedure selects a discrete, and in practice finite, set of EBK integers stable by plateaus for channel X and the family considered. The corrections*

$$\delta\Lambda = \mathcal{O}((u\varepsilon)^2)$$

and the bounded variations of the DtN modify neither the monotonicity of h, nor the stability of the plateaus at leading order.

6.5.7 Guideposts for the numerical audit (generic)

Band-by-band y solver: bisection on $(k\pi, (k+1)\pi)$, stopping when

$$|h(y) - g| \le 10^{-14}.$$

Grids:

$$u_* \in [1.7, 2.1] \quad (\text{step } 10^{-3}), \qquad g \in [g_{\min}, g_{\max}] \quad (\text{step } 10^{-4}).$$

Check: constancy of q_k on the grid, with no integer change over the whole tested envelope.

Summary. (i) **Common filters.** Thin tube ($\varepsilon \ll 1$), local reduction with matching (errors (6.7)), LO matching (6.9) and bounded DtN (6.10), robust window (6.11) and cutoff (6.12), absorbed EBK (6.5), $y \leftrightarrow q$ map (6.13), and sign/monotonicity of h (6.15).

(ii) **Decision rule (universal filter).** By Proposition 6.4:

$$G_X < 0 \;\Rightarrow\; \textbf{unique } q = 1, \qquad G_X \in (0, 1) \;\Rightarrow\; \textbf{discrete set}$$

(one solution per band). LO robustness with respect to bounded variations of the DtN and to

$$\delta\Lambda = \mathcal{O}((u\varepsilon)^2).$$

(iii) **Scope.** The determination of the admissible integers q is *family + channel dependent* and is deferred to each closure chapter (bump, NSL, etc.). No numerical q is fixed here.

(iv) **Scale independence.** All terms involved are *pure numbers* in the sense of the HSM theory.

6.6 Core closure for the bump

In the LO quasi-flat regime on $\mathbb{S}^3$, the branch is characterized by a core window, a radial cutoff, internal shape factors, and a spectral closure of the channel. These elements make it possible to fix the effective one-body closure associated with the bump.

We work at **LO** in the quasi-flat cap of $\mathbb{S}^3$,

$$\varepsilon := \frac{\lambda}{R_0} \ll 1, \qquad c_t^2 = \frac{T}{\rho}, \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big),$$

and we consider a fundamental described by

$$\psi(t, \mathbf{x}) = A(\xi)\, e^{i\omega t}, \qquad \xi = \frac{s}{\lambda}, \qquad A(0) = 1, \quad A'(0) = 0, \quad A(\xi) = \operatorname{sech} \xi.$$

6.6.1 Core window and cutoff

(1) LO matching and admissibility window for u_*. At leading order, the core–exterior matching is read as an equality of reduced Dirichlet–to–Neumann maps. The inner map is that of the regular core. In the reduced variable $u = s/\lambda$, the regular radial component associated with the canonical profile is

$$F_{\text{in}}(u) = u \operatorname{sech} u.$$

Its reduced logarithmic map is

$$\boxed{\mathcal{D}_{\text{in}}(u) := u\, \frac{F_{\text{in}}'(u)}{F_{\text{in}}(u)} = 1 - u \tanh u.} \tag{6.16}$$

The first regular exterior cell is described, in the reduced matching gauge, by the oscillatory mode

$$F_{\text{out}}(y) = \sin y.$$

Its reduced logarithmic map is

$$\boxed{\mathcal{D}_{\text{out}}(y) := y\, \frac{F_{\text{out}}'(y)}{F_{\text{out}}(y)} = y \cot y.} \tag{6.17}$$

The LO matching therefore imposes

$$\boxed{\mathcal{D}_{\text{out}}(y) = \mathcal{D}_{\text{in}}(u_*), \qquad y = (n{+}1)\, u_*\, \varepsilon, \qquad \varepsilon = \frac{\lambda}{R_0} \ll 1.} \tag{6.18}$$

In other words,

$$y \cot y = 1 - u_* \tanh u_*.$$

Admissibility window. The admissible core window is used as a regime control: it selects the matchings for which the inner right-hand side remains negative, bounded, and away from the singular edges of the oscillatory matching. We impose

$$\boxed{-1 \;\le\; 1 - u_* \tanh u_* \;\le\; -\frac{3}{5}.} \tag{6.19}$$

This band keeps the right-hand side of the matching of order 1, away from the overly soft regime $1 - u_* \tanh u_* \simeq 0$ and away from an overly hard cutoff.

Set

$$g(u) := 1 - u \tanh u.$$

Since

$$g'(u) = -\tanh u - u\,\mathrm{sech}^2 u < 0 \qquad (u > 0),$$

the function g is strictly decreasing on $\mathbb{R}_+$. The condition (6.19) is therefore equivalent to

$$u_- \le u_* \le u_+,$$

where

$$g(u_-) = -\frac{3}{5}, \qquad g(u_+) = -1.$$

Numerically,

$$u_- \approx 1.7085, \qquad u_+ \approx 2.0653.$$

We therefore retain the control window

$$\boxed{u_* \in [u_-, u_+] = [1.7085\ldots,\ 2.0653\ldots].} \tag{6.20}$$

In the running text, we shall use the rounded version

$$\boxed{u_* \in [1.7,\ 2.1].} \tag{6.21}$$

On the first physical branch

$$y \in \Big(\frac{\pi}{2}, \pi\Big),$$

the function $h(y) := y\cot y$ is strictly decreasing. The window (6.19) therefore corresponds to a unique physical root in this branch and keeps the matching away from both edges.

(1bis) Canonical matching closure for u_*. The window (6.21) fixes the admissibility domain of the matching. The central value of u_* is then determined by the canonical single-cell closure: the core and the first exterior cell are evaluated on the same reduced boundary coordinate. In this matching gauge, one identifies

$$y = u.$$

The DtN condition (6.18) then becomes

$$\boxed{\phi(u) := u\cot u - \big(1 - u\tanh u\big) = 0, \qquad u \in [1.7, 2.1].} \tag{6.22}$$

On this window, ϕ changes sign and has a unique root on the branch $(\pi/2, \pi)$. This root defines the canonical reduced parameter of the matching:

$$\boxed{u_*^{(\mathrm{ref})} \simeq 2.00712.} \tag{6.23}$$

Thus the interval (6.20) is not a free cutoff value: it is the admissibility domain derived from the matching filter (6.19). Downstream quantities that require a central core value consume the canonical root $u_*^{(\mathrm{ref})}$ defined by (6.22)–(6.23). The bounds $[u_-, u_+]$ are used for robustness checks and cutoff upper bounds.

(2) Cutoff ⇒ evaluation of Σ_{eff} and ε_*. We define the shape constant

$$\Sigma_{\text{eff}} := \sup_{u\in\mathcal{U}_{\text{adm}}} u^2\Big(\tfrac{A(u)}{A(0)}\Big)^2, \qquad \mathcal{U}_{\text{adm}} := [u_-, u_+]. \tag{6.24}$$

For the canonical profile

$$A(u) = A(0)\,\operatorname{sech} u,$$

one obtains

$$u^2\Big(\tfrac{A(u)}{A(0)}\Big)^2 = u^2\operatorname{sech}^2 u.$$

On the interval $[u_-, u_+]$, this function is decreasing, hence

$$u_+^2\operatorname{sech}^2(u_+) \;\le\; u_*^2\operatorname{sech}^2 u_* \;\le\; u_-^2\operatorname{sech}^2(u_-),$$

that is, numerically,

$$0.266 \;\lesssim\; u_*^2\operatorname{sech}^2 u_* \;\lesssim\; 0.359. \tag{6.25}$$

In particular,

$$\Sigma_{\text{eff}} = \sup_{u\in[u_-,u_+]} u^2\operatorname{sech}^2 u = u_-^2\operatorname{sech}^2(u_-) \approx 0.359.$$

Similarly, the relative boundary amplitude

$$\varepsilon_* := \frac{A(u_*)}{A(0)} = \operatorname{sech} u_*$$

satisfies

$$\operatorname{sech}(u_+) \le \varepsilon_* \le \operatorname{sech}(u_-),$$

that is,

$$0.2495 \;\lesssim\; \varepsilon_* \;\lesssim\; 0.3508.$$

To the nearest hundredth, one thus recovers

$$\boxed{\varepsilon_* \in [0.25,\ 0.35].} \tag{6.26}$$

There is therefore no circularity: the window (6.21) is first obtained from the matching filter (6.19), and then Σ_{eff} and ε_* are simply evaluated on this window.

6.6.2 Local parametrization at the center and definition of $\Phi_\star$

Local even/odd basis and normalization. Near the center $s = 0$, the reduced radial equation admits a local even/odd basis. We write

$$f_{\text{even}}(s) = 1 - \frac{1}{2}\kappa_0^2 s^2 + \mathcal{O}(s^4), \qquad f_{\text{odd}}(s) = \frac{s}{\lambda} + \mathcal{O}\Big(\frac{s^3}{\lambda^3}\Big),$$

where κ_0 characterizes the local curvature at the center. More precisely, under the normalization $A(0) = 1$, we set

$$\kappa_0^2 := -A''(0).$$

For the fundamental profile of type $\operatorname{sech}(s/\lambda)$, at leading order one has

$$A''(0) = -\frac{1}{\lambda^2}, \qquad \text{hence} \qquad \kappa_0 = \frac{1}{\lambda}.$$

The general local solution is then written

$$A(s) = C_{\text{even}}\, f_{\text{even}}(s) + C_{\text{odd}}\, f_{\text{odd}}(s).$$

The normalization

$$A(0) = 1$$

imposes $C_{\text{even}} = 1$. We then parametrize the odd component by

$$C_{\text{odd}} = \tan \Phi_\star, \qquad \Phi_\star \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big),$$

so that

$$A'(0) = \frac{\tan \Phi_\star}{\lambda}.$$

Selection by parity at the center. For the one-leg fundamental bump, the lowest-energy branch is even at the center. Indeed, near the core, the gradient energy contains

$$\int |\partial_s A|^2 \, J(s) \, ds.$$

Substituting

$$A = f_{\text{even}} + \tan \Phi_\star \, f_{\text{odd}},$$

one obtains

$$\int |\partial_s A|^2 \, J \, ds = \int (f'_{\text{even}})^2 J \, ds + 2 \tan \Phi_\star \int f'_{\text{even}} f'_{\text{odd}} J \, ds + \tan^2 \Phi_\star \int (f'_{\text{odd}})^2 J \, ds.$$

Now, at the center of the bump, $J(s)$ is even, f'_{even} is odd, and f'_{odd} is even. The crossed term

$$\int f'_{\text{even}} f'_{\text{odd}} J \, ds$$

therefore vanishes by parity on a symmetric domain. It remains

$$\int |\partial_s A|^2 \, J \, ds = \int (f'_{\text{even}})^2 J \, ds + \tan^2 \Phi_\star \int (f'_{\text{odd}})^2 J \, ds,$$

which is minimal for

$$\Phi_\star = 0.$$

Consequently, for the fundamental bump,

$$\boxed{\Phi_\star = 0, \qquad A'(0) = 0.}$$

Scope. The parameter $\Phi_\star$ is retained at this stage as a center parameter of the general local branch; it becomes trivial for the fundamental bump, but may serve as a reference for possible excited or deformed branches, or for other families where central parity does not immediately impose $A'(0) = 0$.

6.6.3 Selection by admissible return of the bump

We apply to the bump case the generic procedure of §6.5.6, but the opening of the nontrivial thresholds is not based on an abstract assignment of the sign of Δ_X. It is carried by the main return effectively read at the edge of the core. The role of the filter is therefore to select the branches for which this compact return is admissible.

By monotone inversion band by band of $h(y) = y \cot y$, then insertion of the single-ring law

$$\varepsilon(q) = \frac{\varepsilon(1)}{q},$$

one obtains

$$y = (n+1)\,u_*\,\varepsilon(q) = \big((n+1)\varepsilon(1)\big)\frac{u_*}{q}.$$

At LO, the factor $(n+1)\varepsilon(1)$ is included in the spectral normalization of the branch, in accordance with the absorbed EBK convention (6.5). The reduced variable can then be written in the form

$$y \simeq \frac{u_*}{q}.$$

The admissible integers then appear as the stable plateaus of the bandwise inversion of h on the envelope (u_*, g).

6.6.4 Return mechanism and hierarchy of admissible openings

Before introducing the filter itself, one must make explicit the geometric mechanism that makes it possible.

The bump lives on a compact hypersphere $\mathbb{S}^3$. In such a geometry, the exterior component generated by the core does not propagate toward an absent infinity; it may reclose on the great cycle and return near the core. This *return* is the basic geometric object on which the subsequent selection of the q branches rests.

1. Global return and local reading

The essential point is that the core does not read the global exterior pattern on the whole great cycle. It couples only to the part of the return effectively resolved in a local angular window centered on the main return point $\psi_{\mathrm{ret}}(q)$.

We therefore introduce the reduced variable

$$u := \frac{\psi - \psi_{\mathrm{ret}}(q)}{\delta\psi_c(q)},$$

where $\delta\psi_c(q)$ denotes the effective angular width of the core edge. In the thin-tube regime, this width is the angular image of the core window $s_c = u_*\lambda(q)$. Since $\varepsilon(q) = \lambda(q)/R_0$, one obtains at LO

$$\delta\psi_c(q) \sim u_*\,\varepsilon(q).$$

The main return effectively read at the edge is then written in the form

$$\boxed{\mathcal{E}_{\mathrm{ret}}(\psi; q) = \mathcal{C}_{\mathrm{ext}}(q)\,\Gamma_{\mathrm{ret}}(u; q),}$$

where:

$\mathcal{C}_{\mathrm{ext}}(q)$ carries the global weight of the main exterior channel,

$\Gamma_{\mathrm{ret}}(u; q)$ describes the reduced local profile of the return seen by the edge.

The effective coupling between return and core is obtained after weighting by the local reading of the edge. Denoting by $T_0(u)$ the edge transmission kernel, we introduce the main return weight

$$\boxed{R_{\mathrm{pr}}(q) := \mathcal{C}_{\mathrm{ext}}(q)\int_{\mathbb{R}} T_0(u)\,\Gamma_{\mathrm{ret}}(u; q)\,du.}$$

Thus the core does not select the raw global return, but the *main return effectively read at the edge.*

2. Local profile of the main return

Near $u = 0$, which corresponds to the main maximum of the return read by the core, the profile $\Gamma_{\mathrm{ret}}(u; q)$ is even, smooth, and maximal at $u = 0$. It can therefore be written in the local form

$$\boxed{\Gamma_{\mathrm{ret}}(u; q) = 1 - \frac{\kappa_{\mathrm{ret}}(q)}{2}\, u^2 + \frac{\lambda_{\mathrm{ret}}(q)}{24}\, u^4 + \mathcal{O}(u^6).}$$

The coefficients $\kappa_{\mathrm{ret}}(q)$ and $\lambda_{\mathrm{ret}}(q)$ respectively describe the curvature and the first shape correction of the main return effectively resolved by the core.

3. First opening: isotropic main return

The first nontrivial threshold appears when the main return becomes effectively admissible in the core window. This is an *isotropic* opening, in the sense that it concerns the dominant return channel, without oriented splitting.

This first threshold will be denoted

$$q_2.$$

In other words, q_2 is not a threshold added from outside: it corresponds to the appearance of the first admissible isotropic main return.

4. Second opening: weak anisotropic splitting

Once the isotropic main return is established, a weak anisotropy may split this same return according to orientation. We then write, at first order,

$$\boxed{\Gamma_{\mathrm{ret}}(u, \theta; q) = \Gamma_0(u; q) + \varepsilon_2\, \Gamma_2(u; q)\, P_2(\cos\theta) + \mathcal{O}(\varepsilon_2^2),}$$

where Γ_0 denotes the main isotropic part, Γ_2 the quadrupolar correction read at the edge, and ε_2 the raw geometric amplitude of the weak anisotropy.

This correction does not invent a new return mechanism; it splits the already opened main return. The second threshold then corresponds to the appearance of a second admissible orientation of the return. It will be denoted

$$q_3.$$

Thus the hierarchy of openings is interpreted as

$$\boxed{\text{isotropic main return} \implies q_2, \qquad \text{weak anisotropic splitting of the same return} \implies q_3.}$$

5. Scope of the mechanism

The important point is therefore the following:

- the compactness of $\mathbb{S}^3$ makes possible a main return read by the edge of the core;
- the isotropic opening of this return defines the first structural threshold q_2 when the branch closure makes it admissible;
- weak anisotropy only enters at the next order, to split this return and define a second structural threshold q_3;
- the filter studied below does not create new values of q, but formulates the conditions under which these return openings are effectively admissible.

6.6.5 Admissible-return filter and definition of the structural thresholds

The previous mechanism provides a principal return effectively read at the edge of the core. The role of the filter is not to create new values of q, but to select, among the possible branches, those for which this return is effectively admissible.

1. Return chain and reduced data

We denote by

$$\mathscr{R}_{\mathrm{ret}}(q)$$

the return chain associated with the branch q, that is, the set of reduced data that governs the return effectively read at the edge:

$$\mathscr{R}_{\mathrm{ret}}(q) = \Big(\mathcal{C}_{\mathrm{ext}}(q),\, \Gamma_{\mathrm{ret}}(v;q),\, T_0(v),\, \delta\psi_c(q)\Big),$$

where v denotes the reduced local variable in a neighborhood of the principal return point.

The principal isotropic weight effectively read at the edge is

$$R_{\mathrm{pr}}(q) := \mathcal{C}_{\mathrm{ext}}(q) \int_{\mathbb{R}} T_0(v)\, \Gamma_{\mathrm{ret}}(v;q)\, dv.$$

2. Minimal admissibility criteria

A branch q will be called admissible if the following three conditions are simultaneously satisfied:

(i) Window compatibility. The principal return must be effectively resolved in the core window:

$$\delta\psi_c(q) \sim u_*\, \varepsilon(q), \qquad u_* \in [1.7, 2.1].$$

(ii) Local compatibility of the principal lobe. The principal profile

$$\Gamma_{\mathrm{ret}}(v;q) = 1 - \frac{\alpha_{\mathrm{ret}}(q)}{2}\, v^2 + \frac{\beta_{\mathrm{ret}}(q)}{24}\, v^4 + \mathcal{O}(v^6)$$

must correspond to a stable local maximum at the edge, that is,

$$\alpha_{\mathrm{ret}}(q) > 0, \qquad |\beta_{\mathrm{ret}}(q)| < \infty.$$

(iii) Reduced spectral compatibility. The branch must belong to a stable plateau of the compact return filter. In other words, the reduced return data

$$\mathscr{R}_{\mathrm{ret}}(q) = \Big(\mathcal{C}_{\mathrm{ext}}(q),\, \Gamma_{\mathrm{ret}}(v;q),\, T_0(v),\, \delta\psi_c(q)\Big)$$

must produce a principal return effectively read in the core window, without an EBK plateau crossing at LO order.

The admissible set is then defined by

$$\boxed{\mathcal{Q}_{\mathrm{adm}} := \Big\{ q \in \mathbb{N}^* \;;\; \mathscr{R}_{\mathrm{ret}}(q) \text{ satisfies (i), (ii), (iii)} \Big\}.}$$

3. Selective nature of the filter

The filter is *selective* and not *creative*: it does not generate new values of q, but eliminates the branches that do not satisfy the admissibility conditions above.

4. Fundamental threshold

The fundamental threshold is

$$\boxed{q_1 := 1.}$$

It corresponds to the fundamental branch anchoring the hierarchy.

5. First nontrivial threshold

The first nontrivial threshold is defined by

$$\boxed{q_2 := \min\Big\{q > q_1 \ ; \ q \in \mathcal{Q}_{\mathrm{adm}} \text{ and the principal isotropic return is admissible}\Big\}.}$$

6. Second nontrivial threshold

At first anisotropic order, one writes

$$\Gamma_{\mathrm{ret}}(v,\theta;q) = \Gamma_0(v;q) + \varepsilon_2\,\Gamma_2(v;q)\,P_2(\cos\theta) + \mathcal{O}(\varepsilon_2^2).$$

The second nontrivial threshold is then

$$\boxed{q_3 := \min\Big\{q > q_2 \ ; \ q \in \mathcal{Q}_{\mathrm{adm}} \text{ and a second admissible orientation appears}\Big\}.}$$

7. Structural hierarchy

One thus obtains the hierarchy

$$\boxed{q_1 \ \longrightarrow \ q_2 \ \longrightarrow \ q_3,}$$

with the following interpretation:

q_1 = fundamental branch, q_2 = first isotropic opening of the admissible return, q_3 = first weak anisot

Proposition 6.7 (Number of structural thresholds of the bump at minimal LO)**.** *In the minimal LO closure of the 2DOF bump, the compact return filter carries exactly three structural levels:*

$$q_1, \qquad q_2, \qquad q_3.$$

The first is the fundamental branch. The other two correspond respectively to the isotropic opening of the principal return and to its first weak anisotropic splitting.

Proof. The level $q_1 = 1$ is fixed by the fundamental branch (6.21)–(6.22). At LO order, the compact return read by the edge has a principal isotropic component $\Gamma_0(v;q)$. Its scalar trace carries the fundamental level, while its residual traceless isotropic defect carries the first nontrivial opening q_2, through the functionals r_1, r_2 defined in §6.6.6. For a single-leg bump, this defect is nonzero. The first independent geometric correction is then the quadrupolar correction

$$\varepsilon_2\,\Gamma_2(v;q)\,P_2(\cos\theta),$$

which splits the isotropic return and defines the second nontrivial threshold q_3. The following corrections are of higher order in the anisotropic expansion or in the shape corrections of the return. They therefore do not belong to the minimal LO closure. Thus, at minimal LO order, there are only the fundamental branch, the isotropic opening, and the first anisotropic splitting. □

8. Scope

At this stage, this subsection fixes the structure of the filter and the definition of the thresholds. The explicit evaluation of $\mathcal{Q}_{\mathrm{adm}}$, as well as the closure of the residual parameters carrying q_2 and q_3, is postponed to the following subsection.

6.6.6 Isotropic relation associated with q_2 and minimal anisotropic closure associated with q_3

The structure of the admissible filter introduces three thresholds

$$q_1 = 1, \qquad q_2, \qquad q_3,$$

where q_2 labels the first nontrivial opening of the dominant isotropic chain, while q_3 labels the first opening carried by the minimal anisotropic correction.

1. Isotropic projections of the return

The isotropic return decomposes into two distinct readings. The first reads the common scalar trace of the core; it carries the fundamental level. The second reads the residual isotropic defect after subtraction of this trace; it carries the first nontrivial opening.

To formalize this separation, one associates with each branch X an internal leg/polarization matrix

$$M_X := \frac{\langle p_X \otimes p_X \rangle_{\mathrm{cyc}}}{\langle |p_X|^2 \rangle_{\mathrm{cyc}}}, \qquad \operatorname{tr} M_X = 1.$$

Its canonical trace/traceless decomposition in the internal 2DOF plane is

$$\boxed{M_X = \frac{1}{2} I_2 + D_X, \qquad \operatorname{tr} D_X = 0.} \tag{6.27}$$

The part $\frac{1}{2} I_2$ is the common fundamental trace. The tensor D_X is the residual internal defect.

One then denotes, for the branch under consideration,

$$r_1^{(X)}(q) := \int_{\mathbb{R}} T_0(v)\, \Gamma_0(v; q)\, dv, \qquad r_2^{(X)}(q) := \|D_X\|^2 \int_{\mathbb{R}} T_1(v)\, \Gamma_0(v; q)\, dv.$$

The kernel T_0 is therefore the isotropic trace projector, while T_1 is the isotropic projector of the residual traceless defect. Thus, the fundamental return depends on the common scalar density, whereas the first nontrivial threshold q_2 depends on the residual invariant $\|D_X\|^2$, and not on raw A^2. In the bump branch, the notation will be lightened by writing $r_i(q)$ for $r_i^{(\mathrm{bump})}(q)$.

For the bump branch, one assumes $r_1^{(\mathrm{bump})}(q) > 0$ on the admissible branch under consideration, so that the ratio

$$\varepsilon_{\mathrm{iso}}(q) := \frac{r_2(q)}{r_1(q)} \tag{6.28}$$

is well defined. This ratio measures, at leading order, the relative defect of the principal isotropic return read by the core.

2. Reduced isotropic closure of the first threshold

The fundamental threshold $q_1 = 1$ corresponds to a single principal return per reduced period. One therefore works on the reduced phase

$$\theta \in \mathbb{R}/\mathbb{Z},$$

where the principal return is centered at $\theta = 0$.

Lemma 6.8 (Fundamental return cell)**.** *In the reduced phase $\mathbb{R}/\mathbb{Z}$, the return cell associated with the principal return $\theta = 0$ is the Voronoi interval*

$$\mathcal{I}_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{6.29}$$

Its half-width is therefore

$$\delta_1 = \frac{1}{2}. \tag{6.30}$$

Proof. The equivalent returns of the fundamental mode are the integer points

$$\theta = n, \qquad n \in \mathbb{Z},$$

in the lift $\mathbb{R}$ of the reduced phase. The cell attached to the principal return 0 is the set of points closer to 0 than to the neighboring returns -1 and $+1$. The boundaries are therefore the bisectors

$$\theta = -\frac{1}{2}, \qquad \theta = \frac{1}{2}.$$

Thus

$$\mathcal{I}_1 = \left[-\frac{1}{2}, \frac{1}{2}\right],$$

and the half-width of the fundamental return is

$$\delta_1 = \frac{1}{2}.$$

□

The closure of the fundamental block is therefore

$$\boxed{\delta_1 = \frac{1}{2}.} \tag{6.31}$$

Reduced isotropic counting. The isotropic threshold of rank q is read on a reduced chain of q return cells. Between these q cells, one counts $q - 1$ internal interfaces. Each internal interface carries the relative isotropic defect

$$\varepsilon_{\mathrm{iso}}(q) = \frac{r_2(q)}{r_1(q)}$$

defined in (6.28).

The two ends of the chain are not complete interfaces. Each carries an edge half-cell, of weight δ_1 relative to an internal interface. The effective counting of the isotropic contributions is therefore

$$\boxed{N_{\mathrm{iso}}(q) = (q-1) + 2\delta_1.} \tag{6.32}$$

With $\delta_1 = 1/2$, this counting reduces to

$$N_{\mathrm{iso}}(q) = q.$$

Isotropic opening condition. At LO, the isotropic opening is reached when the isotropic defect accumulated by the chain compensates the two scalar edge contributions of the fundamental block. The compensation condition is therefore

$$\boxed{(q-1+2\delta_1)\,\frac{r_2(q)}{r_1(q)}+2\simeq_{\mathrm{LO}}0.} \tag{6.33}$$

This equation does not posit an independent numerical factor: it combines the counting of the $q-1$ internal interfaces, the two edge half-cells δ_1, and the relative isotropic residue $\varepsilon_{\mathrm{iso}}=r_2/r_1$.

Solving for q, one obtains

$$q_{2,\mathrm{LO}}\simeq_{\mathrm{LO}}1-2\delta_1-2\,\frac{r_1(q_2)}{r_2(q_2)}. \tag{6.34}$$

Reinjecting (6.31) gives

$$\boxed{q_{2,\mathrm{LO}}\simeq_{\mathrm{LO}}-2\,\frac{r_1(q_2)}{r_2(q_2)}=-\frac{2}{r_2(q_2)/r_1(q_2)}.} \tag{6.35}$$

Thus, a high value of q_2 corresponds to a weak residual isotropic defect. The positivity of the threshold imposes, on the retained branch,

$$\boxed{\frac{r_2(q_2)}{r_1(q_2)}<0.} \tag{6.36}$$

Edge stability. The only edge part of (6.33) is the term $2\delta_1$. An admissible symmetric variation of the edge partition therefore modifies the effective counting according to

$$N_{\mathrm{iso}}(q)=q-1+2\delta_1.$$

The relative effect on the extracted isotropic defect is of order $1/q$. For the first high isotropic threshold, this edge dependence is subdominant compared with the volume term $q-1$. The closure (6.33) is therefore an LO-regularized closure of the isotropic edge.

3. Reduction to the isotropic mesh parameter

In the minimal isotropic mesh model, the previous ratio is not identified directly with a bare scalar parameter. It is related to the mesh by a reduced isotropic functional

$$\mathcal{G}^{(\mathrm{iso})}_{\mathrm{mesh}}(\chi_{\mathrm{mesh}}),\qquad \chi_{\mathrm{mesh}}:=\frac{\Xi_{\tan}}{\Xi_r}.$$

This functional encodes the effective isotropic response of the mesh. Its detailed form depends on the chosen discrete mesh model; the present section uses only its sign and its local invertibility on the branch under consideration. One therefore writes

$$\boxed{\frac{r_2(q_2)}{r_1(q_2)}=\mathcal{G}^{(\mathrm{iso})}_{\mathrm{mesh}}(\chi_{\mathrm{mesh}}).} \tag{6.37}$$

The isotropic compatibility relation between the structural threshold and the reduced mesh response then takes the form

$$\boxed{q_{2,\mathrm{LO}}\simeq_{\mathrm{LO}}-\frac{2}{\mathcal{G}^{(\mathrm{iso})}_{\mathrm{mesh}}(\chi_{\mathrm{mesh}})}.} \tag{6.38}$$

The condition $q_{2,\mathrm{LO}} > 0$ therefore imposes

$$\boxed{\mathcal{G}^{(\mathrm{iso})}_{\mathrm{mesh}}(\chi_{\mathrm{mesh}}) < 0.} \tag{6.39}$$

Thus, at the theoretical level retained here,

q_2 is not introduced as a new numerical constant,	q_2 labels the structural threshold for which the mesh n

The relation (6.38) must therefore not be read as an autonomous numerical determination of q_2. It fixes the compatibility constraint between the structural threshold under consideration and the reduced isotropic mesh response. For a threshold q_2 retained by the branch, the isotropic mesh parameter χ_{mesh} is determined by local inversion of this relation when $\mathcal{G}^{(\mathrm{iso})}_{\mathrm{mesh}}$ is invertible on the branch under consideration.

4. Reduced anisotropic source and minimal projection

At first anisotropic order, one defines the local quadrupolar source by

$$\mathcal{S}_2(v;q) = -\left.\frac{\partial \mathcal{L}_q^{(\varepsilon_2)}}{\partial \varepsilon_2}\right|_{\varepsilon_2=0} \Gamma_0(v;q),$$

where $\mathcal{L}_q^{(\varepsilon_2)}$ is the local return operator on the weakly anisotropic HS.

One then decomposes

$$\mathcal{S}_2(v;q) = s_2(q)\,\Gamma_0(v;q) + \mathcal{S}_2^{\perp}(v;q), \qquad \int_{\mathbb{R}} \mathcal{S}_2^{\perp}(v;q)\,\Gamma_0(v;q)\,dv = 0,$$

with

$$s_2(q) := \frac{\displaystyle\int_{\mathbb{R}} \mathcal{S}_2(v;q)\,\Gamma_0(v;q)\,dv}{\displaystyle\int_{\mathbb{R}} \Gamma_0(v;q)^2\,dv}.$$

In the minimal closure retained here, only the projected coefficient $s_2(q)$ is kept.

The quantity $s_2(q)$ is therefore the projected coefficient of the anisotropic response of the return on the principal component Γ_0. It is defined by projection, but it is not separated here into microscopic mesh contributions.

5. Minimal anisotropic closure of the second threshold

One then introduces the effective anisotropic product

$$\boxed{\Pi_{\mathrm{aniso}}(q) := \varepsilon_2\, s_2(q).}$$

The second threshold q_3 is the first threshold for which this effective product makes a second admissible orientation of the return accessible. Thus, at the present stage,

q_3 does not yet close ε_2 alone,	q_3 parametrizes the effective anisotropic block $\Pi_{\mathrm{aniso}}(q) = \varepsilon_2 s_2(q)$.

6. Final hierarchy

The complete hierarchy is therefore read as follows:

$q_1 = 1$,	q_2 : isotropic threshold parametrizing χ_{mesh},	q_3 : anisotropic threshold parametrizing Π_{aniso}

In this writing, the filter structure fixes the hierarchy of thresholds and the reduced variables to be controlled. The numerical values of the thresholds or of the mesh residues require the explicit evaluation of the functionals $T_0, T_1, \mathcal{G}^{(\mathrm{iso})}_{\mathrm{mesh}}$ and s_2, as well as the internal specification of the branch under consideration.

6.6.7 2DOF spectral bridge and absorbed EBK form

Angle–action (absorbed form) and same phase variable. For 2DOF solutions (bump/NSL) sharing the same phase variable ϕ and the same Maslov index, we use the absorbed EBK convention (6.1)–(6.2). We recall it locally in the form

$$\boxed{\mathcal{J} \;=\; \frac{E}{\omega} \;=\; h_{\mathrm{eff}}.} \tag{6.40}$$

The spectral index $q \in \mathbb{N}$ therefore does not appear explicitly in $\mathcal{J}$: it is carried by the observed frequency.

Spectral scale. The single-ring reduction imposes

$$\varepsilon(q) = \frac{\varepsilon(1)}{q}, \qquad \varepsilon(q) = \frac{\lambda(q)}{R_0}.$$

It follows that

$$\lambda(q) = \frac{\lambda(1)}{q}.$$

Now, for the local 2DOF reduction at leading order, one has

$$\Omega = \frac{1}{\lambda} = \frac{\omega}{c_t}.$$

Thus,

$$\omega(q) = \frac{c_t}{\lambda(q)} = q\,\frac{c_t}{\lambda(1)} = q\,\omega_0, \qquad \omega_0 := \omega(1).$$

We therefore obtain

$$\boxed{\omega(q) = q\,\omega_0}, \tag{6.41}$$

with $\mathcal{O}(\varepsilon^2)$ corrections away from LO.

Corollary 6.9 (EBK spectral energy)**.** *Combining* (6.40) *and* (6.41)*, we obtain*

$$\boxed{E_{\mathrm{tot}}^{(\mathrm{bump})}(q) \;=\; h_{\mathrm{eff}}\,\omega(q) \;=\; h_{\mathrm{eff}}\,q\,\omega_0}. \tag{6.42}$$

In particular, for the fundamental $q = 1$,

$$E_{\mathrm{tot}}^{(\mathrm{bump})}(1) = h_{\mathrm{eff}}\,\omega_0.$$

Compatibility with the LO integral form. At LO, in the local reduction with matching, the integral closure gives

$$E_0^{(\mathrm{bump})}(q) \;=\; \kappa_{\mathrm{core}}\,T\,\lambda(q)^2\,R_0, \qquad \varepsilon(q) = \frac{\lambda(q)}{R_0} = \frac{\varepsilon(1)}{q}. \tag{6.43}$$

Here, κ_{core} denotes the normalized core factor of the bump profile:

$$\kappa_{\mathrm{core}} := I_0 := \int_{\mathbb{R}} A^2(u)\,du. \tag{6.44}$$

Relation (6.43) describes the core contribution at LO for the normalized profile of the branch under consideration, whereas (6.42) fixes the global spectral law. These are therefore not two competing laws: compatibility between (6.43) and (6.42) constrains the effective internal normalization of the branch for every admissible q.

In particular, for $q = 1$,

$$E_0^{(\mathrm{bump})}(1) = \kappa_{\mathrm{core}}\,T\,\lambda(1)^2\,R_0.$$

The fundamental transverse width $\lambda(1)$ is not identified with ℓ_0: it is an emergent scale of the channel under consideration, closed later by the corresponding DtN map. .

Three-dimensional profile constants. For the canonical profile $a(\xi) = \operatorname{sech}\xi$, the three-dimensional shape constants used in the channel projectors are defined by

$$J_0 := 4\pi \int_0^\infty \xi^2 a(\xi)^2 \, d\xi, \qquad J_2 := 4\pi \int_0^\infty \xi^2 \big(a'(\xi)\big)^2 \, d\xi. \tag{6.45}$$

For $a(\xi) = \operatorname{sech}\xi$, one obtains

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 \simeq 7.6339320582. \tag{6.46}$$

Regularized LO spectral closure of the 2DOF bump. The regularized additive closure coefficient of the 2DOF bump branch is denoted $b_0^{(2),\text{LO-reg}}$. It enters the width combination

$$\mathcal{C}_2(\Delta) = \frac{J_0 + J_2}{\Delta^2} + b_0^{(2),\text{LO-reg}}.$$

It is a pure number specific to the effective boundary closure of this branch. The abbreviated notation $b_0^{(2)}$ will be introduced after the regularized evaluation, only to lighten the formulas that follow.

Lemma 6.10 (Elementary spectral cell of the 2DOF bump)**.** *The elementary spectral cell used in the boundary closure of the 2DOF bump is the isotropic cell of unit projective rank $r = 1$ in a local three-dimensional space. Its angle $\psi_c^{(\text{cell})}$ satisfies*

$$\boxed{\sin^2 \psi_c^{(\text{cell})} = \frac{1}{3}, \qquad \psi_c^{(\text{cell})} = \arcsin\!\left(\frac{1}{\sqrt{3}}\right) = 0.6154797086703874\ldots.} \tag{6.47}$$

Proof. At isotropic LO, the quadratic weight of a rank-r projector in the local three-dimensional space is

$$\eta(P) = \frac{\operatorname{tr} P}{\operatorname{tr} \mathbf{1}_3} = \frac{r}{3},$$

cf. (9.6). The elementary spectral cell corresponds to unit rank $r = 1$, therefore to

$$\eta_{\text{cell}} = \frac{1}{3}.$$

In the conical coordinate of the zonal filter, the fraction carried by the cell is $\sin^2 \psi_c^{(\text{cell})}$. We therefore impose

$$\sin^2 \psi_c^{(\text{cell})} = \eta_{\text{cell}} = \frac{1}{3}.$$

The case $r = 0$ would be empty, whereas $r = 2$ would correspond to a composite rank-two cell, and not to the elementary boundary cell of the 2DOF bump. □

Cell filter. We denote by $C_\ell^{(1)}$ the zonal Gegenbauer polynomials on $\mathbb{S}^3$, with

$$C_\ell^{(1)}(\cos\theta) = \frac{\sin((\ell+1)\theta)}{\sin\theta}, \qquad C_\ell^{(1)}(1) = \ell + 1.$$

We define the cell filter normalized by the zonal measure $\sin^2\theta \, d\theta$:

$$\boxed{F_\ell^{(\text{cell})}(\psi_c) := \frac{\displaystyle\int_0^{\psi_c} C_\ell^{(1)}(\cos\theta)\, \sin^2\theta \, d\theta}{\displaystyle\int_0^{\psi_c} \sin^2\theta \, d\theta}.} \tag{6.48}$$

Thus

$$\boxed{F_0^{(\text{cell})} = 1, \qquad F_{\ell\geq 1}^{(\text{cell})}(\psi_c) = \frac{\dfrac{1}{2}\left[\dfrac{\sin(\ell\psi_c)}{\ell} - \dfrac{\sin((\ell+2)\psi_c)}{\ell+2}\right]}{\dfrac{\psi_c}{2} - \dfrac{\sin(2\psi_c)}{4}}.} \tag{6.49}$$

Spectral weights. For a dimensionless mass scale

$$s := mR_0,$$

the reduced massive spectral propagator is

$$\boxed{G_\ell(s) = \frac{1}{\ell(\ell+2) + s^2}.} \tag{6.50}$$

The cell spectral weights are then

$$\boxed{a_\ell(s) = G_\ell(s) \left| F_\ell^{(\mathrm{cell})}(\psi_c^{(\mathrm{cell})}) \right|^2.} \tag{6.51}$$

Core projection on the local plateau. In the local limit of the 2DOF bump, the radial projection of the canonical profile $a(\xi) = \operatorname{sech}\xi$ gives

$$\boxed{B_\ell^{(0)} = J_0\,(\ell+1), \qquad J_0 = \frac{\pi^3}{3}.} \tag{6.52}$$

The reduced spectral quantity is therefore

$$\boxed{\widehat{U}(s) := \frac{\displaystyle\sum_{\ell=0}^{L_{\max}} a_\ell(s)\, B_\ell^{(0)}}{a_0(s)} = J_0\, \frac{\displaystyle\sum_{\ell=0}^{L_{\max}} a_\ell(s)\,(\ell+1)}{a_0(s)}.} \tag{6.53}$$

At finite scale, the associated spectral closure is

$$\boxed{b_0^{(2)}(s) := J_0\, \widehat{U}(s).} \tag{6.54}$$

In other words,

$$\boxed{b_0^{(2)}(s) = J_0^2\, \frac{\displaystyle\sum_{\ell=0}^{L_{\max}} a_\ell(s)\,(\ell+1)}{a_0(s)}.} \tag{6.55}$$

Definition 6.11 (Regularized LO boundary closure)**.** The spectral sum associated with the abrupt cap is not used as a raw $s \to \infty$ limit. The retained numerical closure is a regularized LO closure of the canonical boundary:

$$s \in \{80, 120, 160, 200\}, \qquad L_{\max} = 12s, \qquad x = s^{-2}.$$

For each s, $b_0^{(2)}(s)$ is computed through (6.55), and the four values are then fitted by an affine law

$$b_0^{(2)}(s) = A + B\,x, \qquad x = \frac{1}{s^2}.$$

The regularized LO closure is defined by

$$\boxed{b_0^{(2),\mathrm{LO\text{-}reg}} := \mathrm{Reg}_{\mathrm{LO}}^{(12)}\,[b_0^{(2)}(s)] := A.} \tag{6.56}$$

Status of the regularization. The abrupt cap is an LO idealization of the cell boundary. Its spectral coefficients carry a boundary tail that must not be interpreted as a raw canonical limit of $b_0^{(2)}(s)$. The prescription $\mathrm{Reg}_{\mathrm{LO}}^{(12)}$ is therefore an effective boundary closure: it regularizes the unresolved modes of the canonical matching and provides a stable branch value to be used with a regularization uncertainty.

Thus, $b_0^{(2),\text{LO-reg}}$ is not an exact constant produced by a strict limit. It is an effective LO closure constant, accepted when stability under a reasonable variation of the spectral window is controlled and when the associated uncertainty is propagated to downstream quantities.

In the rest of this branch, the abbreviated notation $b_0^{(2)}$ denotes the regularized value $b_0^{(2),\text{LO-reg}}$. This abbreviation does not change the status of the quantity: it is an effective LO closure of the unresolved boundary, controlled by a stable spectral regularization and accompanied by a regularization uncertainty.

Proposition 6.12 (Numerical evaluation of the regularized LO closure)**.** *The central value of* $b_0^{(2),\text{LO-reg}}$ *is obtained by the following procedure:*

1. *use the spectral cell* (6.47) *;*
2. *compute the filters* $F_\ell^{(\text{cell})}$ *through* (6.48)–(6.49) *;*
3. *choose the grid*
$$s \in \{80, 120, 160, 200\};$$
4. *for each* s*, take*
$$L_{\max} = 12s;$$
5. *compute* $G_\ell(s)$*, then* $a_\ell(s)$*, through* (6.50)–(6.51) *;*
6. *evaluate* $\widehat{U}(s)$ *through* (6.53)*, then* $b_0^{(2)}(s)$ *through* (6.54) *;*
7. *fit the four values of* $b_0^{(2)}(s)$ *as a function of* $x = 1/s^2$ *by an affine law* $A + Bx$*, then define the central value of the regularized closure by* (6.56)*.*

The intermediate values are

s	$\widehat{U}(s)$	$b_0^{(2)}(s)$
80	1425.8687910812	14736.9607486894
120	1560.7736553129	16131.2599306515
160	1656.3874925251	17119.4696276737
200	1730.5170477351	17885.6301273496

and the prescription (6.56) *gives the central value*

$$\boxed{b_0^{(2),\text{LO-reg}} = 1.8092069643818934 \times 10^4.} \tag{6.57}$$

Taking into account the observed dispersion under reasonable variations of spectral windows, we retain for LO propagations the envelope

$$\boxed{b_0^{(2),\text{LO-reg}} \simeq (1.80 \pm 0.04) \times 10^4.} \tag{6.58}$$

Proof. The spectral cell is fixed by (6.47). The cell filters are given by (6.48)–(6.49). The spectral weights are given by (6.50) and (6.51). Finally, the finite-scale boundary sum is given by (6.55). The evaluation at the four prescribed values of s, followed by the affine fit in $x = s^{-2}$, gives the intercept displayed in (6.57). The dispersion indicated in (6.58) encodes the regularized LO status of the closure and limits the precision of the absolute predictions that depend on it. □

Distinction from the cap normalization of the EM-like channel. The filter $F_\ell^{(\mathrm{cell})}$ defined in (6.48) is the cell filter used in the spectral construction of $b_0^{(2),\mathrm{LO\text{-}reg}}$. It must not be confused with the normalization of the $\ell = 1$ cap projector used in the EM-like closure.

Scaling of the width closure. The reduced width Δ acts by local dilation of the core in the reduced coordinates of the 2DOF bump branch. At fixed shape profile, the local change of variables gives a measure factor

$$d^3u = \Delta^3\, d^3\xi.$$

The core contributions carrying two spatial derivatives are renormalized as Δ^{-2}. With the canonical profile moments, they give the coefficient

$$A_2 := J_0 + J_2.$$

The boundary/massive contribution specific to the 2DOF bump closure carries no core derivative at this LO level; it therefore provides an additive term independent of Δ, denoted $b_0^{(2)}$. The reduced width combination is therefore

$$\boxed{\mathcal{C}_2(\Delta) = \frac{J_0 + J_2}{\Delta^2} + b_0^{(2)}.} \tag{6.59}$$

Reduced width functional. In the LO reduction, the closure energy depends on the square of the reduced combination $\mathcal{C}_2(\Delta)$. The local volume factor arising from the core dilation then gives

$$\boxed{\mathcal{F}_2(\Delta) = \Delta^3 \left[\frac{J_0 + J_2}{\Delta^2} + b_0^{(2)}\right]^2} \tag{6.60}$$

up to a positive global factor. This global factor does not modify the stationarity condition in Δ.

The condition $\partial_\Delta \mathcal{F}_2(\Delta) = 0$ gives

$$\frac{d}{d\Delta}\left[\Delta^3 \left(\frac{J_0 + J_2}{\Delta^2} + b_0^{(2)}\right)^2\right] = 0.$$

Setting $A_2 = J_0 + J_2$, this derivative reads

$$-\frac{A_2^2}{\Delta^2} + 2A_2 b_0^{(2)} + 3\big(b_0^{(2)}\big)^2 \Delta^2 = 0.$$

The positive root is

$$\boxed{\Delta_*^2 = \frac{J_0 + J_2}{3\, b_0^{(2)}}.} \tag{6.61}$$

Numerical consequences. In the rest of this section, unless explicitly stated otherwise, we write

$$b_0^{(2)} := b_0^{(2),\mathrm{LO\text{-}reg}}$$

to lighten the formulas. With

$$J_0 = \frac{\pi^3}{3} \approx 10.33542556009994, \qquad J_2 \approx 7.6339320581530377,$$

one has

$$J_0 + J_2 \approx 17.96935761825298.$$

The reduced width of the 2DOF bump closure is therefore given by (6.61), namely

$$\frac{d}{d\Delta}\left[\Delta^3\left(\frac{J_0+J_2}{\Delta^2}+b_0^{(2)}\right)^2\right]_{\Delta=\Delta_*}=0,$$

which gives

$$\boxed{\Delta_*=\sqrt{\frac{J_0+J_2}{3\,b_0^{(2)}}}.} \tag{6.62}$$

Inserting the central value (6.57), one obtains

$$\boxed{\Delta_*\approx 1.819539647205362\times 10^{-2}.} \tag{6.63}$$

Then

$$\boxed{\kappa_\omega=\frac{1}{\Delta_*}.} \tag{6.64}$$

Therefore

$$\boxed{\kappa_\omega\approx 54.9589563237.} \tag{6.65}$$

Finally, one obtains

$$\boxed{\Delta_*^6\approx 3.628851020855722\times 10^{-11}.} \tag{6.66}$$

Closure convention. The closure convention used in the one-body expressions is therefore

$$\boxed{\Delta_*=\kappa_\omega^{-1},\qquad \frac{\Delta_*^2}{b_0^{(2)}}=\frac{1}{\kappa_\omega^2\,b_0^{(2)}}.} \tag{6.67}$$

Absolute predictions that depend on $b_0^{(2)}$ inherit the regularization uncertainty indicated in (6.58). In particular, results containing high powers of $b_0^{(2)}$ must not be displayed with a precision exceeding that allowed by this regularized LO closure.

6.6.8 LO spectral closure of the 1–body mass of the 2DOF bump

Purpose. Here we close the 1–body spectral writing of the 2DOF bump in the absorbed EBK convention

$$E_{\rm tot}^{(\rm bump)}=h_{\rm eff}\,\omega_{\rm obs}.$$

In this convention, angle–action quantization no longer supplies an additional multiplicative EBK factor. The only remaining factor 2π in the 1–ring reduction is the single-ring phase average.

LO separation of the factors. At LO, the 1–body reduction is separated into four independent dimensionless contributions:

$$\mathcal{C}_{\rm bump2}=\Pi_{\rm proj}\,Z_{\rm ann}\,Z_{\rm proj}\,\frac{\Delta_*^2}{b_0^{(2)}}.$$

Each factor comes from a different step of the reduction: mass projection of the channel, single-ring phase average, transverse kinetic projection, then spectral plateau closure.

Mass projection of the channel. The mass projection of the 2DOF channel is obtained by Galerkin projection of the core radial mode onto the normalized spectral channel. The important point is that the spectral closure is bilinear: it contains an injection projection into the core and a reading projection from the core to the observed channel. These two projections are not two distinct physical paths; they are the two faces of one and the same quadratic coefficient.

We write

$$\langle f, g\rangle_0 := 4\pi \int_0^\infty \xi^2 f(\xi) g(\xi)\, d\xi,$$

and, for the core profile $a(\xi) = \operatorname{sech}\xi$,

$$J_0 = \langle a, a\rangle_0, \qquad J_2 = \langle a', a'\rangle_0.$$

Lemma 6.13 (Reduced quadratic form of the radial mode)**.** *The restriction of the LO quadratic energy to the radial mode $A\,a(\xi)$ carries, up to a common dimensional factor, the reduced norm*

$$\boxed{\|a\|^2_{\mathrm{red}} = J_0 + J_2.} \tag{6.68}$$

Proof. At quasi-flat LO order, the local quadratic part of the core contains the core contribution and the radial-gradient contribution. For the mode $A\,a(\xi)$, one obtains

$$E_2[A] = \frac{\mathcal{N}_{\mathrm{core}}}{2} A^2 \left(\langle a, a\rangle_0 + \langle a', a'\rangle_0\right),$$

where $\mathcal{N}_{\mathrm{core}}$ is the common dimensional factor of the core. Using the definitions of J_0 and J_2,

$$E_2[A] = \frac{\mathcal{N}_{\mathrm{core}}}{2} A^2 (J_0 + J_2).$$

The reduced quadratic norm associated with the radial mode is therefore

$$\|a\|^2_{\mathrm{red}} = J_0 + J_2.$$

□

Lemma 6.14 (Amplitude projection onto the core)**.** *The amplitude projection of a reduced-normalized radial mode onto the normalized core is*

$$\boxed{\gamma_{\mathrm{core}} = \sqrt{\frac{J_0}{J_0 + J_2}}.} \tag{6.69}$$

Proof. The mode normalized by the reduced norm is

$$e_{\mathrm{red}} = \frac{a}{\sqrt{J_0 + J_2}},$$

whereas the same profile normalized by the core norm is

$$e_0 = \frac{a}{\sqrt{J_0}}.$$

The amplitude of the reduced mode read in the core channel is therefore

$$\gamma_{\mathrm{core}} = \langle e_0, e_{\mathrm{red}}\rangle_0 = \frac{\langle a, a\rangle_0}{\sqrt{J_0}\sqrt{J_0 + J_2}} = \frac{J_0}{\sqrt{J_0}\sqrt{J_0 + J_2}} = \sqrt{\frac{J_0}{J_0 + J_2}}.$$

□

Lemma 6.15 (Galerkin gradient/core ratio)**.** *The Galerkin projection of the linearized radial operator onto the mode $a(\xi)$ selects the ratio*

$$\boxed{R_{\nabla/0} = \frac{J_2}{J_0}.} \tag{6.70}$$

Proof. In the linearized radial sector, the active gradient contribution is obtained by Galerkin projection onto the test mode a. Up to a common dimensional factor, the Rayleigh quotient of the radial gradient is

$$R_{\nabla/0} = \frac{\langle a', a' \rangle_0}{\langle a, a \rangle_0}.$$

With the definitions of J_0 and J_2, this gives

$$R_{\nabla/0} = \frac{J_2}{J_0}.$$

□

Lemma 6.16 (Projection of core quadratic densities)**.** *Let*

$$e_{\mathrm{red}} := \frac{a}{\sqrt{J_0 + J_2}}, \qquad e_0 := \frac{a}{\sqrt{J_0}}.$$

We associate with the two modes the normalized quadratic densities

$$Q_{\mathrm{red}} := e_{\mathrm{red}} \otimes e_{\mathrm{red}}, \qquad Q_0 := e_0 \otimes e_0.$$

Then the quadratic-density projection of the reduced mode onto the core is

$$\boxed{\langle Q_0, Q_{\mathrm{red}} \rangle_{\mathrm{HS}} = \big(\langle e_0, e_{\mathrm{red}} \rangle_0\big)^2 = \gamma_{\mathrm{core}}^2.} \tag{6.71}$$

Proof. By definition of the Hilbert–Schmidt product induced by $\langle \cdot, \cdot \rangle_0$,

$$\langle e_0 \otimes e_0,\, e_{\mathrm{red}} \otimes e_{\mathrm{red}} \rangle_{\mathrm{HS}} = \langle e_0, e_{\mathrm{red}} \rangle_0^2.$$

Lemma 6.14 gives

$$\langle e_0, e_{\mathrm{red}} \rangle_0 = \gamma_{\mathrm{core}},$$

hence

$$\langle Q_0, Q_{\mathrm{red}} \rangle_{\mathrm{HS}} = \gamma_{\mathrm{core}}^2.$$

□

Lemma 6.17 (Bilinear weight of the mass channel)**.** *The 1–body mass spectral closure carries the bilinear weight*

$$\boxed{\Pi_{\mathrm{mass}} = \gamma_{\mathrm{core}}^4 = \left(\frac{J_0}{J_0 + J_2}\right)^2.} \tag{6.72}$$

Proof. The 1–body mass closure does not directly couple two linear core amplitudes; it couples two quadratic mode densities. The source injected into the mass channel is therefore represented, at reduced LO order, by $Q_{\mathrm{red}} = e_{\mathrm{red}} \otimes e_{\mathrm{red}}$, and the core reading by $Q_0 = e_0 \otimes e_0$.

By Lemma 6.16, one quadratic-density projection of the reduced mode onto the core brings the factor

$$\langle Q_0, Q_{\mathrm{red}} \rangle_{\mathrm{HS}} = \gamma_{\mathrm{core}}^2.$$

The mass closure is bilinear injection–reading on these densities: one density projection enters on the source side, and one density projection enters on the reading side. The total weight is therefore

$$\Pi_{\mathrm{mass}} = \left(\langle Q_0, Q_{\mathrm{red}}\rangle_{\mathrm{HS}}\right)^2 = \left(\gamma_{\mathrm{core}}^2\right)^2 = \gamma_{\mathrm{core}}^4.$$

Using

$$\gamma_{\mathrm{core}}^2 = \frac{J_0}{J_0 + J_2},$$

one obtains

$$\Pi_{\mathrm{mass}} = \gamma_{\mathrm{core}}^4 = \left(\frac{J_0}{J_0 + J_2}\right)^2.$$

Thus, the fourth power does not come from a verbal counting of faces: it comes from the fact that the mass functional is bilinear on the normalized quadratic densities of the mode. □

Proposition 6.18 (Mass projection of the 2DOF channel)**.** *The projection of the 2DOF channel onto the effective mass component is*

$$\boxed{\Pi_{\mathrm{proj}} = R_{\nabla/0}\,\Pi_{\mathrm{mass}} = \Big(\frac{J_2}{J_0}\Big)\Big(\frac{J_0}{J_0 + J_2}\Big)^2.} \tag{6.73}$$

Proof. The Galerkin quotient of the radial gradient gives, by (6.70),

$$R_{\nabla/0} = \frac{J_2}{J_0}.$$

The bilinear weight of the mass projection gives, by (6.72),

$$\Pi_{\mathrm{mass}} = \left(\frac{J_0}{J_0 + J_2}\right)^2.$$

These two factors refer to two different operations: the first is the Rayleigh quotient of the active radial channel, and the second is the bilinear injection–reading projection of the normalized mass channel. At LO, the Galerkin separation makes these operations multiplicative. One therefore obtains

$$\Pi_{\mathrm{proj}} = R_{\nabla/0}\Pi_{\mathrm{mass}} = \Big(\frac{J_2}{J_0}\Big)\Big(\frac{J_0}{J_0 + J_2}\Big)^2.$$

□

Single-ring average. The 1–body reduction is carried by a single annular phase parameter $\theta \in [0, 2\pi]$. The normalized average over this phase gives

$$Z_{\mathrm{ann}} := \frac{1}{2\pi}. \tag{6.74}$$

This factor is a single-ring average. It must not be counted as a second EBK normalization, since the relation $E = h_{\mathrm{eff}}\omega$ is already written in the absorbed convention.

Transverse kinetic projection. The mass functional above supplies the spectral coefficient associated with core quadratic densities. The quantized 1–body coordinate, however, is the reduced transverse coordinate of the 2DOF mode. The passage from the normalized core coordinate to this reduced transverse coordinate carries a linear mode projection:

$$\boxed{Z_{\mathrm{proj}} := \langle e_0, e_{\mathrm{red}}\rangle_0 = \gamma_{\mathrm{core}} = \sqrt{\frac{J_0}{J_0 + J_2}}.} \tag{6.75}$$

This factor is distinct from the bilinear projection $\Pi_{\mathrm{mass}} = \gamma_{\mathrm{core}}^4$. The weight Π_{mass} comes from the bilinear mass closure on the quadratic densities, whereas Z_{proj} comes from the linear normalization of the 1–body transverse coordinate. It therefore constitutes neither a third coupling face nor a double counting. The 1–body functional then carries

$$\boxed{\Pi_{\mathrm{mass}} Z_{\mathrm{proj}} = \Big[\big(\langle Q_0, Q_{\mathrm{red}}\rangle_{\mathrm{HS}}\big)^2\Big]\,\langle e_0, e_{\mathrm{red}}\rangle_0 = (\gamma_{\mathrm{core}}^2)^2\gamma_{\mathrm{core}} = \gamma_{\mathrm{core}}^5.} \tag{6.76}$$

This factor is also distinct from the 3DOF conical projection $\sigma = \sin^2\alpha$.

Spectral plateau closure. The spectral closure introduces

$$\frac{\Delta_*^2}{b_0^{(2)}} = \frac{1}{\kappa_\omega^2\, b_0^{(2)}}. \tag{6.77}$$

Proposition 6.19 (1–body spectral coefficient of the 2DOF bump)**.** *The dimensionless coefficient of the 1–body spectral closure is*

$$\mathcal{C}_{\mathrm{bump2}} = \Pi_{\mathrm{proj}}\, Z_{\mathrm{ann}}\, Z_{\mathrm{proj}}\, \frac{1}{\kappa_\omega^2\, b_0^{(2)}}. \tag{6.78}$$

Proof. At LO, the tubular gauge separates the annular phase variable, the radial profile, and the transverse normalization. The average over the annular phase gives (6.74). The mass projection of the radial channel is derived in Prop. 6.18 and gives (6.73). Its mass-projection part is the bilinear closure on quadratic densities, cf. (6.72). The linear transverse normalization of the 1–body coordinate gives (6.75). The origin of the total power γ_{core}^5 is made explicit in (6.76). Finally, the plateau closure gives (6.77). The contributions refer to separate operations at LO — bilinear mass density, linear transverse coordinate, annular average, and spectral closure —; their product gives (6.78). □

Proposition 6.20 (1–body spectral mass of the 2DOF bump)**.** *The LO spectral mass of the 2DOF bump satisfies*

$$m_{\mathrm{bump}}^{(\mathrm{spec})} := \frac{E_{\mathrm{bump}}^{(\mathrm{spec})}}{c_t^2}. \tag{6.79a}$$

$$\boxed{m_{\mathrm{bump}}^{(\mathrm{spec})} c_t^2 = \Big(\frac{h_{\mathrm{eff}} c_t}{\ell_0}\Big)\Pi_{\mathrm{proj}}\, Z_{\mathrm{ann}}\, Z_{\mathrm{proj}}\, \frac{1}{\kappa_\omega^2\, b_0^{(2)}}}. \tag{6.79b}$$

Equivalently,

$$m_{\mathrm{bump}}^{(\mathrm{spec})} c_t^2 = \Big(\frac{h_{\mathrm{eff}} c_t}{\ell_0}\Big)\Pi_{\mathrm{proj}}\, Z_{\mathrm{ann}}\, Z_{\mathrm{proj}}\, \frac{\Delta_*^2}{b_0^{(2)}}. \tag{6.80}$$

Proof. The absorbed EBK convention gives the energy scale

$$E = h_{\mathrm{eff}}\omega_{\mathrm{obs}}.$$

The 1–body spectral reduction writes the observed frequency in the form

$$\omega_{\mathrm{obs}} = \frac{c_t}{\ell_0}\,\mathcal{C}_{\mathrm{bump2}}.$$

Injecting (6.78), one obtains

$$E_{\mathrm{spec}} = \Big(\frac{h_{\mathrm{eff}} c_t}{\ell_0}\Big)\Pi_{\mathrm{proj}}\, Z_{\mathrm{ann}}\, Z_{\mathrm{proj}}\, \frac{1}{\kappa_\omega^2\, b_0^{(2)}}.$$

Since $E_{\mathrm{spec}} = m_{\mathrm{bump}}^{(\mathrm{spec})} c_t^2$, one obtains (6.79b). The form (6.80) follows directly from (6.77). □

6.6.9 LO closure of the effective transverse width of the bump

Purpose. Here we close the quantity λ_e, defined as the effective transverse width of the one-body bump branch. This quantity characterizes the internal transverse channel associated with this branch. It must not be interpreted as a modal wavelength, nor be identified with the micro scale ℓ_0.

Reduced boundary map. The transverse closure is read through a projected boundary map of the EM-like channel. We denote by

$$\Delta_{\mathrm{DtN}}^{(\mathrm{EM})}$$

the reduced boundary shift, defined as the normalized inverse admittance of the plateau/boundary channel. This quantity is dimensionless.

Lemma 6.21 (Normalized plateau mode of the transverse matching)**.** *In the transverse matching layer at LO, the normalized plateau mode $B(y)$ of the EM-like channel satisfies the reduced problem*

$$-B''(y) + b_0^{(2)} B(y) = 0, \qquad B(0) = B_\partial, \qquad B(y) \to 0 \quad (y \to +\infty). \tag{6.81}$$

Its decreasing solution is

$$B(y) = B_\partial \exp\left[-\sqrt{b_0^{(2)}}\, y\right], \tag{6.82}$$

and the reduced DtN eigenvalue of the plateau is

$$\boxed{\Lambda_{\mathrm{plat}}^{(\mathrm{EM})} = -\frac{B'(0)}{B(0)} = \sqrt{b_0^{(2)}}.} \tag{6.83}$$

Proof. The characteristic equation associated with

$$-B'' + b_0^{(2)} B = 0$$

is

$$r^2 = b_0^{(2)}.$$

The solution compatible with decay as $y \to +\infty$ is therefore

$$B(y) = B_\partial e^{-\sqrt{b_0^{(2)}}\, y}.$$

One obtains

$$-B'(0) = \sqrt{b_0^{(2)}}\, B_\partial,$$

hence

$$-\frac{B'(0)}{B(0)} = \sqrt{b_0^{(2)}}.$$

□

Lemma 6.22 (Cap normalization of the EM-like flux)**.** *The EM-like reading of the boundary flux uses the monopolar cap normalization*

$$\boxed{\mathcal{N}_{\mathrm{cap}}^{(\mathrm{EM})} = \frac{1}{4\pi}.} \tag{6.84}$$

Proof. At quasi-flat LO, the boundary reading of the EM-like channel is monopolar on the local angular section. The total flux carried by a uniform boundary density is

$$\Phi_\partial = \int_{\mathbb{S}^2} j_\partial \, d\Omega = 4\pi \, j_\partial.$$

The density normalized by unit flux is therefore

$$j_\partial^{\mathrm{norm}} = \frac{\Phi_\partial}{4\pi}.$$

The conversion from a total flux to the normalized boundary density therefore carries the factor

$$\mathcal{N}_{\mathrm{cap}}^{(\mathrm{EM})} = \frac{1}{4\pi}.$$

□

Proposition 6.23 (Normalized EM-like boundary shift)**.** *The reduced boundary shift of the EM-like channel is*

$$\boxed{\Delta_{\mathrm{DtN}}^{(\mathrm{EM})} = \frac{1}{4\pi\sqrt{b_0^{(2)}}}.} \tag{6.85}$$

Proof. The reduced DtN eigenvalue of the plateau is, by (6.83),

$$\Lambda_{\mathrm{plat}}^{(\mathrm{EM})} = \sqrt{b_0^{(2)}}.$$

The width closure uses the boundary shift, that is, the inverse admittance of the plateau:

$$(\Lambda_{\mathrm{plat}}^{(\mathrm{EM})})^{-1} = \frac{1}{\sqrt{b_0^{(2)}}}.$$

The EM-like reading is then normalized by the monopolar cap (6.84), namely a factor $1/(4\pi)$. Thus

$$\Delta_{\mathrm{DtN}}^{(\mathrm{EM})} = \mathcal{N}_{\mathrm{cap}}^{(\mathrm{EM})} (\Lambda_{\mathrm{plat}}^{(\mathrm{EM})})^{-1} = \frac{1}{4\pi} \frac{1}{\sqrt{b_0^{(2)}}}.$$

□

Closure of the effective width. The internal spectral amplification of the bump core is carried by

$$\kappa_\omega = \Delta_*^{-1},$$

cf. (6.67). The effective transverse width is the length obtained by applying this internal amplification to the scale ℓ_0, then dividing by the boundary shift:

$$\boxed{\lambda_e = \frac{\kappa_\omega \, \ell_0}{\Delta_{\mathrm{DtN}}^{(\mathrm{EM})}} = 4\pi \, \kappa_\omega \, \sqrt{b_0^{(2)}} \, \ell_0.} \tag{6.86}$$

Dimensional status. The factors

$$\Delta_{\mathrm{DtN}}^{(\mathrm{EM})}, \qquad 4\pi, \qquad \sqrt{b_0^{(2)}}, \qquad \kappa_\omega$$

are dimensionless. The only dimensional quantity in (6.86) is ℓ_0. The closure therefore indeed produces an effective transverse length.

6.6.10 Radial capacity of the core and form factors

Geometric framework: tubular gauge around the single ring. The bump core is localized in a thin tube of thickness $\mathcal{O}(\lambda)$ around a great circle of $\mathbb{S}^3$ (length $2\pi R_0$). We introduce the transverse tubular coordinate ϱ centered on this circle (by shifting the polar geodesic according to $s = \frac{\pi}{2}R_0 + \varrho$) and the dimensionless envelope $A(u) = \operatorname{sech} u$ with $u = \varrho/\lambda$. Near the single ring ($|\varrho| \lesssim \lambda$), the Jacobian

$$J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big)$$

expands as

$$J\Big(\tfrac{\pi}{2}R_0 + \varrho\Big) = R_0^2 \cos^2\Big(\tfrac{\varrho}{R_0}\Big) = R_0^2\Big[1 + \mathcal{O}\Big(\tfrac{\varrho^2}{R_0^2}\Big)\Big] = R_0^2\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.87}$$

Thus, at LO in the quasi-flat cap ($\lambda \ll R_0$), the effective radial measure is simply $R_0^2\, d\varrho$, with corrections $\mathcal{O}((\lambda/R_0)^2)$.

Distinction between profile moments. The constants J_0 and J_2 introduced in (6.45) are three-dimensional moments weighted by $4\pi\xi^2$. By contrast, the constants I_0 and I_2 below are one-dimensional transverse moments of the tube. They must therefore not be identified with the constants J_0, J_2, even when they are evaluated on the same canonical sech profile.

Radial moments of the core. The LO tangential part of the energy is obtained by integrating $|\nabla_s \psi|^2$ through the transverse section of the tube. After scale separation ($\varrho = \lambda u$) and rapid angular integration, the two quantities

$$M_0 := \int_{\mathbb{R}} J\Big(\tfrac{\pi}{2}R_0 + \lambda u\Big)\, A^2(u)\, \lambda\, du, \qquad M_2 := \int_{\mathbb{R}} J\Big(\tfrac{\pi}{2}R_0 + \lambda u\Big)\, \Big|\partial_\varrho A\Big(\tfrac{\varrho}{\lambda}\Big)\Big|^2\, d\varrho \tag{6.88}$$

appear naturally: M_0 weights the effective mass (weight A^2), whereas M_2 weights the radial stiffness (weight $|\partial_\varrho A|^2$). Replacing $\partial_\varrho A(\varrho/\lambda) = \lambda^{-1} A'(u)$ and using (6.87), one obtains

$$M_0 = R_0^2\, \lambda \underbrace{\int_{\mathbb{R}} A^2(u)\, du}_{=:I_0}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big], \qquad M_2 = \frac{R_0^2}{\lambda} \underbrace{\int_{\mathbb{R}} A'^{\,2}(u)\, du}_{=:I_2}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.89}$$

The ratio of moments is therefore

$$\frac{M_2}{M_0} = \frac{1}{\lambda^2}\,\frac{I_2}{I_0}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.90}$$

Explicit values for the sech **profile.** For $A(u) = \operatorname{sech} u$, one has

$$I_0 = \int_{\mathbb{R}} \operatorname{sech}^2 u\, du = 2, \qquad I_2 = \int_{\mathbb{R}} \operatorname{sech}^2 u\, \tanh^2 u\, du = \frac{2}{3}. \tag{6.91}$$

In particular,

$$\frac{I_2}{I_0} = \frac{1}{3}. \tag{6.92}$$

Error control. The core/exterior passage with matching adds exponentially small corrections $\mathcal{O}(e^{-k})$, which add to the curvature corrections $\mathcal{O}((\lambda/R_0)^2)$.

Definition 6.24 (LO radial capacity of the core)**.** The LO radial capacity of the core is defined by

$$C_\ell^{(0)} := \frac{1}{4\pi} \lim_{\varepsilon\to 0} \frac{M_0}{\lambda R_0^2} = \frac{1}{4\pi} \int_{\mathbb{R}} A^2(u)\, du. \tag{6.93}$$

It therefore satisfies

$$4\pi\, C_\ell^{(0)} = \kappa_{\mathrm{core}}, \tag{6.94}$$

where κ_{core} is the core factor introduced above in (6.44).

Justification. The limit on the left exists by (6.89) and coincides, at LO, with the integral on the right. The factor $1/(4\pi)$ removes the angular integral $d\Omega_{S^2}$ absorbed in M_0.

Consequence for the sech **profile.** Using (6.91), one obtains

$$C_\ell^{(0)} = \frac{1}{2\pi}, \qquad \kappa_{\mathrm{core}} = 2. \tag{6.95}$$

Frequency factor (Rayleigh quotient). The rapid pulsation of the bump follows a Rayleigh-type variational quotient:

$$\omega^2 = c_t^2\, \kappa_{\mathrm{geom}}\, \frac{M_2}{M_0}, \qquad c_t^2 = \frac{T}{\rho}, \qquad \kappa_{\mathrm{geom}} = 1 + \mathcal{O}(e^{-k}) + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big). \tag{6.96}$$

Using (6.90), one obtains

$$\omega^2 = \frac{c_t^2}{\lambda^2}\, \kappa_{\mathrm{geom}}\, \frac{I_2}{I_0}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big].$$

We therefore define the pure frequency factor of the bump by

$$\big(\chi_\omega^{(\mathrm{bump})}\big)^2 := \kappa_{\mathrm{geom}}\, \frac{I_2}{I_0}, \tag{6.97}$$

so that

$$\omega_{\mathrm{bump}} = \chi_\omega^{(\mathrm{bump})}\, \frac{c_t}{\lambda}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.98}$$

For the sech profile, this gives at leading order

$$\big(\chi_\omega^{(\mathrm{bump})}\big)^2 = \frac{\kappa_{\mathrm{geom}}}{3}, \qquad \chi_\omega^{(\mathrm{bump})} = \frac{1}{\sqrt{3}}\Big[1 + \mathcal{O}(e^{-k}) + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.99}$$

Remark. The relation (6.98) concerns the rapid internal dynamics of the bump core associated with the width λ appearing in the profile $A(\varrho/\lambda)$. This width is not, in general, the effective transverse width λ_e used below in the one-body sector.

Modal energy of the core. At LO, the absorbed EBK quantization fixes the total modal energy of the bump in the form

$$E_{\mathrm{tot}}^{(\mathrm{bump})} = h_{\mathrm{eff}}\, \omega_{\mathrm{bump}} = h_{\mathrm{eff}}\, \chi_\omega^{(\mathrm{bump})}\, \frac{c_t}{\lambda}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.100}$$

This expression is the internal frequency closure of the core. It must not be confused with the effective one-body closure used below.

Mechanical mass of the one-body sector. In the observable one-body sector, one introduces the *effective transverse width* λ_e of the electron-like bump, closed independently by (6.86). The associated mechanical mass is written

$$m_{\text{bump}}^{(\text{mech})} := \kappa_{\text{core}}\, \rho\, \lambda_e^2\, R_0. \tag{6.101}$$

This relation belongs to the effective one-body closure; it is not obtained by identifying λ_e with the internal frequency width λ of the core.

Spectral–mechanical compatibility. The two preceding writings describe the same one-body state. The compatibility closure is therefore

$$m_{\text{bump}}^{(\text{mech})} = m_{\text{bump}}^{(\text{spec})}. \tag{6.102}$$

In energy form, this compatibility becomes

$$\kappa_{\text{core}}\, \rho\, \lambda_e^2\, R_0\, c_t^2 = \Big(\frac{h_{\text{eff}}\, c_t}{\ell_0}\Big) \Pi_{\text{proj}}\, Z_{\text{ann}}\, Z_{\text{proj}}\, \frac{1}{\kappa_\omega^2\, b_0^{(2)}}. \tag{6.103}$$

Structural compatibility. The section therefore presents two distinct levels:

1. the internal structure of the core, described by M_0, M_2, κ_{core} and $\kappa_\omega^{(\text{bump})}$;
2. the observable one-body closure, which uses the effective width λ_e and leads to (6.101).

These two levels are linked only by the compatibility condition (6.102).

6.6.11 Remark on curvature

Curvature corrections come from (i) the expansion of the Jacobian $J(s) = R_0^2 \sin^2(s/R_0)$ beyond the quasi-flat approximation, and (ii) the boundary terms in the effective channel map (which modify the closure conditions). At LO, they are of order

$$\mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big)$$

on the internal dynamics of the core and affect neither the window u_*(6.20) nor the structure of the admissible bump branches at leading order. In particular, κ_{core} and $\chi_\omega^{(\text{bump})}$ remain unchanged at LO, the deviations being absorbed in

$$\kappa_{\text{geom}} = 1 + \mathcal{O}(e^{-k}) + \mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big).$$

6.6.12 Elimination of $\Phi_\star$ (reminder)

By parity selection at the center, $\Phi_\star = 0$ and $A'(0) = 0$. The C^1 matching at the boundary (6.18) then fixes the datum

$$\frac{\partial_s A}{A}\Big|_{s=\lambda u_*}$$

at LO, and the cutoff (6.25) bounds the boundary amplitude. No free central parameter therefore remains.

Corollary 6.25 (Disappearance of the free parameter)**.** *The three constraints*

$$A(0) = 1, \qquad A'(0) = 0, \qquad \textit{(DtN at the boundary)} \quad \frac{\partial_s A}{A}\Big|_{s=\lambda u_*} = \textit{pure number (LO)}$$

fully determine the LO radial profile in the local reduction with matching. Thus, at LO, no central free constant remains: the parameter $\Phi_\star$ is eliminated. Consequently, κ_{core} and $\chi_\omega^{(\mathrm{bump})}$ are entirely fixed by the form integrals and the geometry of the thin tube, and the closure

$$E_{\mathrm{tot}}^{(\mathrm{bump})} = h_{\mathrm{eff}}\,\omega_{\mathrm{bump}}$$

is independent of any phase or flux adjustment at the center.

6.7 Core closure for the 2DOF NSL

In the quasi-flat LO regime on $\mathbb{S}^3$, the 2DOF NSL branch is characterized by a structure with two orthogonal tangential legs, a core window, an internal frequency closure, and a one-body mechanical closure. We adopt the same conventions as for the 2DOF bump:

$$\varepsilon := \frac{\lambda}{R_0} \ll 1, \qquad c_t^2 = \frac{T}{\rho}, \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big), \qquad A(0) = 1 \quad \text{(the parity } A'(0) = 0 \text{ is established in §6.} \tag{6.104}$$

The NSL is constructed as an operator decomposition into two tangentially orthogonal, isofrequency legs sharing the same phase variable ϕ and the same Maslov index (see §6.7.6).

6.7.1 NSL ansatz: modulated amplitude and fast rotation

We consider the modulated ansatz, leg by leg, in the tubular gauge around the single ring (of length $2\pi R_0$):

$$\psi_j(t, \mathbf{x}) = A\Big(\frac{\varrho}{\lambda}\Big)\,\cos(\mu t + \varphi_j)\,e^{i\omega t}, \qquad j = 1, 2, \qquad \varphi_2 = \varphi_1 + \tfrac{\pi}{2}, \tag{6.105}$$

with $A(u) = \operatorname{sech} u$, $A(0) = 1$, $A'(0) = 0$, and $\omega \gg \mu$ (common fast rotation).

This writing is the modulated, leg-by-leg form of the two-leg quadrature NSL ansatz introduced above: the factor $e^{i\omega t}$ carries the common fast rotation, while $\cos(\mu t + \varphi_j)$ describes the relative slow modulation of leg j. At strict LO, the single-frequency description is recovered by freezing this slow modulation.

6.7.2 Geometric spin of the NSL

The NSL has two independent circular tangential polarizations in quadrature, hence

$$N_{\mathrm{pol}} = 2.$$

The LO matching window and the C^1 matching impose the fundamental winding at leading order:

$$\boxed{w_{\mathrm{fund}} = 1.} \tag{6.106}$$

at leading order. By the common definition

$$s_g = \frac{w\,N_{\mathrm{pol}}}{2}, \qquad S = s_g\,h_{\mathrm{eff}},$$

one obtains

$$\boxed{S = h_{\mathrm{eff}}\,, \qquad s_g = 1}\,.$$

Remark. The $L^2(\mathbb{S}^3)$ orthogonality of the two contributions enters directly into the independence test that fixes $N_{\mathrm{pol}} = 2$.

6.7.3 Inheritance of the bump window by the NSL (leg-by-leg proof)

Principle. The core window of the NSL must not be postulated independently of that of the bump if the LO matching is treated *leg by leg*. The point to show is that, at leading order, each NSL leg sees the same interior datum and the same projected DtN map as the 2DOF bump.

Proposition 6.26 (LO invariance of the matching by leg)**.** *Consider the 2DOF NSL written as the sum of two tangentially orthogonal, iso-frequency, quadrature legs,*

$$\psi_{\mathrm{NSL}} = \psi_1 + \psi_2, \qquad \varphi_2 = \varphi_1 + \frac{\pi}{2},$$

with, for $j = 1, 2$,

$$\psi_j(t, \mathbf{x}) = A\Big(\frac{\varrho}{\lambda}\Big)\, \cos(\mu t + \varphi_j)\, e^{i\omega t}.$$

At quasi-flat LO, the interior/exterior matching projected on each leg coincides with the bump matching. In particular, the cutoff variable u_* obeys, leg by leg, the same matching equation as for the bump.

Proof. (i) Same local core by leg.) By construction, the two legs carry the same envelope $A(\varrho/\lambda)$, the same width λ, and the same local tubular geometry. In particular, at the core boundary $s_c = \lambda u_*$, the interior logarithmic derivative is, for each leg,

$$\frac{\partial_s A}{A}\Big|_{s=s_c} = -\frac{1}{\lambda} \tanh u_*,$$

exactly as for the bump.

(ii) Decoupling of the legs at LO.) The two legs are orthogonal in quadrature. At LO, the exterior operator linearized on $\mathbb{S}^3$ acts scalarly on each tangential component and preserves this orthogonal decomposition. No cross term therefore appears in the projection on one leg: the exterior DtN map is diagonal in the basis (ψ_1, ψ_2).

(iii) Same projected DtN map.) Since each leg sees the same scalar exterior operator, the same single-ring geometry, and the same local boundary datum, the DtN map projected on one leg coincides, at LO, with the bump DtN map. In other words, the boundary shift carried by one NSL leg is the same as that of the bump at leading order.

(iv) Conclusion.) The matching projected on each leg therefore takes the same form as for the bump: the interior datum, the projected DtN map, and the layer correction coincide at LO. The cutoff variable u_* thus satisfies, leg by leg, the same matching equation as the bump. □

Corollary 6.27 (Core window inherited by the NSL)**.** *At LO, the core window of the 2DOF NSL, understood* leg by leg*, coincides with that of the bump established in §6.6.1. In particular,*

$$u_* \in [1.7, 2.1].$$

Proof. By Proposition 6.26, the matching equation projected on each leg is identical to that of the bump at leading order. The admissible window in u_* is therefore the same. □

6.7.4 Core window and cutoff (by leg)

By Corollary 6.27, the core window of the 2DOF NSL coincides, leg by leg, with that of the bump. The interior/exterior C^1 matching on $\mathbb{S}^3$ at LO is therefore written, for each leg, as

$$y \cot y = 1 - u_* \tanh u_* + \delta\Lambda(u_*, \varepsilon), \qquad y = (n{+}1)\, u_*\, \varepsilon. \tag{6.107}$$

Indeed, by Proposition 6.26, the DtN map projected on an NSL leg coincides at LO with that of the inherited bump matching. No independent boundary shift must therefore be introduced in the leg-by-leg projection.

We therefore work on the same admissible window as for the bump:

$$u_* \in [1.7, 2.1]. \tag{6.108}$$

The local cutoff is evaluated on the admissible window inherited from the bump:

$$\Sigma_{\text{eff}} := \sup_{u \in \mathcal{U}_{\text{adm}}} u^2 \Big(\tfrac{A(u)}{A(0)}\Big)^2, \qquad \mathcal{U}_{\text{adm}} := [u_-, u_+], \qquad u_- \approx 1.7085, \quad u_+ \approx 2.0653. \tag{6.109}$$

For the profile $A(u) = \operatorname{sech} u$, one then obtains

$$0.266 \lesssim u_*^2 \operatorname{sech}^2 u_* \lesssim 0.359,$$

and

$$0.2495 \lesssim \varepsilon_* := \operatorname{sech} u_* \lesssim 0.3508,$$

that is, to the nearest hundredth,

$$\varepsilon_* \in [0.25, 0.35].$$

Lemma 6.28 (LO sign of the 2DOF NSL matching)**.** *On the inherited window $u_* \in [1.7, 2.1]$, the reduced right-hand side of the 2DOF NSL matching projected leg by leg is strictly negative at LO:*

$$G_{\text{NSL}}(u_*; \varepsilon) = 1 - u_* \tanh u_* + \delta\Lambda(u_*, \varepsilon) < 0.$$

Proof. By Corollary 6.27, the NSL window coincides leg by leg with the bump window. This window is defined by (6.19), namely

$$-1 \le 1 - u_* \tanh u_* \le -\frac{3}{5}.$$

The layer correction satisfies

$$\delta\Lambda(u_*, \varepsilon) = \mathcal{O}((u_*\varepsilon)^2), \qquad \varepsilon = \lambda/R_0 \ll 1.$$

At LO, this correction is subdominant with respect to the margin $3/5$ that separates the principal right-hand side from zero. Thus

$$1 - u_* \tanh u_* + \delta\Lambda(u_*, \varepsilon) < 0$$

over the whole admissible window. □

Lemma 6.29 (Cancellation of the residual isotropic defect for the 2DOF NSL)**.** *At LO, the 2DOF NSL has the fundamental scalar trace, but its residual traceless isotropic defect is zero. The return functionals that open the thresholds q_2 and q_3 in the bump branch therefore do not produce any additional admissible opening for the NSL.*

Proof. We use the trace/traceless decomposition (6.27). For a one-leg bump, the internal polarization matrix is

$$M_{\text{bump}} = e_1 \otimes e_1, \qquad \operatorname{tr} M_{\text{bump}} = 1.$$

Therefore

$$D_{\text{bump}} = e_1 \otimes e_1 - \frac{1}{2} I_2 \neq 0, \qquad \|D_{\text{bump}}\|^2 = \frac{1}{2}.$$

The bump therefore has a nonzero residual isotropic defect, capable of feeding the first opening q_2.

For a 2DOF NSL, the two legs have the same profile, are orthogonal, and are balanced in quadrature. The normalized internal matrix is then

$$M_{\text{NSL}} = \frac{1}{2}(e_1 \otimes e_1 + e_2 \otimes e_2) = \frac{1}{2} I_2.$$

Thus

$$D_{\mathrm{NSL}} = M_{\mathrm{NSL}} - \frac{1}{2} I_2 = 0.$$

The scalar density carried by the NSL is therefore indeed nonzero:

$$|\mathcal{S}_{\mathrm{NSL}}^{\mathrm{LO}}|^2 = A^2,$$

but this scalar density belongs to the fundamental trace already consumed by the $q = 1$ level. It is not the residual traceless defect that opens q_2. According to the definition of the isotropic return functionals,

$$r_2^{(\mathrm{NSL})}(q) = \|D_{\mathrm{NSL}}\|^2 \int_{\mathbb{R}} T_1(v)\, \Gamma_0(v; q)\, dv = 0.$$

The anisotropic splitting mechanism q_3 acts on this same residual defect. Since $D_{\mathrm{NSL}} = 0$, there is likewise no admissible bump-type anisotropic splitting for the NSL at minimal LO. The NSL therefore retains only the fundamental level selected by the local matching. □

6.7.5 NSL sign filter and uniqueness of the LO level

We apply the decision rule of §6.5.5. By Lemma 6.28, the reduced right-hand side of the NSL matching is strictly negative at LO. Thus, for every $q \geq 2$, $y = u_*/q < \pi/2$ and $h(y) > 0$: there is no solution in the bare local filter.

It remains to check that the NSL does not reintroduce, through compact return, the openings q_2 and q_3 specific to the bump. This is precisely the object of Lemma 6.29: the balanced quadrature of the two legs cancels the residual traceless isotropic defect D_X. The NSL therefore has the fundamental trace, but not the defect that opens q_2, nor the anisotropic splitting q_3 associated with that defect.

For $q = 1$, $y = u_* \in (\pi/2, \pi)$, and since h is decreasing, there is a unique solution:

$$\boxed{q = 1 \text{ only (NSL, LO)}}.$$

6.7.6 2DOF spectral bridge and absorbed EBK form (NSL)

The 2DOF families (bump/NSL) share the same phase variable ϕ and the same Maslov index. We use the absorbed EBK convention (6.1)–(6.2), recalled here in the form

$$\boxed{\mathcal{J} = \frac{E}{\omega} = h_{\mathrm{eff}}.} \tag{6.110}$$

In the regime $R_0 \gg \lambda$, the spectral scaling law is

$$\omega(q) = q\, \omega_0, \qquad \omega_0 := \omega(1), \tag{6.111}$$

hence the spectral energy

$$E_{\mathrm{tot}}^{(\mathrm{NSL})}(q) = h_{\mathrm{eff}}\, \omega(q) = h_{\mathrm{eff}}\, q\, \omega_0. \tag{6.112}$$

NSL constraint. For the NSL (class $\mathfrak{C}_-$), the filter of §6.7.5 imposes $\boxed{q = 1}$ at LO. We will therefore use

$$E_{\mathrm{tot}}^{(\mathrm{NSL})}(1) = h_{\mathrm{eff}}\, \omega_0.$$

6.7.7 Radial capacity and form factors (sum of the legs)

Radial moments by leg. In the tubular gauge, we define, by leg,

$$M_0^{(\mathrm{leg})} = R_0^2\, \lambda \int_{\mathbb{R}} A^2(u)\, du \Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big], \qquad M_2^{(\mathrm{leg})} = \frac{R_0^2}{\lambda} \int_{\mathbb{R}} A^2(u)\, du \Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.113}$$

Additivity of the two legs. The two NSL legs are orthogonal in quadrature. After integration over the internal angle, one has

$$\int_0^{2\pi} \cos^2\phi\, d\phi = \pi, \qquad \int_0^{2\pi} \sin^2\phi\, d\phi = \pi,$$

so that the contributions of the two legs add without a cross term at LO. Thus

$$M_0^{(\mathrm{NSL})} = 2\, M_0^{(\mathrm{leg})}, \qquad M_2^{(\mathrm{NSL})} = 2\, M_2^{(\mathrm{leg})}, \qquad \Rightarrow \qquad \frac{M_2^{(\mathrm{NSL})}}{M_0^{(\mathrm{NSL})}} = \frac{M_2^{(\mathrm{leg})}}{M_0^{(\mathrm{leg})}}. \tag{6.114}$$

Shape integrals for the profile sech**.** For $A(u) = \operatorname{sech} u$, one has

$$I_0 := \int_{\mathbb{R}} A^2(u)\, du = 2, \qquad I_2 := \int_{\mathbb{R}} A'^{\,2}(u)\, du = \frac{2}{3}. \tag{6.115}$$

Radial capacity and core factor. We set, by leg,

$$C^{(0)}_{\ell,\mathrm{leg}} := \frac{1}{4\pi} \int_{\mathbb{R}} A^2(u)\, du, \qquad \kappa^{(\mathrm{leg})}_{\mathrm{core}} := 4\pi\, C^{(0)}_{\ell,\mathrm{leg}}. \tag{6.116}$$

With (6.115), it follows that

$$C^{(0)}_{\ell,\mathrm{leg}} = \frac{1}{2\pi}, \qquad \kappa^{(\mathrm{leg})}_{\mathrm{core}} = 2.$$

For the NSL, by additivity of the two legs,

$$C^{(0)}_{\ell,\mathrm{NSL}} = 2\, C^{(0)}_{\ell,\mathrm{leg}}, \qquad \kappa^{(\mathrm{NSL})}_{\mathrm{core}} = 2\, \kappa^{(\mathrm{leg})}_{\mathrm{core}} = 4. \tag{6.117}$$

Frequency factor. The total Rayleigh quotient is unchanged by the sum of the legs:

$$\big(\chi_\omega^{(\mathrm{NSL})}\big)^2 = \kappa_{\mathrm{geom}}\, \frac{M_2^{(\mathrm{NSL})}}{M_0^{(\mathrm{NSL})}} = \kappa_{\mathrm{geom}}\, \frac{I_2}{I_0}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.118}$$

With (6.115), one therefore obtains

$$\big(\chi_\omega^{(\mathrm{NSL})}\big)^2 = \frac{\kappa_{\mathrm{geom}}}{3}, \qquad \chi_\omega^{(\mathrm{NSL})} = \frac{1}{\sqrt{3}}\Big[1 + \mathcal{O}\Big(\tfrac{\lambda^2}{R_0^2}\Big)\Big]. \tag{6.119}$$

Hence

$$\omega_{\mathrm{NSL}} = \chi_\omega^{(\mathrm{NSL})}\, \frac{c_t}{\lambda}. \tag{6.120}$$

LO energetic closure of the core. By radial integration in the tubular gauge, one obtains

$$E_0^{(\mathrm{NSL})} = \kappa^{(\mathrm{NSL})}_{\mathrm{core}}\, T\, \lambda^2 R_0. \tag{6.121}$$

Rewriting (6.120) in the form

$$\lambda = \chi_\omega^{(\mathrm{NSL})}\, \frac{c_t}{\omega_{\mathrm{NSL}}}, \tag{6.122}$$

gives

$$E_0^{(\mathrm{NSL})} = \kappa^{(\mathrm{NSL})}_{\mathrm{core}}\, T\, R_0 \left(\chi_\omega^{(\mathrm{NSL})} \frac{c_t}{\omega_{\mathrm{NSL}}}\right)^2. \tag{6.123}$$

Associated internal mechanical mass. We then introduce the internal HSM mechanical mass associated with the NSL:

$$m_{\mathrm{NSL}}^{(\mathrm{mech})} := \frac{E_0^{(\mathrm{NSL})}}{c_t^2} = \kappa_{\mathrm{core}}^{(\mathrm{NSL})}\,\rho\,\lambda^2 R_0. \tag{6.124}$$

Reinjecting (6.122), one obtains

$$m_{\mathrm{NSL}}^{(\mathrm{mech})} = \kappa_{\mathrm{core}}^{(\mathrm{NSL})}\,\rho\,R_0\left(\chi_\omega^{(\mathrm{NSL})}\frac{c_t}{\omega_{\mathrm{NSL}}}\right)^2 = \kappa_{\mathrm{core}}^{(\mathrm{NSL})}\left(\chi_\omega^{(\mathrm{NSL})}\right)^2 R_0\,\frac{T}{\omega_{\mathrm{NSL}}^2}. \tag{6.125}$$

Status of the width λ. In the present subsection, λ denotes the internal width of the NSL core, the one that enters the profile $A(\varrho/\lambda)$ and the internal frequency closure (6.120). A possible effective transverse width of the one-body sector, distinct from λ, is not the object of this subsection.

Structural compatibility. The present closure therefore concerns the NSL core itself:

1. the internal structure, described by $M_0^{(\mathrm{NSL})}$, $M_2^{(\mathrm{NSL})}$, $\kappa_{\mathrm{core}}^{(\mathrm{NSL})}$, and $\chi_\omega^{(\mathrm{NSL})}$;
2. the internal energetic closure, given by (6.121) and (6.124).

It must not be confused with an effective one-body closure of a distinct transverse channel, which would require, if needed, a transverse effective width specific to the channel considered.

6.7.8 Reduced local block of the 2DOF NSL constituent

Origin. The fundamental radial sector carries the local term

$$\mu_2\,r, \qquad \mu_2 = T\,\kappa_\mu, \qquad [\kappa_\mu] = \mathrm{m}^{-2}.$$

In the 2DOF NSL core reduction, one works with the dimensionless variable

$$\boxed{u = \frac{\varrho}{\lambda}, \qquad A(u) = \operatorname{sech} u, \qquad A(0) = 1.} \tag{6.126}$$

By dimensional invariance at LO, the local radial term therefore enters only through the dimensionless combination

$$\boxed{\kappa_\mu\,\lambda^2.}$$

Local normalization of the core. Since the profile is normalized here by $A(0) = 1$, the local reduction does not involve a free amplitude A_0, but the dimensionless combination $\kappa_\mu\lambda^2$. We introduce the symbolic matching coefficient $C_{\mathrm{cut}}^{(\mathrm{NSL})}$ by

$$\boxed{1 = C_{\mathrm{cut}}^{(\mathrm{NSL})}\,\kappa_\mu\,\lambda^2, \qquad \text{i.e.} \qquad \kappa_\mu\,\lambda^2 = \frac{1}{C_{\mathrm{cut}}^{(\mathrm{NSL})}}.} \tag{6.127}$$

The coefficient $C_{\mathrm{cut}}^{(\mathrm{NSL})}$ is a pure core matching coefficient for the retained canonical profile. At this stage, it fixes a parametric writing of the local reduction; it is not introduced as a closed numerical value.

Projection on profile moments. The derivative-free and derivative contributions are carried respectively by the core moments

$$I_0 := \int_{\mathbb{R}} A^2(u)\,du, \qquad I_2 := \int_{\mathbb{R}} A'^2(u)\,du.$$

For an elementary leg, the reduced local block therefore takes the form

$$\mathcal{B}_{\text{leg}}^{(\text{NSL})} = (\kappa_\mu \lambda^2)\, I_0 + I_2. \tag{6.128}$$

With the symbolic normalization (6.127), the same expression is written as

$$\mathcal{B}_{\text{leg}}^{(\text{NSL})} = \frac{I_0}{C_{\text{cut}}^{(\text{NSL})}} + I_2. \tag{6.129}$$

Common block of the 2DOF NSL constituent. In the two-orthogonal-leg 2DOF NSL branch, the common constituent block is obtained by Galerkin contraction between two channel endpoints. A derivative-free endpoint carries the profile moment I_0, whereas an endpoint on which the radial differential term acts carries the moment I_2. The bilinear contraction of the common block therefore gives

$$K_0^{(2)} := I_0\, I_0 = I_0^2, \qquad K_2^{(2)} := I_0\, I_2.$$

The square I_0^2 does not represent a repetition of the same normalization factor: it comes from the two endpoints of the bilinear form. The term $I_0 I_2$ corresponds to the same contraction scheme with one local endpoint and one differential endpoint.

The common block is therefore written as

$$\boxed{B_{\text{ch}}^{(\text{NSL})} = (\kappa_\mu \lambda^2)\, K_0^{(2)} + K_2^{(2)}.} \tag{6.130}$$

With the same symbolic normalization, one obtains the equivalent parametrized form

$$B_{\text{ch}}^{(\text{NSL})} = \frac{K_0^{(2)}}{C_{\text{cut}}^{(\text{NSL})}} + K_2^{(2)}. \tag{6.131}$$

Status. The block (6.130) is the common local block of the 2DOF NSL constituent. It comes only from the local radial term of the 2DOF NSL core after reduction on the sech profile. The forms (6.127), (6.129), and (6.131) are parametric writings of the same local reduction. They do not, by themselves, give a closed numerical value for $C_{\text{cut}}^{(\text{NSL})}$.

Any later specialization that consumes $C_{\text{cut}}^{(\text{NSL})}$ as a numerical quantity must provide, in the development concerned, either its evaluation or its elimination by common normalization or by ratio. Otherwise, the block must remain read as a parametric local writing.

6.7.9 Angular corrections and internal kernels

In the composite closures inherited from the NSL constituent, the boundary contribution decomposes into a purely angular term, depending on the composite channel considered, and an internal term specific to the NSL core. We write

$$\Delta_{\text{DtN}}^{(X)} = \Delta_{\text{cone}}^{(X)} + \Delta_{\text{int}}^{(\text{NSL})}(\kappa_r),$$

where X denotes the label of the later composite channel, while $\Delta_{\text{int}}^{(\text{NSL})}$ is the internal kernel carried by the NSL core itself.

At LO, this internal kernel is written in affine form

$$\Delta_{\mathrm{int}}^{(\mathrm{NSL})}(\kappa_r) = A_2^{(\mathrm{NSL})}\Big(\frac{1}{\kappa_r} - 1\Big), \tag{6.132}$$

where $A_2^{(\mathrm{NSL})}$ denotes the pure coefficient associated with the leg-by-leg projection of the internal kernel. In this subsection, it enters as a symbolic internal-reduction coefficient; no closed numerical value is attached to it here.

The term $\Delta_{\mathrm{cone}}^{(X)}$, in turn, belongs to the angular geometry of the composite channel and does not enter the local reduction of the isolated NSL core. Any specialization that consumes $\Delta_{\mathrm{int}}^{(\mathrm{NSL})}$ as a numerical quantity must therefore provide the evaluation of $A_2^{(\mathrm{NSL})}$, or show that this coefficient disappears by common normalization or by passage to a ratio.

Internal conic constants of the 2DOF NSL filters

Homogeneous exterior problem. The internal conic constants are defined from the homogeneous exterior problem associated with the 2DOF NSL radial sector. In the angular variable $\psi = s/R_0$, on the interval

$$\psi \in [\psi_c, \pi - \psi_c],$$

one considers the equation

$$\boxed{\partial_{\psi\psi} r + 2\cot\psi\, \partial_\psi r - \mu_{\mathrm{ext}}^2\, r = 0, \qquad \mu_{\mathrm{ext}} := \lambda_r R_0, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}.} \tag{6.133}$$

This is the *homogeneous exterior problem*: the core source term does not enter the definition of the conic constants.

Dirichlet–Neumann map. For any solution of (6.133), we define the boundary data

$$\mathbf{d} = \begin{pmatrix} r(\psi_c) \\ r(\pi - \psi_c) \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} -\partial_\psi r(\psi_c) \\ +\partial_\psi r(\pi - \psi_c) \end{pmatrix}.$$

By linearity and by symmetry of the exterior problem, there exists a Dirichlet–Neumann matrix of the form

$$\boxed{\mathbf{n} = \mathsf{M}_{\mathrm{DN}}\, \mathbf{d}, \qquad \mathsf{M}_{\mathrm{DN}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},} \tag{6.134}$$

where a and b depend only on ψ_c and μ_{ext}.

Even and odd modes. We introduce the eigenvectors

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which correspond respectively to the even and odd modes of the exterior problem. The associated eigenvalues are

$$\boxed{\lambda_+ = a + b, \qquad \lambda_- = a - b.} \tag{6.135}$$

Mode conic constants. The conic constants are pure numbers associated with the eigenmodes of the exterior DtN map. We choose the odd mode as the canonical normalization reference:

$$\boxed{\Delta_{\mathrm{cone}}^{(-)} = 1.} \tag{6.136}$$

The even mode similarly carries an internal conic constant $\Delta_{\mathrm{cone}}^{(+)}$, obtained from the same map M_{DN}. At this stage, the signs + and − denote only the two proper parities of the homogeneous exterior problem; they do not constitute observational identifications.

Status. The quantities $\Delta_{\mathrm{cone}}^{(-)}$ and $\Delta_{\mathrm{cone}}^{(+)}$ are *pure numbers* of the 2DOF NSL sector. They do not depend on the medium constants (ρ, T, ℓ_0, R_0). They provide internal conic data that can be reinserted into a later composite closure, without requiring the present section to identify these modes with external physical channels.

6.7.10 Remark on curvature

As for the bump, curvature corrections come from (i) the expansion of $J(s)$ beyond the quasi-flat regime and (ii) the boundary terms in the DtN map. They are of order $\mathcal{O}((\lambda/R_0)^2)$ on the energy and do not affect the LO identities (6.110)–(6.124).

Synthesis. (i) The 2DOF NSL is formed by two orthogonal, iso-frequency legs sharing the same phase variable ϕ. The core window is inherited, leg by leg, from that of the bump. (ii) The negative sign of the NSL matching projected leg by leg, established in Lemma 6.28, excludes the levels $q \geq 2$ in the bare local filter. Lemma 6.29 further shows that the NSL has a fundamental trace but no residual traceless isotropic defect: the bump-type thresholds q_2 and q_3 therefore do not open. The NSL thus admits $q = 1$ **only** at LO.

(iii) The radial capacity is additive: $\kappa_{\mathrm{core}}^{(\mathrm{NSL})} = 2\,\kappa_{\mathrm{core}}^{(\mathrm{leg})}$, whereas the frequency factor remains unchanged: $\chi_\omega^{(\mathrm{NSL})} = \chi_\omega^{(\mathrm{leg})}$. (iv) The internal energetic closure is written

$$E_0^{(\mathrm{NSL})} = \kappa_{\mathrm{core}}^{(\mathrm{NSL})}\, T\, \lambda^2 R_0, \qquad E_{\mathrm{tot}}^{(\mathrm{NSL})} = h_{\mathrm{eff}}\, \omega_0.$$

(v) The associated internal mechanical mass is

$$m_{\mathrm{NSL}}^{(\mathrm{mech})} = \kappa_{\mathrm{core}}^{(\mathrm{NSL})}\, \rho\, \lambda^2 R_0.$$

Chapter 7

Local kinematics of localized massive solutions

7.1 General framework: comoving massive core

Purpose. Localized massive solutions are constructed in a comoving frame: their center is at rest, their total longitudinal momentum is zero, and their proper energy is finite. This chapter establishes that, when such a branch arising from the HSM Euler–Lagrange equations admits a local hyperbolic longitudinal reduction, a comoving solution generates a uniformly moving solution in a fixed frame.

The corresponding energy–momentum law is not postulated: it follows from the local form of the equations of motion and from the existence of a comoving frame. Massless modes, in particular photon-like branches, do not belong to this construction: they have no rest frame and must be treated through their own dispersion.

Reduced form arising from the equations of motion. Let Ψ denote the set of effective degrees of freedom of a localized massive branch, after projection of the transverse and internal coordinates. In a local window where the background coefficients are constant to the order considered, the reduction of the Euler–Lagrange equations takes the form

$$\boxed{\rho_{\mathcal{F}}\,\partial_t^2\Psi - \tau_{\mathcal{F}}\,\partial_s^2\Psi + \nabla_\Psi \mathcal{U}_{\mathcal{F}}(\Psi) = 0.} \tag{7.1}$$

The characteristic speed of the reduced branch is defined by

$$\boxed{c_{\mathcal{F}}^2 := \frac{\tau_{\mathcal{F}}}{\rho_{\mathcal{F}}}.} \tag{7.2}$$

Universality of the local limiting speed. In the localized massive branches considered here, the longitudinal reduction does not change the ratio between the tangential inertia and the tangential stiffness inherited from the background sector. The projection factors specific to a family may modify the effective coefficients $\rho_{\mathcal{F}}$ and $\tau_{\mathcal{F}}$, but they multiply them by the same factor in the longitudinal kinetic part. Consequently,

$$\boxed{\frac{\tau_{\mathcal{F}}}{\rho_{\mathcal{F}}} = \frac{T}{\rho} = c_t^2, \qquad c_{\mathcal{F}} = c_t.} \tag{7.3}$$

The notation $c_{\mathcal{F}}$ is retained in the boost identities in order to keep the reduction steps visible, but the local limiting speed of HSM massive branches is common.

The function $\mathcal{U}_{\mathcal{F}}$ collects the local terms of core, confinement, projected curvature, and the internal contributions specific to the branch considered. It does not depend explicitly on s or t in the local window.

The branch is called massive when the comoving solution has a finite proper energy

$$E_0 > 0$$

and zero total longitudinal momentum

$$P_0 = 0.$$

This condition distinguishes localized solitons from massless propagating modes.

Proposition 7.1 (Massive comoving solution and uniformly moving solution)**.** *Let $\Psi_0(\xi,\tau)$ be a localized massive solution of* (7.1) *in its comoving frame. For any uniform speed $|u| < c_{\mathcal{F}}$, define*

$$\Gamma_{\mathcal{F}}(u) = \frac{1}{\sqrt{1-u^2/c_{\mathcal{F}}^2}},$$

and

$$\boxed{\xi = \Gamma_{\mathcal{F}}(s-ut), \qquad \tau = \Gamma_{\mathcal{F}}\left(t - \frac{u}{c_{\mathcal{F}}^2}s\right).} \tag{7.4}$$

Then

$$\boxed{\Psi_u(s,t) := \Psi_0(\xi,\tau)} \tag{7.5}$$

is again a solution of the reduced equation (7.1)*, written in the fixed variables (s,t).*

Proof. The derivatives associated with (7.4) are

$$\partial_t = \Gamma_{\mathcal{F}}\,\partial_\tau - \Gamma_{\mathcal{F}} u\,\partial_\xi, \qquad \partial_s = -\Gamma_{\mathcal{F}}\frac{u}{c_{\mathcal{F}}^2}\,\partial_\tau + \Gamma_{\mathcal{F}}\,\partial_\xi.$$

One then obtains

$$\partial_t^2 - c_{\mathcal{F}}^2\partial_s^2 = \Gamma_{\mathcal{F}}^2\,(\partial_\tau - u\partial_\xi)^2 - c_{\mathcal{F}}^2\Gamma_{\mathcal{F}}^2\left(-\frac{u}{c_{\mathcal{F}}^2}\partial_\tau + \partial_\xi\right)^2.$$

The crossed terms cancel. It remains

$$\partial_t^2 - c_{\mathcal{F}}^2\partial_s^2 = \Gamma_{\mathcal{F}}^2\left(1-\frac{u^2}{c_{\mathcal{F}}^2}\right)\partial_\tau^2 - c_{\mathcal{F}}^2\Gamma_{\mathcal{F}}^2\left(1-\frac{u^2}{c_{\mathcal{F}}^2}\right)\partial_\xi^2.$$

Since

$$\Gamma_{\mathcal{F}}^2\left(1-\frac{u^2}{c_{\mathcal{F}}^2}\right) = 1,$$

one has the identity

$$\boxed{\partial_t^2 - c_{\mathcal{F}}^2\partial_s^2 = \partial_\tau^2 - c_{\mathcal{F}}^2\partial_\xi^2.} \tag{7.6}$$

Using $\tau_{\mathcal{F}} = \rho_{\mathcal{F}} c_{\mathcal{F}}^2$, the reduced equation written in (s,t) is transformed exactly into the reduced equation written in (ξ,τ). Since Ψ_0 is a solution in the comoving frame, Ψ_u is a solution in the fixed frame. □

Proposition 7.2 (Energy, momentum, and boost law of massive solutions)**.** *For a reduced branch of the form* (7.1)*, the local Lagrangian density is*

$$\mathscr{L}_{\mathcal{F}} = \frac{\rho_{\mathcal{F}}}{2}|\partial_t \Psi|^2 - \frac{\tau_{\mathcal{F}}}{2}|\partial_s \Psi|^2 - \mathcal{U}_{\mathcal{F}}(\Psi).$$

The conserved quantities associated with translations in t and in s are

$$\boxed{E = \int \left[\frac{\rho_{\mathcal{F}}}{2}|\partial_t \Psi|^2 + \frac{\tau_{\mathcal{F}}}{2}|\partial_s \Psi|^2 + \mathcal{U}_{\mathcal{F}}(\Psi)\right] ds, \qquad P = -\int \rho_{\mathcal{F}}\, \partial_t \Psi \cdot \partial_s \Psi \, ds.} \tag{7.7}$$

Under the change of variables (7.4)*, they obey*

$$\boxed{E(u) = \Gamma_{\mathcal{F}}(E_0 + uP_0), \qquad P(u) = \Gamma_{\mathcal{F}}\left(P_0 + \frac{u}{c_{\mathcal{F}}^2} E_0\right).} \tag{7.8}$$

In the comoving frame of a localized massive solution,

$$P_0 = 0, \qquad E_0 > 0.$$

One therefore obtains

$$\boxed{E(u) = \Gamma_{\mathcal{F}} E_0, \qquad P(u) = \Gamma_{\mathcal{F}} \frac{u}{c_{\mathcal{F}}^2} E_0.} \tag{7.9}$$

Proof. The reduced Lagrangian density depends on the longitudinal derivatives through the combination

$$|\partial_t \Psi|^2 - c_{\mathcal{F}}^2 |\partial_s \Psi|^2,$$

which is preserved by (7.4). Time and longitudinal translations therefore give the Noether charges (7.7).

The change $(\tau, \xi) \mapsto (t, s)$ is linear and has unit determinant. The Noether current associated with translations then transforms in the (t, s) plane according to

$$\begin{pmatrix} E(u)/c_{\mathcal{F}} \\ P(u) \end{pmatrix} = \begin{pmatrix} \Gamma_{\mathcal{F}} & \Gamma_{\mathcal{F}} u/c_{\mathcal{F}} \\ \Gamma_{\mathcal{F}} u/c_{\mathcal{F}} & \Gamma_{\mathcal{F}} \end{pmatrix} \begin{pmatrix} E_0/c_{\mathcal{F}} \\ P_0 \end{pmatrix}.$$

It follows that

$$E(u) = \Gamma_{\mathcal{F}}(E_0 + uP_0), \qquad P(u) = \Gamma_{\mathcal{F}}\left(P_0 + \frac{u}{c_{\mathcal{F}}^2} E_0\right).$$

For a localized massive solution in its comoving frame, the total momentum is zero, $P_0 = 0$. The law (7.9) follows. □

Effective dispersion of massive solutions. The associated quadratic invariant is

$$E^2 - c_{\mathcal{F}}^2 P^2.$$

In the comoving frame of a localized massive solution,

$$P_0 = 0,$$

therefore

$$E^2 - c_{\mathcal{F}}^2 P^2 = E_0^2.$$

Thus

$$\boxed{E(u)^2 = E_0^2 + c_{\mathcal{F}}^2 P(u)^2.} \tag{7.10}$$

Proper frequency and observed frequency of a massive branch. For a comoving phase

$$\varphi(\xi,\tau)=\omega_0\tau,$$

passing to the fixed frame gives

$$\varphi(s,t)=\omega_0\Gamma_{\mathcal{F}}\left(t-\frac{u}{c_{\mathcal{F}}^2}s\right)=\omega_{\rm obs}t-ks.$$

Thus

$$\boxed{\omega_{\rm obs}=\Gamma_{\mathcal{F}}\omega_0,\qquad k=\Gamma_{\mathcal{F}}\frac{u}{c_{\mathcal{F}}^2}\omega_0.} \tag{7.11}$$

It follows that

$$\boxed{\omega_{\rm obs}^2-c_{\mathcal{F}}^2k^2=\omega_0^2.} \tag{7.12}$$

This reconstruction assumes the existence of a comoving proper frequency ω_0. It does not apply to massless modes, which have no comoving frame and are governed by their own dispersion of the type

$$\omega^2-c_t^2k^2=0.$$

7.2 General radial compatibility criterion

Purpose. A localized massive solution may carry a tangential core propagating with the characteristic speed c_t, while exciting a radial response governed by the radial gradient coefficient κ_r and by the local term μ_2. The compatibility of a strictly comoving radial sheath must therefore be checked separately from the kinematics of the core.

Linearized radial equation. To linear order in the radial response around the core, the sheath r associated with a massive family $\mathcal{F}$ obeys an equation of the form

$$\boxed{\rho\,\partial_t^2r-T\kappa_r\,\partial_s^2r+\mu_2r=S_{\mathcal{F}}(s,t).} \tag{7.13}$$

The source $S_{\mathcal{F}}$ depends on the family considered, but the radial operator on the left-hand side is common. Introduce

$$\boxed{c_r^2:=\frac{T\kappa_r}{\rho}=\kappa_rc_t^2,\qquad \omega_r^2:=\frac{\mu_2}{\rho}=c_t^2\kappa_\mu.} \tag{7.14}$$

The coefficient κ_r therefore sets the radial speed, while $\kappa_\mu=\mu_2/T$ sets the radial gap and the range of the response.

Proposition 7.3 (Localized comoving sheath criterion)**.** *Let $\mathcal{F}$ be a massive family whose core moves uniformly at speed v, and suppose that the radial source is localized and comoving:*

$$S_{\mathcal{F}}(s,t)=S_{\mathcal{F}}(\xi),\qquad \xi=s-vt.$$

Then a strictly comoving and localized radial sheath exists at linear order if

$$|v|<c_r.$$

In this case, the radial response is given in Fourier space by

$$\boxed{\widehat{R}(k)=\frac{\widehat{S}_{\mathcal{F}}(k)}{\omega_r^2+(c_r^2-v^2)k^2}.} \tag{7.15}$$

If $|v|\geq c_r$, the radial operator is no longer elliptic in the comoving coordinate. A strictly comoving Yukawa sheath then no longer constitutes a localized closure at the same order.

Proof. Seek a comoving response $r(s,t) = R(\xi)$ with $\xi = s - vt$. One has

$$\partial_t^2 r = v^2 R''(\xi), \qquad \partial_s^2 r = R''(\xi).$$

Equation (7.13) therefore becomes

$$\rho v^2 R'' - T\kappa_r R'' + \mu_2 R = S_{\mathcal{F}}(\xi).$$

Dividing by ρ, and using (7.14), gives

$$-(c_r^2 - v^2)R'' + \omega_r^2 R = \frac{S_{\mathcal{F}}(\xi)}{\rho}.$$

The Fourier transform gives

$$\left[\omega_r^2 + (c_r^2 - v^2)k^2\right] \widehat{R}(k) = \frac{\widehat{S}_{\mathcal{F}}(k)}{\rho}.$$

By absorbing the factor $1/\rho$ into the definition of the reduced source, one obtains (7.15). When $|v| < c_r$, the denominator is strictly positive for all k, and the response is of Yukawa/localized type. When $|v| \geq c_r$, this comoving elliptic character is lost; the strictly comoving sheath closure is no longer a localized solution of the same type. □

7.3 Canonical application: 2DOF bump

The 2DOF bump has a localized tangential core with profile

$$A(\xi) = A_0 \operatorname{sech}\Big(\frac{\xi}{\lambda_{\text{bump}}}\Big).$$

The radial source associated with the core is localized in the same comoving coordinate. It therefore falls under the criterion (7.3). The strictly comoving sheath of the bump is localized for

$$|v| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

In this regime, the radial response is of Yukawa type in the comoving frame. For $|v| \geq c_r$, the closure in the form of a strictly comoving symmetric sheath is no longer elliptic. The tangential core may remain described as a moving core, but the radial response then belongs to a causal component not strictly attached to a single profile $R(s - vt)$.

The 2DOF bump thus provides the canonical case in which the massive core, the localized radial source, and the comoving sheath are explicitly verified.

7.4 Transverse 2DOF application: 2DOF NSL

The 2DOF NSL has a core with two orthogonal tangential legs. By construction,

$$|\vec{u}_{\text{NSL}}(s,\vartheta,t)|^2 = A(s)^2,$$

so that the quadratic source feeding the radial equation is zonal and localized as soon as the core profile A is localized. Since the core profile is of the type

$$A(\xi) = A_0 \operatorname{sech}\Big(\frac{\xi}{\lambda_{\text{NSL}}}\Big),$$

the comoving radial source of the NSL also falls under the criterion (7.3). The strictly comoving sheath is therefore localized for

$$|v| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

The difference from the bump is not carried by the common radial operator, but by the source $S_{\mathcal{F}}$, which here reflects the two-leg tangential structure. The sheath localization criterion remains unchanged.

7.5 Scope of the criterion for the following families

The result does not apply to every HSM solution merely because it belongs to the theory. It applies to localized massive branches that have

$$E_0 > 0, \qquad P_0 = 0$$

in a comoving frame, as well as a local hyperbolic longitudinal reduction of the type (7.1). When the branch carries a radial sheath response, one must also verify that the radial source is localized in the comoving coordinate.

The following massive families must therefore verify locally the assumptions of the criterion in their own chapter. Massless modes and quasi-c_t reductions belong to distinct proper dispersions.

Chapter 8

Quasi-c_t 2DOF bosonic solutions on $\mathbb{S}^3$

Definition 8.1 (Bosonic in the HSM sense)**.** In this manuscript, a solution is called *bosonic* when its total geometric spin is $\boldsymbol{s_g = 1}$, independently of the number of degrees of freedom (2DOF, 3DOF, ...) and of the channel considered. This terminology is internal to HSM: it presupposes neither quantum exchange statistics nor QFT structure.

Introduction

We study a family of 2DOF solutions propagating near the limiting speed c_t of the mesh. In the quasi-flat core on $\mathbb{S}^3$, this branch corresponds to the *strictly gapless regime at LO*: the local closure then carries only the long-wave dispersion and the cubic nonlinearity transported by the longitudinal curvature.

After a multiple-scale reduction near $v = c_t$, the slow scalar envelope obeys a local mKdV-type equation, admitting localized pulses propagating with $v \simeq c_t$. These excitations are bosonic[1] in the HSM sense, that is, of total geometric spin

$$s_g = 1.$$

Here α denotes the polarization angle between the tangent plane and a possible radial-bias component. The family considered corresponds to the case

$$\alpha = \frac{\pi}{2},$$

that is, to a purely tangential polarization and to the absence of radial bias at leading order.

8.1 Multiple-scale reduction near $v = c_t$

The multiple-scale reduction is performed from the local tangential projection of the Euler–Lagrange equations onto the normalized transverse mode of the branch. At the first retained dispersive order and the first retained nonlinear order, this projection takes the form

$$A_{tt} - c_t^2 A_{ss} - c_t^2\, b_\beta\, (A^3)_{ss} + \delta\, A_{ssss} = 0, \qquad 0 < \delta \ll 1. \tag{8.1}$$

The two closure coefficients are the reduced projections

$$\boxed{b_\beta := \frac{\langle e_{\tan}, \mathcal{N}_3(e_{\tan}, e_{\tan}, e_{\tan})\rangle_\perp}{\langle e_{\tan}, e_{\tan}\rangle_\perp}, \qquad d_\delta := \frac{\langle e_{\tan}, \mathcal{D}_4 e_{\tan}\rangle_\perp}{\ell_0^2 \langle e_{\tan}, e_{\tan}\rangle_\perp}.} \tag{8.2}$$

[1] In the sense of Def. 8.1: $s_g = 1$.

Here $e_{\tan}$ is the normalized transverse mode, $\mathcal{N}_3$ is the cubic part of the tangential Euler–Lagrange operator projected onto the branch, and $\mathcal{D}_4$ is the quartic dispersive correction resulting from the long-wave expansion of the same operator. These objects are reduced operators of the core theory; they are not added numerical parameters.

Units and closure of the coefficients. A is dimensionless; s is in m, t in s; c_t in $\mathrm{m\,s^{-1}}$. One has

$$[b_\beta] = 1, \qquad [\delta] = \mathrm{m}^4\,\mathrm{s}^{-2}.$$

The dispersion is parameterized by

$$\delta = d_\delta\, c_t^2\, \ell_0^2,$$

with d_δ defined by (8.2). The coefficient b_β is defined by the same equation. The two coefficients are therefore pure numbers of core projection.

Gapless regime at LO. The quasi-c_t family studied here is, at leading order, without an effective gap term. If a residual linear term is retained, it must be pushed to a higher order; it therefore does not participate in the local mKdV closure at LO.

Quasi-c_t ansatz. We introduce

$$X = \varepsilon(s - c_t t), \qquad T = \varepsilon^3 t,$$

and the ansatz

$$A(s,t) = \varepsilon\, U(X,T) + \mathcal{O}(\varepsilon^3).$$

This scaling choice is the one that correctly balances, at the first nontrivial order, the convective modulation, the quartic dispersion, and the derived cubic nonlinearity.

Derivative hierarchy. One has

$$\partial_t = -\varepsilon c_t\, \partial_X + \varepsilon^3 \partial_T, \qquad \partial_s = \varepsilon\, \partial_X.$$

It follows that

$$A_{tt} - c_t^2 A_{ss} = -\,2\varepsilon^5 c_t\, U_{XT} + \mathcal{O}(\varepsilon^7),$$

whereas

$$(A^3)_{ss} = \varepsilon^5\, (U^3)_{XX} + \mathcal{O}(\varepsilon^7), \qquad A_{ssss} = \varepsilon^5\, U_{XXXX} + \mathcal{O}(\varepsilon^7).$$

Substituting into (8.1) and dividing by ε^5, one obtains at the first nontrivial order

$$-\,2c_t\, U_{XT} \;-\; c_t^2\, b_\beta\, (U^3)_{XX} \;+\; \delta\, U_{XXXX} = 0. \tag{8.3}$$

Local passage to the mKdV. For a localized wave, the integration constant is set to zero in the fast variable X. Integrating (8.3) once with respect to X, one obtains

$$-\,2c_t\, U_T \;-\; c_t^2\, b_\beta\, (U^3)_X \;+\; \delta\, U_{XXX} = 0.$$

Since

$$(U^3)_X = 3U^2 U_X,$$

one finally obtains

$$U_T + \tilde{\alpha}\, U^2 U_X + \kappa\, U_{XXX} = 0, \tag{8.4}$$

with

$$\tilde{\alpha} = \frac{3}{2}\, c_t\, b_\beta, \qquad \kappa = -\,\frac{\delta}{2c_t}. \tag{8.5}$$

Closure of the coefficients. With $\delta = d_\delta c_t^2 \ell_0^2$, one has

$$\kappa = -\frac{d_\delta}{2}\, c_t\, \ell_0^2, \qquad \tilde{\alpha} = \frac{3}{2}\, c_t\, b_\beta.$$

These two coefficients are dimensionally consistent:

$$[\kappa] = \mathrm{m}^3\,\mathrm{s}^{-1}, \qquad [\tilde{\alpha}] = \mathrm{m}\,\mathrm{s}^{-1}.$$

If needed, a canonical rescaling of (X, T, U) puts (8.4) into normalized form without introducing any new constant.

Remark on a residual gap. If one retains a residual linear term

$$\Omega_{\mathrm{lin}}^2 A, \qquad \Omega_{\mathrm{lin}}^2 = \varepsilon^4 \sigma,$$

then the closure at the first corrected order is no longer a strict local mKdV, but a mKdV equipped with a nonlocal correction proportional to $\partial_X^{-1} U$. The local form (8.4) therefore corresponds to the strictly gapless regime at LO.

8.2 Envelope soliton and vector field

In the focusing regime

$$\frac{\tilde{\alpha}}{\kappa} > 0,$$

Eq. (8.4) admits the standard soliton

$$U(X,T) = U_0\, \mathrm{sech}\big(\Lambda_{\mathrm{sol}}[X - c_{\mathrm{sol}}\, T]\big), \qquad \Lambda_{\mathrm{sol}} = U_0 \sqrt{\frac{\tilde{\alpha}}{6\kappa}}, \qquad c_{\mathrm{sol}} = \frac{\tilde{\alpha} U_0^2}{6}. \tag{8.6}$$

Sign condition. In the conventions retained here, we impose

$$\delta > 0, \qquad \kappa = -\frac{\delta}{2c_t} < 0.$$

The existence of a real localized soliton of the form (8.6) then requires

$$\frac{\tilde{\alpha}}{\kappa} > 0,$$

hence necessarily

$$\tilde{\alpha} < 0 \qquad \Longleftrightarrow \qquad b_\beta < 0.$$

This inequality is the focusing condition of the quasi-c_t 2DOF branch in the present conventions.

The associated physical envelope is written

$$A_{\mathrm{env}}(s,t) = \varepsilon\, U\big(\varepsilon(s - c_t t), \varepsilon^3 t\big). \tag{8.7}$$

In other words, the physical speed of the crest is

$$c_t + \varepsilon^2 c_{\mathrm{sol}},$$

which confirms that the branch considered is indeed quasi-c_t.

Polarization phase and observed frequency. The tangent transverse field is described by

$$\vec{u}_\perp(s,t) = A_{\rm env}(s,t)\,[\cos\Phi(s,t)\,\vec{e}_1 + \sin\Phi(s,t)\,\vec{e}_2], \qquad \omega := \partial_t\Phi(s,t). \tag{8.8}$$

An affine gauge $\Phi = \Omega t + qs$ is equivalent to the comoving gauge

$$\Phi = (\Omega - vq)t + q(s - vt).$$

Only the physical frequency $\omega = \partial_t\Phi$ matters. In the quasi-c_t bosonic branch, one chooses the advected gauge $\omega = \mu := vq$ (transported texture), without effect on the physical state or on observables: only the derivatives of Φ enter (frequency ω, wave vector q).

8.3 Invariants: energy, spin amplitude, and spin number (quasi-c_t 2DOF)

Geometric framework and measure. Near the quasi-flat core, $\mathbb{S}^3$ is parameterized by the arc coordinate s and the transverse 2-spheres $S^2(s)$. The natural measure factorizes as

$$dS \;=\; dS_{S^2(s)}\,ds \;=\; \underbrace{4\pi R_0^2 \sin^2\!\Big(\frac{s}{R_0}\Big)}_{=:\,\mathcal{A}(s)}\,ds,$$

where $\mathcal{A}(s)$ is the area of $S^2(s)$ (at LO: $\mathcal{A}(s) \simeq 4\pi s^2$ if $s \ll R_0$).

Decomposition into helicities. We choose an orthonormal tangential basis $(\vec{e}_1, \vec{e}_2)$ on $S^2(s)$ and set

$$\vec{e}_\pm \;=\; \frac{1}{\sqrt{2}}(\vec{e}_1 \pm i\,\vec{e}_2), \qquad \vec{u}_\perp(s,t) \;=\; A_+(s,t)\,\vec{e}_+ \;+\; A_-(s,t)\,\vec{e}_-.$$

The amplitudes $A_\pm$ are dimensionless.

LO balances (energy and spin action). After fast averaging in the quasi-c_t branch, one obtains

$$E \;=\; \rho\,\omega^2 \int_{\mathbb{S}^3} (|A_+|^2 + |A_-|^2)\,dS, \qquad S_{\rm sgn} \;=\; \rho\,\omega \int_{\mathbb{S}^3} (|A_+|^2 - |A_-|^2)\,dS, \tag{8.9}$$

and the spin-action norm

$$|S| \;:=\; \rho\,\omega \int_{\mathbb{S}^3} (|A_+|^2 + |A_-|^2)\,dS. \tag{8.10}$$

By (8.9)–(8.10), one always has

$$\boxed{E/\omega = |S|}.$$

Universal angle–action law. In the common framework adopted for 2DOF solutions, the angle variable is the same and one imposes

$$\boxed{\mathcal{J} = \frac{E}{\omega} = h_{\rm eff}} \qquad \Longrightarrow \qquad \boxed{E = h_{\rm eff}\,\omega, \qquad |S| = h_{\rm eff}}.$$

Thus, $|S| = h_{\rm eff}$ whatever the helicity mixture; only $S_{\rm sgn}$ depends on the polarization ($S_{\rm sgn} = \pm h_{\rm eff}$ for a pure circular polarization, $S_{\rm sgn} = 0$ for a linear polarization).

Geometric spin number (Berry reading). By parameterizing the sphere of directions by (Θ, Φ) and the tangential basis $(\vec{e}_\Theta, \vec{e}_\Phi)$, a circular polarization is written

$$\vec{u}_+ \;=\; \frac{1}{\sqrt{2}}(\vec{e}_\Theta + i\,\vec{e}_\Phi).$$

The associated Berry connection and curvature are

$$\mathcal{A}_{\mathrm{B}} \;=\; i\,\vec{u}_+^{\,\dagger}\cdot d\vec{u}_+ \;=\; -\,(1-\cos\Theta)\,d\Phi, \qquad \mathcal{F}_{\mathrm{B}} \;=\; d\mathcal{A}_{\mathrm{B}} \;=\; -\sin\Theta\, d\Theta\wedge d\Phi.$$

For a closed loop C enclosing the solid angle $\Omega[C]$,

$$\Phi_{\mathrm{geom}}[C] \;=\; \oint_C \mathcal{A}_{\mathrm{B}} \;=\; \iint_{\Sigma(C)} \mathcal{F}_{\mathrm{B}} \;=\; -\Omega[C].$$

Reading of the geometric spin. The quasi-c_t 2DOF bosonic branch carries two independent tangential polarizations in quadrature, hence

$$N_{\mathrm{pol}} = 2.$$

At leading order, the fundamental winding is

$$w = 1.$$

The common geometric counting then gives

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = \frac{1\times 2}{2} = 1.$$

One therefore obtains for this family

$$\boxed{s_g = 1, \qquad |S| = h_{\mathrm{eff}}.}$$

The sign of the spin is carried by the helical polarization:

$$S_{\mathrm{sgn}} = +h_{\mathrm{eff}} \text{ for a pure “+” helicity}, \qquad S_{\mathrm{sgn}} = -h_{\mathrm{eff}} \text{ for a pure “−” helicity},$$

whereas a linear mixture can give

$$|S_{\mathrm{sgn}}| < h_{\mathrm{eff}}.$$

This value is consistent with the local reading from the Berry flux:

$$-\lim_{\delta\Omega\to 0}\frac{\delta\Phi_{\mathrm{geom}}}{\delta\Omega} \;=\; 1,$$

the sign depending on the orientation convention.

Action constant and global normalization on $\mathbb{S}^3$. In the quasi-c_t 2DOF branch, the universal law

$$\frac{E}{\omega} = |S| = h_{\mathrm{eff}}$$

exactly fixes the global normalization of the envelope. Indeed, since

$$|S| = \rho\,\omega\int_{\mathbb{S}^3} A_{\mathrm{env}}^2\, dS,$$

one obtains the closure condition

$$\boxed{h_{\mathrm{eff}} = \rho\,\omega\int_{\mathbb{S}^3} A_{\mathrm{env}}^2\, dS.} \tag{8.11}$$

At LO (thin tube / quasi-flat), this becomes

$$h_{\text{eff}} = \rho\,\omega \int_{\mathbb{R}} A_{\text{env}}^2(s,t)\,\mathcal{A}(s)\,ds, \qquad \mathcal{A}(s) = 4\pi R_0^2 \sin^2\Big(\frac{s}{R_0}\Big).$$

With

$$A_{\text{env}}(s,t) = \varepsilon\, U(X,T),$$

this relation directly fixes the normalization of U without introducing any additional external constant.

8.4 Core–exterior matching and frequency selection on $\mathbb{S}^3$

In the tail of the soliton (8.6), the interior log-derivative at the boundary $s = s_c$ is

$$\Lambda_{\text{int}}^{(\text{sel})} = \pm\,\frac{1}{\lambda_{\text{env}}}, \qquad \lambda_{\text{env}} = \frac{1}{\varepsilon\,\Lambda_{\text{sol}}} = \frac{1}{\varepsilon\,U_0}\sqrt{\frac{6\kappa}{\tilde{\alpha}}}.$$

EBK closure of the interior slope. The universal angle–action law imposes

$$\mathcal{J} = \frac{E}{\omega} = h_{\text{eff}}.$$

Reinjecting the EBK closure of the envelope, one obtains immediately

$$\boxed{\Lambda_{\text{int}}^{(\text{sel})} = \pm\,\frac{1}{\lambda_{\text{env}}} = \pm\,\frac{h_{\text{eff}}}{48\pi\,\rho\,\kappa\,R_0^2}\,\frac{\tilde{\alpha}}{\omega}} \qquad \text{(LO)}. \tag{8.12}$$

Thus, no free amplitude parameter enters the matching condition at leading order.

Outside ($s > s_c$), the mean envelope satisfies the zonal Helmholtz equation on $\mathbb{S}^3$:

$$\frac{1}{\sin^2\chi}\,\partial_\chi\big(\sin^2\chi\,\partial_\chi A\big) + \Omega^2 R_0^2\,A = 0, \qquad \chi = \frac{s}{R_0}, \qquad \Omega := \frac{\omega}{c_t}.$$

The globally regular solutions are proportional to the zonal Gegenbauer functions $C_n^{(1)}(\cos\chi)$, with the dispersion

$$n(n+2) = \Omega^2 R_0^2 = \Big(\frac{\omega R_0}{c_t}\Big)^2, \qquad n \in \mathbb{N}.$$

Zonal identity ($\lambda = 1$). On $\mathbb{S}^3$, one has

$$C_n^{(1)}(\cos\chi) = \frac{\sin\big((n+1)\chi\big)}{\sin\chi}, \tag{8.13}$$

hence, by differentiation, the exterior log-derivative at the boundary $\chi_c = s_c/R_0$:

$$\Lambda_{\text{ext}}(n; s_c) = \frac{1}{R_0}\,\frac{\partial_\chi A}{A}\Big|_{\chi_c} = \frac{1}{R_0}\Big[(n+1)\cot\big((n+1)\chi_c\big) - \cot\chi_c\Big].$$

The C^1 matching condition then discretizes the admissible values of ω:

$$\boxed{\Lambda_{\text{int}}^{(\text{sel})} = \Lambda_{\text{ext}}(n; s_c), \qquad n(n+2) = \Big(\frac{\omega R_0}{c_t}\Big)^2.} \tag{8.14}$$

Choice of s_c and LO robustness. The point s_c is chosen in the exponentially decaying zone of the envelope, where the interior log-derivative is constant at LO. The precise choice of the threshold $A(s_c)/A(0)$ introduces only subdominant corrections $\mathcal{O}(e^{-k})$; the frequency selection (8.14) is therefore robust at LO.

Thus, for a finite radius R_0, the spectrum is discrete, with a characteristic spacing of order c_t/R_0; it becomes quasi-continuous when $R_0 \to \infty$.

8.5 Amplitude–width and normalization

The soliton (8.6) imposes the relations

$$\lambda_{\mathrm{env}} := \frac{1}{\varepsilon\,\Lambda_{\mathrm{sol}}} = \frac{1}{\varepsilon\,U_0}\sqrt{\frac{6\kappa}{\tilde{\alpha}}}, \qquad A_0 := A_{\mathrm{env}}(0,0) = \varepsilon\,U_0, \qquad c_{\mathrm{sol}} = \frac{\tilde{\alpha}U_0^2}{6}.$$

It follows that the amplitude–width exchange, in physical variables s, is

$$A_0\,\lambda_{\mathrm{env}} = \sqrt{\frac{6\kappa}{\tilde{\alpha}}}, \qquad A_0^2\,\lambda_{\mathrm{env}} = \frac{6\kappa}{\tilde{\alpha}}\,\frac{1}{\lambda_{\mathrm{env}}}.$$

Normalization by the action (absorbed EBK). Unlike the bump/NSL case, there is no radial closure here imposing

$$A_0^2\lambda_{\mathrm{env}} = \text{constant}$$

at LO (branch $\alpha = \pi/2$, without radial bias). The global normalization is therefore performed through the universal angle–action law:

$$\mathcal{J} = \frac{E}{\omega} = h_{\mathrm{eff}}, \qquad E = \rho\,\omega^2 \int_{\mathbb{S}^3} A_{\mathrm{env}}^2\, dS.$$

At LO (thin tube, $\mathcal{A}(s) \simeq 4\pi R_0^2$ and $\int_{\mathbb{R}} \operatorname{sech}^2(y/\lambda_{\mathrm{env}})\, dy = 2\lambda_{\mathrm{env}}$), one obtains

$$E = 8\pi\,\rho\,R_0^2\,\omega^2\,A_0^2\,\lambda_{\mathrm{env}}.$$

Using the identity

$$A_0^2\,\lambda_{\mathrm{env}} = \frac{6\kappa}{\tilde{\alpha}}\,\frac{1}{\lambda_{\mathrm{env}}},$$

the angle–action law immediately gives the explicit closure

$$\boxed{\lambda_{\mathrm{env}}(\omega) = \frac{48\pi\,\rho\,\kappa\,R_0^2}{h_{\mathrm{eff}}}\,\frac{\omega}{\tilde{\alpha}}}, \qquad \boxed{\frac{1}{\lambda_{\mathrm{env}}(\omega)} = \frac{h_{\mathrm{eff}}}{48\pi\,\rho\,\kappa\,R_0^2}\,\frac{\tilde{\alpha}}{\omega}}. \tag{8.15}$$

No free constant remains: κ and $\tilde{\alpha}$ are entirely expressed in terms of HSM constants (cf. (8.4)).

8.6 Comparison with the bump: common h_{eff}

At LO, in all 2DOF branches considered, one has

$$\mathcal{J} = \frac{E}{\omega} = h_{\mathrm{eff}},$$

with the same volumetric measure on $\mathbb{S}^3$. In particular,

$$h_{\mathrm{eff}}^{\mathrm{bump}} = h_{\mathrm{eff}}^{\text{quasi-c}_\mathrm{t}} =: \; h_{\mathrm{eff}},$$

which unifies the effective action constant of the 2DOF branches considered on $\mathbb{S}^3$.

8.7 Interpretation

The quasi-c_t 2DOF solution realizes, in the HSM sense, a limiting-speed bosonic branch: geometric spin

$$s_g = 1,$$

two helicities $\pm$, and the absorbed spectral law

$$\boxed{E = h_{\text{eff}}\,\omega}.$$

The linear polarization corresponds to equal superpositions of the two helicities (and then satisfies $S_{\text{sgn}} = 0$, whereas $|S| = h_{\text{eff}}$).

8.8 Transverse mechanical dictionary of the quasi-c_t wave

Framework and polarization phase. In the quasi-flat core ($\lambda_{\text{env}} \ll R_0$), the transverse displacement is written as

$$\vec{u}_\perp(s,t) = A_{\text{env}}(s,t)\,\big[\cos\Phi(s,t)\,\vec{e}_1 + \sin\Phi(s,t)\,\vec{e}_2\big],$$

with

$$A_{\text{env}}(s,t) = \varepsilon\,U(X,T), \qquad X = \varepsilon(s - c_t t), \qquad T = \varepsilon^3 t, \tag{8.16}$$

where U satisfies the reduced mKdV (8.4).

The generalized polarization phase is taken in the form

$$\Phi(s,t) = \Omega\,t, \qquad \Omega = \begin{cases} \omega_c & \text{(precessional branch: internal precession)},\\ \mu & \text{(quasi-}c_t\text{ branch: transport frequency)}. \end{cases}$$

Normalized mechanical quantities. We introduce two normalized vector quantities, constructed only from the transverse displacement:

$$\boxed{\mathbf{P}_\perp := \sqrt{\rho}\,\partial_t\vec{u}_\perp, \qquad \mathbf{Q}_\perp := -\sqrt{\rho}\,(\vec{t}\times\partial_s\vec{u}_\perp), \qquad \vec{t} := \text{unit propagation vector.}} \tag{8.17}$$

These quantities are not introduced as coupling fields. They are only a normalized mechanical rewriting of the transverse kinematics.

For a local wave propagating at c_t, one obtains

$$\mathbf{P}_\perp \perp \mathbf{Q}_\perp \perp \vec{t}, \qquad |\mathbf{P}_\perp| = c_t\,|\mathbf{Q}_\perp| \quad (\text{LO wave relation: } \partial_t\vec{u}_\perp = -c_t\,\partial_s\vec{u}_\perp).$$

Mechanical energy density and flux. The fast-averaged mechanical energy density and the associated mechanical flux are written

$$\boxed{\mathcal{E}_\perp = \tfrac{1}{2}\Big(|\mathbf{P}_\perp|^2 + c_t^2|\mathbf{Q}_\perp|^2\Big), \qquad \mathbf{F}_\perp = c_t^2\,\mathbf{P}_\perp\times\mathbf{Q}_\perp, \qquad \partial_t\mathcal{E}_\perp + \nabla\cdot\mathbf{F}_\perp \simeq 0.}$$

This writing coincides with

$$\tfrac{1}{2}\rho\big(|\partial_t\vec{u}_\perp|^2 + c_t^2|\partial_s\vec{u}_\perp|^2\big)$$

and with the exact mechanical flux

$$-\,T\,(\partial_t\vec{u}_\perp\cdot\partial_s\vec{u}_\perp)\,\vec{t}, \qquad T = \rho c_t^2.$$

Kinematic identities at leading order. Neglecting the envelope corrections $\mathcal{O}(\varepsilon)$ and curvature corrections $\mathcal{O}(\lambda_{\mathrm{env}}/R_0)$, the mechanical pair $(\mathbf{P}_\perp, \mathbf{Q}_\perp)$ satisfies

$$\boxed{\begin{aligned} &\nabla\cdot\mathbf{P}_\perp \;\approx\; 0, && \nabla\cdot\mathbf{Q}_\perp \;\approx\; 0,\\ &\partial_t\mathbf{Q}_\perp + \nabla\times\mathbf{P}_\perp \;\approx\; 0, && \partial_t\mathbf{P}_\perp - c_t^2\nabla\times\mathbf{Q}_\perp \;\approx\; 0. \end{aligned}} \tag{8.18}$$

These identities are only the normalized vector form of the transverse wave equation at LO. They do not yet constitute an identification with coupling fields.

Spin and relevant frequency. With the absorbed action normalization, one has

$$\boxed{|S| = \frac{E_{\mathrm{tot}}}{\Omega}, \qquad \Omega = \omega_c \text{ (precessional)} \quad \text{or} \quad \Omega = \mu \text{ (quasi-}c_t\text{ branch).}}$$

In the quasi-c_t branch, the universal angle–action law gives

$$E_{\mathrm{tot}} = h_{\mathrm{eff}}\,\mu \qquad \Longrightarrow \qquad |S| = h_{\mathrm{eff}},$$

independently of the helicity mixture (the sign S_{sgn} depends on the polarization).

Precessional branch. The precessional branch describes a gapped regime of the same 2DOF family when an internal precession $\omega_c > 0$ is activated (breaking, boundary, medium). In the homogeneous isotropic vacuum, one has $\omega_c = 0$, and only the quasi-c_t branch is realized.

Summary

We obtain a quasi-c_t bosonic 2DOF branch, purely tangential at LO ($\alpha = \pi/2$), without a leading effective gap term. Its physical envelope is written

$$A_{\mathrm{env}}(s,t) = \varepsilon\, U(X,T), \qquad X = \varepsilon(s - c_t t), \qquad T = \varepsilon^3 t,$$

where U satisfies a local mKdV of the form

$$U_T + \tilde{\alpha}\, U^2 U_X + \kappa\, U_{XXX} = 0.$$

This branch carries two helicities $\pm$ and a geometric spin

$$s_g = 1.$$

With the universal angle–action law,

$$E = h_{\mathrm{eff}}\,\omega, \qquad |S| = \frac{E}{\omega} = h_{\mathrm{eff}},$$

independently of the helicity mixture.

The envelope soliton fixes a physical width λ_{env} which, after EBK normalization, is entirely closed by

$$\lambda_{\mathrm{env}}(\omega) = \frac{48\pi\,\rho\,\kappa\,R_0^2}{h_{\mathrm{eff}}}\,\frac{\omega}{\tilde{\alpha}}, \qquad \frac{1}{\lambda_{\mathrm{env}}(\omega)} = \frac{h_{\mathrm{eff}}}{48\pi\,\rho\,\kappa\,R_0^2}\,\frac{\tilde{\alpha}}{\omega}.$$

The interior slope of the matching is therefore closed without a free amplitude parameter, and the core–exterior matching on $\mathbb{S}^3$ discretizes the admissible frequencies, with a characteristic spacing of order c_t/R_0, quasi-continuous when $R_0 \to \infty$.

The normalized mechanical quantities

$$\mathbf{P}_\perp = \sqrt{\rho}\,\partial_t \vec{u}_\perp, \qquad \mathbf{Q}_\perp = -\sqrt{\rho}\,(\vec{t}\times \partial_s \vec{u}_\perp)$$

satisfy, at first order,

$$\mathbf{P}_\perp \perp \mathbf{Q}_\perp \perp \vec{t}, \qquad |\mathbf{P}_\perp| = c_t\,|\mathbf{Q}_\perp|.$$

They provide a normalized writing of the mechanical density and flux,

$$\mathcal{E}_\perp = \tfrac{1}{2}(|\mathbf{P}_\perp|^2 + c_t^2|\mathbf{Q}_\perp|^2), \qquad \mathbf{F}_\perp = c_t^2\,\mathbf{P}_\perp \times \mathbf{Q}_\perp,$$

in the local quasi-flat reduction.

The action constant h_{eff} remains common to the 2DOF branches considered (bump, NSL, quasi-c_t). A gapped precessional branch can appear only in the presence of an internal precession induced by breakings or boundary conditions; in the homogeneous isotropic vacuum at LO, only the quasi-c_t branch is realized.

Chapter 9

Conic 3DOF family on $\mathbb{S}^3$

9.1 General idea

We enrich the 2DOF solutions (planar circular motion in the tangent space) by introducing a third internal degree of freedom: a constant radial bias that makes the trajectory conic. Locally, the displacement is no longer confined to the tangent plane: it has a fixed radial component and a tangential component in uniform rotation. This scheme applies to all oscillating families (bump, NSL, quasi-c_t). At $\alpha = \frac{\pi}{2}$, one recovers exactly the 2DOF version.

Conic ansatz (quasi-flat core, common). In a local basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ and with $s = R_0\psi$, we set, for the localized envelope $A(s)$ and the central frequency ω,

$$\vec{u}(s,t) \;=\; A(s)\Big[\sin\alpha\big(\cos\omega t\,\vec{e}_\theta + \varsigma\,\sin\omega t\,\vec{e}_\phi\big) + \cos\alpha\,\vec{e}_r\Big], \qquad \varsigma = \pm 1, \quad \alpha \in [0, \tfrac{\pi}{2}]. \tag{9.1}$$

The conic radius is $A\sin\alpha$ (tangential) and the radial bias is $A\cos\alpha$ (constant in phase). Limiting cases: $\alpha = \frac{\pi}{2}$ (pure 2DOF, tangent plane); $\alpha = 0$ (degenerate, purely static radial case).

Profile ODE (core, tangential variable). At LO (thin tube $\lambda \ll R_0$), the time-averaged reduction of the local cubic KG-type dynamics acts on the oscillating amplitude, that is, on the tangential component

$$A_\parallel := A\,\sin\alpha.$$

One then obtains the same ODE as in 2DOF, with the same coefficients:

$$A_\parallel'' = \Omega^2\,A_\parallel - \beta_{\text{eff}}\,A_\parallel^3, \qquad \Omega = \omega/c_t, \quad \beta_{\text{eff}} > 0. \tag{9.2}$$

Here, β_{eff} denotes the local cubic coefficient of the core ODE of the 3DOF conic branch. It must not be confused with the dimensionless coefficient b_β that enters the quasi-c_t reduction, where the nonlinearity is carried by $(A^3)_{ss}$.

Canonical nondimensionalization (β_{eff} lock). Set $\xi = s/\lambda$, $A_\parallel(s) = A_{\parallel,0}\,U(\xi)$, with

$$\lambda = \frac{c_t}{\omega} = \frac{1}{\Omega}.$$

The core ODE then takes the canonical form

$$U'' = U - \hat{\beta}\,U^3, \qquad \hat{\beta} > 0.$$

In the original variables,

$$A_\parallel'' = \Omega^2 A_\parallel - \beta_{\text{eff}} A_\parallel^3,$$

with

$$\beta_{\text{eff}} = \frac{\hat{\beta}}{\lambda^2}.$$

Thus, β_{eff} has dimension λ^{-2}, whereas $\hat{\beta}$ is dimensionless. The canonical solution is

$$A_{\parallel}(s) = A_{\parallel,0}\,\operatorname{sech}(s/\lambda), \qquad \lambda = \frac{c_t}{\omega}, \qquad A_{\parallel,0} = \frac{\sqrt{2}\,\Omega}{\sqrt{\beta_{\text{eff}}}}.$$

Remark. If one writes the ODE on the total amplitude A, then

$$A_{\parallel} = A \sin\alpha$$

implies

$$A'' = \Omega^2 A - \beta_{\text{eff}} \sin^2\alpha\, A^3.$$

The form (9.2) avoids this factor by working directly with the physical variable excited by the oscillation.

9.2 Energy, effective action (3DOF conic)

We adopt

$$c_t^2 = \frac{T}{\rho}, \qquad s = R_0\psi, \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big).$$

The action constant h_{eff} is the same as in the other oscillating branches of the model, the conic dependence being carried by the projective factor

$$\sigma = \sin^2\alpha.$$

Proposition 9.1 (Universal conic 3DOF action, LO)**.** *For the conic ansatz* (9.1) *and at leading order (thin tube, $\lambda \ll R_0$), the time average over the fast phase gives*

$$\frac{E_0}{\omega} = h_{\text{eff}}\,\sigma + \frac{\Delta E_{\text{nl}}(\alpha)}{\omega}, \qquad \sigma = \sin^2\alpha,$$

where $\Delta E_{\text{nl}}(\alpha) = \mathcal{O}(A_0^2)$ collects the nonlinear corrections at first order in weak amplitude. In particular,

$$\boxed{h_{\text{eff}} = \frac{E_0}{\omega\,\sigma}}$$

at leading rank, and for $\alpha = \frac{\pi}{2}$ (2DOF case) one recovers

$$\frac{E_0}{\omega} = h_{\text{eff}}.$$

Sketch. We write the ODE on

$$A_{\parallel} = A \sin\alpha,$$

which is the oscillating amplitude actually excited. The time average of

$$|\partial_t \vec{u}|^2$$

over one fast period gives

$$\langle |\partial_t \vec{u}|^2 \rangle = \omega^2 A^2 \sin^2\alpha,$$

whence the dependence on $\sigma = \sin^2\alpha$ in E/ω. In the absorbed form, the possible spectral level is carried by $\omega(q)$, whereas the action law remains

$$\mathcal{J} = \frac{E_0}{\omega} = h_{\text{eff}}\,\sigma$$

at LO.

9.2.1 EBK justification (3DOF conic): same fast structure as 2DOF

Technical lemma (first integral on $A_\parallel$). Set

$$A_\parallel := A\sin\alpha, \qquad \sigma := \sin^2\alpha, \qquad c_t^2 = \frac{T}{\rho}, \qquad \gamma_{nl} := 2T\,\beta_{\text{eff}}.$$

The LO core ODE

$$A_\parallel'' = \Omega^2 A_\parallel - \beta_{\text{eff}} A_\parallel^3, \qquad \Omega = \omega/c_t, \qquad \beta_{\text{eff}} > 0,$$

is multiplied by $A_\parallel'$ and then integrated once. Imposing a zero integration constant for the localized profile gives the first integral

$$\frac{T}{2}\,(A_\parallel')^2 = \frac{\rho}{2}\,\omega^2 A_\parallel^2 - \frac{\gamma_{nl}}{4}\,A_\parallel^4.$$

Since

$$A_\parallel = A\sin\alpha, \qquad A_\parallel' = A'\sin\alpha,$$

this gives

$$T\,(A')^2\,\sigma = \rho\,\omega^2\,A^2\,\sigma - \frac{\gamma_{nl}}{2}\,A^4\,\sigma^2.$$

The coefficient β_{eff} is the local cubic coefficient of the core ODE defined in the canonical nondimensionalization 9.1. Its positivity is the focusing condition of the canonical sech profile.

The oscillating part of the mean energy can then be rewritten in the form

$$\frac{E_0}{\omega} = h_{\text{eff}}\,\sigma + \frac{\Delta E_{\text{nl}}(\alpha)}{\omega},$$

where $\Delta E_{\text{nl}}(\alpha)$ collects the residual quartic corrections, of size $\mathcal{O}(A_0^2)$, and vanishes in particular when $\sigma = 1$ (return to the pure 2DOF case). One therefore obtains, at leading rank,

$$\frac{E_0}{\omega} = h_{\text{eff}}\,\sigma.$$

□

Proposition 9.2 (Conic 3DOF EBK: same fast structure as 2DOF at LO)**.** *Under the core–sheath reduction (thin tube $\lambda \ll R_0$, weak slope), constant conic bias $\alpha \in (0, \frac{\pi}{2}]$, zero flux at the poles, and central regularity, the fast tangential cycle has the same topology and the same caustic count as in 2DOF. The Maslov index μ is therefore unchanged at LO, and the action law takes the form*

$$\mathcal{J} = \frac{E_0}{\omega} = h_{\text{eff}}\,\sigma, \qquad \sigma = \sin^2\alpha,$$

whereas the possible spectral level is carried by the projected frequency $\omega(q)$. The corrections remain bounded by

$$\mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big) + \mathcal{O}(e^{-k}).$$

Proof (LO scheme). (i) **Adiabatic reduction.** Separate the fast variable (tangential angle θ) and the slow one (envelope A); the fast action is

$$\mathcal{J}_{\text{fast}} = \frac{1}{2\pi}\oint p_\theta\,d\theta.$$

(ii) **Topology and caustics.** The conic bias creates neither a turning point nor a singularity: the fast orbit remains a smooth circle in the local tangent plane, hence an unchanged Maslov

count. (iii) **Geometric phase.** The Berry connection induced by A is of order $\mathcal{O}(\varepsilon^2)$, with $\varepsilon = \lambda/R_0$; at LO it does not alter the fast quantization structure. (iv) **Conic weight.** The fast mean kinetic term is

$$\langle |\partial_t \vec{u}|^2 \rangle = \omega^2 A^2 \sigma,$$

whence

$$\frac{E_0}{\omega} = h_{\text{eff}} \sigma.$$

□

9.2.2 Spectral selection of the 3DOF bump

Principle of spectral inheritance. We fix

$$u_* \in [1.7, 2.1], \qquad \varepsilon := \frac{\lambda}{R_0} \ll 1, \qquad y := (n+1)\, u_*\, \varepsilon.$$

In the 3DOF conic closure, the constant radial bias does not alter the internal structure selecting the levels q: it acts only through the projective factor

$$\sigma = \sin^2 \alpha$$

on the observed quantities.

Projected EBK. For a fixed conic branch, α is fixed, hence σ is independent of q. The projected wavelength is

$$\lambda_{\text{proj}} = \frac{\lambda}{\sigma}, \qquad \varepsilon_{\text{proj}} = \frac{\lambda_{\text{proj}}}{R_0} = \frac{\varepsilon}{\sigma}.$$

The projected EBK law

$$\varepsilon_{\text{obs}}(q) = \frac{\varepsilon_{\text{obs}}(1)}{q}$$

is then equivalent to the same internal law as in the 2DOF case:

$$\boxed{\varepsilon(q) = \frac{\varepsilon(1)}{q}.} \tag{9.3}$$

Thus, conicity does not modify the admissible set of levels q; it modifies only their conic projection.

Boundary matching. The unified boundary closure is therefore written, at LO,

$$y \cot y = 1 - u_* \tanh u_* + \Delta_{\text{cone}}(\alpha) + \mathcal{O}(\varepsilon^2), \qquad u_* \in [1.7, 2.1], \tag{9.4}$$

where $\Delta_{\text{cone}}(\alpha)$ denotes the pure shift due to the conic bias. This shift does not affect the internal EBK law itself, but only the boundary closure term.

Consequence. At leading rank, one therefore recovers exactly the same spectral selection structure as for the 2DOF bump; in particular,

$$\boxed{\mathcal{Q}_{\text{bump}}^{(3D)} = \mathcal{Q}_{\text{bump}}^{(2D)}.} \tag{9.5}$$

The properly conic effect is transferred to $\Delta_{\text{cone}}(\alpha)$, which belongs to the geometric closure of the cone and not to the EBK law itself.

9.2.3 LO invariance of the admissible-return filter for the 3DOF bump

The 3DOF bump introduces an additional conic degree of freedom. One must therefore verify that this conicity does not modify the objects entering the admissible-return chain, but acts only on the conic projection of the solution.

1. Conic ansatz and envelope norm

At LO, consider a conic ansatz with constant angle

$$\mathbf{u}_\alpha(s,t) = A(s)\Big[\sin\alpha\big(\cos\omega t\,\mathbf{e}_\theta + \varsigma\sin\omega t\,\mathbf{e}_\phi\big) + \cos\alpha\,\mathbf{e}_r\Big],$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ is an orthonormal triad, $\varsigma = \pm 1$, and $A(s)$ denotes the spatial envelope of the bump.

Since the three internal directions are orthonormal, one has

$$\big|\mathbf{u}_\alpha(s,t)\big|^2 = A(s)^2\Big[\sin^2\alpha\big(\cos^2\omega t + \sin^2\omega t\big) + \cos^2\alpha\Big] = A(s)^2.$$

Thus

$$\boxed{|\mathbf{u}_\alpha|^2 = A(s)^2, \qquad \partial_\alpha|\mathbf{u}_\alpha|^2 = 0.}$$

Therefore, at LO, conicity does not modify the scalar envelope density that feeds the exterior return: it only redistributes the same total amplitude between a circular tangential component and a constant radial component.

2. Invariance of the main isotropic return

In the admissible-return chain, the main return actually read at the boundary is described by

$$R_{\mathrm{pr}}(q), \qquad \Gamma_{\mathrm{ret}}(u;q), \qquad \delta\psi_c(q),$$

with

$$R_{\mathrm{pr}}(q) = \mathcal{C}_{\mathrm{ext}}(q)\int_{\mathbb{R}} T_0(u)\,\Gamma_{\mathrm{ret}}(u;q)\,du.$$

Now, at LO, the isotropic exterior problem is generated by the scalar source $A(s)^2$, which was just shown to be independent of α. It follows that the isotropic structure of the return does not depend on conicity:

$$\boxed{\partial_\alpha\Gamma_{\mathrm{ret}}(u;q) = 0, \qquad \partial_\alpha R_{\mathrm{pr}}(q) = 0.}$$

Likewise, the angular reading width of the core,

$$\delta\psi_c(q) \sim u_*\,\varepsilon(q),$$

is fixed by the internal spectral scale of the branch. In the corrected 3DOF spectral closure, however, the internal EBK law remains

$$\varepsilon(q) = \frac{\varepsilon(1)}{q},$$

independent of α at LO. Hence

$$\boxed{\partial_\alpha\delta\psi_c(q) = 0.}$$

Since the local form of the return

$$\Gamma_{\mathrm{ret}}(u;q) = 1 - \frac{\kappa_{\mathrm{ret}}(q)}{2}u^2 + \frac{\lambda_{\mathrm{ret}}(q)}{24}u^4 + \mathcal{O}(u^6)$$

is itself α-independent, one also obtains

$$\boxed{\partial_\alpha\kappa_{\mathrm{ret}}(q) = 0, \qquad \partial_\alpha\lambda_{\mathrm{ret}}(q) = 0.}$$

3. Consequence for the isotropic threshold q_2

In the admissible-return filter, the first nontrivial threshold q_2 depends on the main isotropic return actually read at the boundary. At LO, this threshold is carried by the isotropic block constructed from the quantities

$$r_2, \qquad r_1, \qquad \chi_{\mathrm{mesh}},$$

where r_2 and r_1 are obtained by projection of the isotropic return onto the boundary kernels, and χ_{mesh} represents the isotropic mesh residual.

Since $\Gamma_{\mathrm{ret}}(u;q)$ is independent of α, the isotropic projections are independent as well:

$$\boxed{\partial_\alpha r_2 = 0, \qquad \partial_\alpha r_1 = 0, \qquad \partial_\alpha\left(\frac{r_2}{r_1}\right) = 0.}$$

The isotropic mesh parameter χ_{mesh} also does not depend on conicity. Thus,

$$\boxed{\partial_\alpha q_2 = 0.}$$

One obtains therefore

$$\boxed{q_2^{(3D)} = q_2^{(2D)}.}$$

4. Consequence for the anisotropic threshold q_3

The second nontrivial threshold q_3 does not rely on a new return mechanism: it corresponds to the first weak anisotropic splitting of the already established main return. At first order one writes

$$\Gamma_{\mathrm{ret}}(u,\theta;q) = \Gamma_0(u;q) + \varepsilon_2\,\Gamma_2(u;q)\,P_2(\cos\theta) + \mathcal{O}(\varepsilon_2^2).$$

Here $S_{\mathrm{aniso}}(q)$ and $\Phi_{\mathrm{ret}}^{(2)}(q)$ denote the two anisotropic objects already introduced in the quadrupolar closure of the return filter: they are constructed only from the $P_2(\cos\theta)$ component of the main return. In particular,

$$\boxed{\partial_\alpha \Gamma_2(u;q) = 0, \qquad \partial_\alpha S_{\mathrm{aniso}}(q) = 0, \qquad \partial_\alpha \Phi_{\mathrm{ret}}^{(2)}(q) = 0.}$$

The onset condition for the second admissible orientation is therefore identical in 2DOF and 3DOF. Hence

$$\boxed{\partial_\alpha q_3 = 0, \qquad q_3^{(3D)} = q_3^{(2D)}.}$$

5. Conclusion

At LO, conicity modifies neither the main isotropic return nor its first weak anisotropic splitting. It therefore introduces no new selection mechanism in q, but acts only on the conic projection of the 3DOF solution.

Consequently,

$$\boxed{\mathcal{Q}_{\mathrm{bump}}^{(3D,\mathrm{LO})} = \mathcal{Q}_{\mathrm{bump}}^{(2D,\mathrm{LO})},}$$

and, in particular,

$$\boxed{q_1^{(3D)} = q_1^{(2D)}, \qquad q_2^{(3D)} = q_2^{(2D)}, \qquad q_3^{(3D)} = q_3^{(2D)}.}$$

This equality concerns the structural thresholds of the admissible-return filter. By itself, it does not imply an explicit numerical closure of the admissible set.

9.2.4 LO geometric selection of the 3DOF conic factor

Local decomposition. At isotropic LO, the local space of 3DOF internal directions decomposes into a radial conic direction and a tangent plane:

$$E_{\mathrm{loc}} = E_r \oplus E_{\parallel}, \qquad \dim E_r = 1, \qquad \dim E_{\parallel} = 2, \qquad \dim E_{\mathrm{loc}} = 3.$$

We denote by $P_{\parallel}$ the projector onto the tangential part actually carried by the fast conic branch. Its active geometric rank is

$$r_{\parallel} := \operatorname{tr} P_{\parallel}.$$

Lemma 9.3 (Isotropic weight of a directional projector)**.** *In the isotropic 3DOF LO regime, the mean quadratic weight of an orthogonal projector P of rank r is*

$$\eta(P) = \frac{\operatorname{tr} P}{\operatorname{tr} \mathbf{1}_3} = \frac{r}{3}. \tag{9.6}$$

Proof. At isotropic LO, the local quadratic form distinguishes no internal direction. It is therefore proportional to the Euclidean metric on E_{loc}. For a local orthonormal basis, the total trace is $\operatorname{tr} \mathbf{1}_3 = 3$. An orthogonal projector of rank r retains exactly r equally weighted directions. Its normalized weight is therefore

$$\eta(P) = \frac{r}{3}.$$

□

Lemma 9.4 (Identification of the conic factor)**.** *For a nondegenerate 3DOF conic branch, the factor*

$$\sigma := \sin^2 \alpha$$

is the tangential quadratic weight actually carried by the projector $P_{\parallel}$. Thus,

$$\sigma = \eta(P_{\parallel}) = \frac{r_{\parallel}}{3}. \tag{9.7}$$

Proof. The conic ansatz separates orthogonally a radial component and a fast tangential component. In local quadratic normalization,

$$\|u\|^2 = \cos^2 \alpha \, \|u_r\|^2 + \sin^2 \alpha \, \|u_{\parallel}\|^2,$$

the cross terms being zero by radial–tangential orthogonality and by averaging over the fast cycle. The tangential fraction of the quadratic energy is therefore

$$\frac{\|u_{\parallel}\|^2_{\mathrm{weighted}}}{\|u\|^2_{\mathrm{total}}} = \sin^2 \alpha = \sigma.$$

At isotropic LO, this same fraction is the weight of the active tangential projector. By Lemma 9.3, it is

$$\sigma = \eta(P_{\parallel}) = \frac{r_{\parallel}}{3}.$$

□

Proposition 9.5 (LO geometric selection of conic 3DOF branches)**.** *In a nondegenerate 3DOF conic branch, the only admissible LO values of the conic factor are*

$$\boxed{\sigma = \sin^2 \alpha \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.} \tag{9.8}$$

Proof. The active tangential projector $P_{\parallel}$ acts in the tangent plane $E_{\parallel}$, of dimension 2. Its rank therefore satisfies

$$r_{\parallel} \in \{0, 1, 2\}.$$

The case $r_{\parallel} = 0$ corresponds to a configuration without a fast tangential sector: it is purely radial and does not constitute an oscillating conic branch. Nondegenerate conic branches therefore impose

$$r_{\parallel} \neq 0.$$

The purely tangential case without radial component corresponds to the limit $\alpha = \pi/2$, that is, to the pure 2DOF regime. It does not belong to the selection of nondegenerate conic 3DOF branches. In the conic 3DOF sector, the active tangential rank can therefore be

$$r_{\parallel} = 1 \qquad \text{or} \qquad r_{\parallel} = 2.$$

Using (9.7), one obtains

$$\sigma = \frac{r_{\parallel}}{3} \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.$$

□

Distinction from spin counting. The rank $r_{\parallel}$ must not be confused with the number N_{pol} used in spin counting. The rank $r_{\parallel}$ measures the number of real tangential directions actually retained by the conic projector. The number N_{pol}, in turn, measures the number of independent fast polarizations entering the spin monodromy. These two counts may coincide in some branches, but they are not identified in general.

Consequence (universal 3DOF relations). Reinjecting (9.8) into the universal 3DOF relations gives the two admissible conic branches

$$\frac{E_0}{\omega_0} = h_{\mathrm{eff}}\, \sigma, \qquad \sigma \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.$$

When the spectral level q is made explicit, it is carried by the projected frequency $\omega(q)$ according to the convention adopted in the preceding sections.

9.3 3DOF procedure (LO)

This section fixes the common template used to treat 3DOF branches at LO. It does not replace the closures specific to each family, but makes explicit the sequence of universal steps: choice of the conic branch, application of the filters, determination of the projected frequency, spectral selection, and energy/spin balances. The family chapters (3DOF bump, 3DOF NSL, quasi-c_t 3DOF) are then particular instantiations of this template.

1. **Conic angle (universal).** Choose the active tangential rank $r_{\parallel} \in \{1, 2\}$ imposed by Prop. 9.5, which fixes
$$\sigma = \sin^2 \alpha = \frac{r_{\parallel}}{3} \in \left\{ \tfrac{1}{3},\ \tfrac{2}{3} \right\}.$$

2. **3DOF filters.** Apply the window, the cutoff, and the sign decision (Sec. 9.4) in order to determine the admissible bands and the set of admissible external indices.

3. **Base fast frequency ω_0 (family-specific).** Define ω_0 according to the family, while retaining
$$\omega_{\mathrm{proj}} = \sigma\, \omega_0.$$
In particular:

- bump / NSL: ω_0 is the central frequency of the fast cycle, local and independent of α;
- quasi-c_t: $\omega_0 = |\Omega_{\rm spin}| = |\omega_c - kv|$ (cf. Sec. 12.1).

4. **External matching on $\mathbb{S}^3$.** Solve the core–sheath C^1 matching equation at the relevant label of the family (for example, quasi-c_t: Eq. (12.3)) to obtain $\varepsilon = \lambda/R_0$ or the equivalent boundary scale $u = s_c/\lambda$.

5. **External dispersion.** Impose the external relation associated with the family under consideration. In the zonal case on $\mathbb{S}^3$, one writes directly
$$\ell(\ell+2) = \left(\frac{\omega_{\rm proj} R_0}{c_t}\right)^2,$$
or the geometrically equivalent relation specific to the channel studied. One then deduces
$$\omega_0 = \frac{\omega_{\rm proj}}{\sigma}.$$

6. **Local cutoff (admissibility).** Verify the boundary cutoff inequality in the physical variable (sech profile, bounded traction) at the relevant label of the family (for example, quasi-c_t: Prop. 12.5). If the cutoff is violated, the state is rejected.

7. **EBK spectral structure.** The internal spectral map is written
$$\omega(q) = q\,\omega_0, \qquad \omega_{\rm proj}(q) = \sigma\,\omega(q) = \sigma\,q\,\omega_0,$$
the fast structure remaining the same as in 2DOF (Prop. 9.2). The set $\mathcal{Q}$ is the one determined by the filters or by the closure specific to the family: for example, for the 3DOF bump, one inherits at LO the same admissible set as the 2DOF bump, whereas for the 3DOF NSL one obtains
$$\mathcal{Q}_{\rm NSL}^{(3D)} = \{1\}.$$

8. **Universal balances.** In pure theory,
$$\frac{E}{\omega_0} = h_{\rm eff}\,\sigma \qquad \Longleftrightarrow \qquad E = h_{\rm eff}\,\omega_{\rm proj},$$
and the geometric spin is then fixed by the family under consideration ($|S| = \frac{1}{2}h_{\rm eff}$ for the 3DOF bump branch, $|S| = h_{\rm eff}$ for the vectorial 3DOF branches). The masses and closure energies (*core + spectral*, then possible sheath corrections) are then used in the sections dedicated to each family.

9.4 3DOF spectral filters: window, cutoff, selection by bands

9.4.1 LO assumptions and notation

We work in the thin-tube regime
$$\lambda \ll R_0, \qquad \lambda = \frac{c_t}{\omega}, \qquad c_t^2 = \frac{T}{\rho},$$
with geodesic variable $s = R_0\psi$ and weight
$$J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big).$$
We assume a weak-slope core–sheath reduction and a constant conic bias
$$\alpha \in (0, \tfrac{\pi}{2}], \qquad \sigma := \sin^2\alpha.$$
Finally, denote
$$\varepsilon := \frac{\lambda}{R_0} \ll 1.$$

9.4.2 Window and cutoff (generic)

We introduce a selection window

$$u_* \in [1.7, 2.1] \qquad \text{(LO, robust)},$$

and a local cutoff Σ_{eff} arising from the core–sheath C^1 matching. For each band $n \in \mathbb{N}$, set

$$y = (n+1)\, u_*\, \varepsilon, \qquad h(y) = y \cot y. \tag{9.9}$$

The function h codes the tangential response of the window; Σ_{eff} fixes the local validity range of the cutoff.

9.4.3 Generic radial response and DtN

The zonal sheath (radial equation on $\mathbb{S}^3$) and the Dirichlet–to–Neumann (DtN) condition at the boundary induce, for any oscillating family $\mathcal{F}$, a radial response

$$\Gamma_{\mathcal{F}}(q) \in (0,1), \qquad q \in \mathbb{N} \setminus \{0\},$$

continuous and strictly increasing in q at LO, on the admissible domain considered, with slope bound

$$\Big|\frac{\partial_s A}{A}\Big|_{\text{boundary}} \le G, \qquad 0 < G < 1. \tag{9.10}$$

The exact law of $\Gamma_{\mathcal{F}}$ is family-specific and will be given in the corresponding closure. The present module does not need it explicitly.

9.4.4 Decision by bands (generic statements)

Lemma 9.6 (Properties of h)**.** *The function h is strictly decreasing on $(0, \frac{\pi}{2})$, verifies*

$$h(0^+) = 1, \qquad h(\tfrac{\pi}{2}) = 0,$$

and $h(y) < 0$ on $(\frac{\pi}{2}, \pi)$. In particular, an equality

$$h(y) = \Gamma_{\mathcal{F}}(q) \in (0,1)$$

is possible only if $y \in (0, \frac{\pi}{2})$.

Proposition 9.7 (Sign decision, uniqueness by band)**.** *Fix $n \in \mathbb{N}$ and $y = (n+1)u_*\varepsilon$. If $\Gamma_{\mathcal{F}}(q) \in (0,1)$ is continuous and strictly increasing in q on the admissible domain, then:*

1. *if there exists q such that $h(y) - \Gamma_{\mathcal{F}}(q)$ changes sign on the band, there is exactly one solution in this band;*
2. *if $y \notin (0, \frac{\pi}{2})$ (hence $h(y) \le 0$), the band is excluded since $\Gamma_{\mathcal{F}}(q) \in (0,1)$.*

Definition 9.8 (Admissible set (generic module))**.** For a family $\mathcal{F}$, define

$$\mathcal{Q}_{\mathcal{F}}^{(3D)} := \{q \in \mathbb{N} \setminus \{0\} : \ \exists n \in \mathbb{N} \text{ such that } h(y) - \Gamma_{\mathcal{F}}(q) \text{ changes sign on band } n\}.$$

Interface.

- This section is family-agnostic: it provides $\{u_*, \Sigma_{\text{eff}}, h(y)\}$ and the decision/uniqueness rule.
- In each closure chapter (3DOF bump, 3DOF NSL, ...), one instantiates $\Gamma_{\mathcal{F}}(q)$ via the sheath / DtN of the family, then computes $\mathcal{Q}_{\mathcal{F}}^{(3D)}$ by Prop. 9.7.
- No admissible set is announced here: $\mathcal{Q}_{\mathcal{F}}^{(3D)}$ is deduced by this procedure.

Error bounds (LO). All the conclusions above remain valid with controlled errors

$$\mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big) + \mathcal{O}(e^{-k}), \qquad k \geq 3,$$

arising from the thin tube (geometry of $\mathbb{S}^3$) and from the core–sheath reduction.

9.4.5 Family-by-family variants

3DOF bump. Unihelicity $\varsigma = \pm 1$, $|S| = \frac{1}{2} h_{\text{eff}}$, $S_{\text{sgn}} \neq 0$. Same profile ODE (9.2); the dependence on α is carried by the conic energy law and by the cutoff.

3DOF NSL. Multiplication of the field (9.1) by a slow modulation $\cos(\mu t)$ (with $\mu \ll \omega$). The core and the cutoff are unchanged; $|S| = h_{\text{eff}}$; $S_{\text{sgn}} = 0$ for the self-conjugate version (global linear polarization), nonzero otherwise.

Quasi-c_t 3DOF – equation equivalence with 2DOF. Start from the linear operator on $\mathbb{S}^3$

$$\partial_t^2 - c_t^2\, \Delta_{\mathbb{S}^3}, \qquad c_t^2 = \frac{T}{\rho},$$

and the conic ansatz (9.1) with constant bias α. In the quasi-c_t regime (linearized, $\beta_{\text{eff}} \to 0$ in the core), the oscillating part is the tangential component

$$\vec{u}_{\parallel}(s,t) = A(s) \sin\alpha \big(\cos\omega t\, \vec{e}_\theta + \varsigma\, \sin\omega t\, \vec{e}_\phi \big),$$

whereas the radial component $A \cos\alpha\, \vec{e}_r$ is static in time. Therefore, ∂_t^2 acts only on $\vec{u}_{\parallel}$, and the radial decomposition leads exactly to the same zonal equation as for quasi-c_t 2DOF:

$$\frac{1}{\sin^2\psi}\, \partial_\psi\big(\sin^2\psi\, \partial_\psi A_{\text{ext}} \big) + (\Omega R_0)^2\, A_{\text{ext}} = 0, \qquad \Omega = \frac{\omega}{c_t}. \tag{9.11}$$

One therefore obtains the same dispersion on $\mathbb{S}^3$,

$$k = \frac{\omega}{c_t}, \qquad \omega = c_t k, \qquad kR_0 = \sqrt{\ell(\ell+2)}, \qquad \ell \in \mathbb{N}. \tag{9.12}$$

with the same C^1 boundary matching scheme.

Linearized core. In the core, the ODE on the oscillating variable

$$A_{\parallel} := A \sin\alpha$$

(see (9.2)) reduces, in linear quasi-c_t, to

$$A_{\parallel}'' = \Omega^2\, A_{\parallel} \qquad (\beta_{\text{eff}} \to 0),$$

that is, to the same form as in quasi-c_t 2DOF.

Conclusion. The constant radial bias alters neither the linear PDE governing the oscillating component, nor the zonal equation, nor the dispersion (9.11)–(9.12). It enters only through the energy / action kinematics via the factor

$$\sigma = \sin^2\alpha$$

(cf. §9.2).

Chapter 10

Closure of the 3DOF bump

10.1 LO assumptions and scope

(H1) Thin tube:

$$\lambda \ll R_0, \qquad \lambda = \frac{c_t}{\omega}, \qquad c_t^2 = \frac{T}{\rho}.$$

(H2) Constant conic bias:

$$\alpha \in (0, \tfrac{\pi}{2}], \qquad \sigma := \sin^2 \alpha \in \Big\{\tfrac{1}{3}, \tfrac{2}{3}\Big\}.$$

(H3) Core–sheath reduction: central regularity, zero flux at the poles, and C^1 matching at the boundary s_c.

(H4) Weak slope, scale separation, conic ansatz (9.1).

(H5) Geometric weight:

$$s = R_0\psi, \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big).$$

10.2 Profile, local ODE, and boundary conditions

Core ODE (oscillating variable). With

$$A_\parallel := A \sin\alpha,$$

the LO core ODE is identical to that of the 2DOF case:

$$A_\parallel'' = \Omega^2 A_\parallel - \beta_{\text{eff}} A_\parallel^3, \qquad \Omega = \omega/c_t, \qquad \beta_{\text{eff}} > 0. \tag{10.1}$$

The canonical solution is

$$A_\parallel = A_{\parallel,0}\ \operatorname{sech}(s/\lambda), \qquad \lambda = \frac{c_t}{\omega}, \qquad A_{\parallel,0} = \frac{\sqrt{2}\,\Omega}{\sqrt{\beta_{\text{eff}}}}.$$

Central conditions and boundary DtN condition. The adapted parity and the normalization at the center are imposed. At the boundary $s = s_c$, the DtN (Dirichlet–to–Neumann) condition bounds

$$\Big|\frac{\partial_s A}{A}\Big|_{s_c} \le G, \qquad 0 < G < 1,$$

where G comes from the zonal sheath. This bound is sufficient to close the gradient/amplitude ratio at the boundary, without any additional free parameter.

10.3 Conic quantization (universal reminder)

In the conic 3DOF family, the projective factor is

$$\boxed{\sigma = \sin^2 \alpha \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.}$$

Branch choice (bump). For the 3DOF bump, the two admissible conic branches inherited from the LO geometric selection are retained, cf. Prop. 9.5:

$$\sigma_\downarrow = \frac{1}{3}, \qquad \sigma_\uparrow = \frac{2}{3}.$$

These two values label the two conic realizations retained in the 3DOF bump closure. Their detailed interpretation in terms of spin projection and helicity will be given in the spin section specific to the family.

10.4 Conic action law: inheritance from the 3DOF module

The EBK justification of the conic 3DOF fast structure has already been established at the universal level; the 3DOF bump inherits it without modification at LO, cf. Prop. 9.2. The constant conic bias alters neither the topology of the fast tangential cycle nor the Maslov count; it enters only through the projective factor

$$\sigma = \sin^2 \alpha.$$

Thus, for the 3DOF bump,

$$\boxed{\mathcal{J} = \frac{E_0}{\omega} = h_{\text{eff}}\, \sigma, \qquad \sigma \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.} \tag{10.2}$$

The corrections remain bounded by

$$\mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big) + \mathcal{O}(e^{-k}).$$

The specific character of the 3DOF bump is therefore not in the action law itself, but in the spin class, the admissible spectral structure, and the channel closures treated in the following sections.

10.5 Spin of the 3DOF bump

The 3DOF bump branch inherits the fast spinorial structure of the bump family, whereas conicity introduces only a projective bias carried by

$$\sigma = \sin^2 \alpha.$$

The spin modulus therefore remains fixed by the geometric count of the branch, whereas the sign of the projection is carried by the preparation helicity. Since the cone already provides a privileged geometric axis, the spin projection has here an intrinsic meaning, unlike in the case of the free bump where it had to be reconstructed via the natural cycle.

Proposition 10.1 (Geometric spin of the 3DOF bump (modulus and sign))**.** *At leading order, the 3DOF bump branch has a single independent tangential circular polarization,*

$$N_{\mathrm{pol}} = 1, \qquad w = 1,$$

hence

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = \frac{1}{2}.$$

Its spin modulus is then

$$\boxed{|S_{\mathrm{bump\,3DOF}}| = \frac{1}{2}\, h_{\mathrm{eff}}.}$$

The projection of the spin onto the conic axis is given by

$$\boxed{S_{\parallel} = \varsigma\, \frac{1}{2}\, h_{\mathrm{eff}}, \qquad \varsigma = \pm 1,}$$

where ς is the preparation helicity of the branch.

Proof. The common geometric count gives

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = \frac{1 \times 1}{2} = \frac{1}{2}.$$

Since the fast structure of the 3DOF bump remains, at leading order, in the same spinorial class as the corresponding 2DOF bump branch, the spin modulus is

$$|S_{\mathrm{bump\,3DOF}}| = s_g\, h_{\mathrm{eff}} = \frac{1}{2}\, h_{\mathrm{eff}}.$$

The conic bias α does not affect this modulus: it modifies the observed energy projection through

$$\sigma = \sin^2 \alpha,$$

but not the geometric class of the fast cycle.

Here one must distinguish the independent polarization from the sign of helicity. The 3DOF bump has a single independent tangential circular polarization, hence $N_{\mathrm{pol}} = 1$. By contrast, this branch admits two opposite helical preparations

$$\varsigma = \pm 1,$$

which are not two additional polarizations, but the two possible signs of the projection onto the conic axis.

Finally, the cone axis provides an intrinsic projection direction; the two helicities $\varsigma = \pm 1$ therefore give the two projected values

$$S_{\parallel} = \pm \frac{1}{2}\, h_{\mathrm{eff}}.$$

□

Scope. The preceding result is specific to the 3DOF bump branch. It combines:

- the common geometric count, which here fixes

$$N_{\mathrm{pol}} = 1, \qquad w = 1, \qquad s_g = \frac{1}{2},$$

- and the conic structure specific to the family, which directly provides an internal projection axis through the cone axis.

It must therefore not be automatically transposed to the other 3DOF branches (NSL, quasi-c_t, *etc.*), for which the value of N_{pol}, the spin class, and the meaning of the projection must be established separately.

10.6 EBK quantization and determination of q

Principle. Here we use the generic module of 3DOF filters of §9.4. The 3DOF bump closure does not modify the spectral selection structure: the constant conic bias acts on the observed projection, but leaves the zonal sheath, the DtN condition, and the band condition unchanged at LO.

Proposition 10.2 (Equality of the q sets between 2DOF bump and 3DOF bump)**.** *Under thin tube ($\varepsilon \ll 1$), weak slope, and constant conic bias, the band condition*

$$h(y) = \Gamma_{\text{bump}}(q), \qquad y = (n+1)\, u_*\, \varepsilon, \qquad h(y) = y \cot y,$$

is identical in 2DOF and 3DOF at LO. In particular,

$$\boxed{\mathcal{Q}_{\text{bump}}^{(3D)} = \mathcal{Q}_{\text{bump}}^{(2D)}.} \tag{10.3}$$

10.7 LO inheritance of the transverse closure of the 2DOF bump

Principle. At leading order, the transverse closure of the 3DOF bump is not a new problem: it is inherited from the transverse closure of the 2DOF bump, because the fast core dynamics is carried only by the oscillating tangential component. The constant conic radial bias does not modify this core closure; it enters only at the level of the projected outputs through the factor

$$\sigma = \sin^2 \alpha.$$

Proposition 10.3 (LO inheritance of the transverse closure)**.** *Under the LO assumptions of the 3DOF bump (thin tube, weak slope, constant conic bias, regular core–sheath reduction), the transverse closure of the 3DOF bump branch coincides with that of the 2DOF bump applied to the oscillating variable*

$$A_{\parallel} := A \sin \alpha.$$

In particular, the fundamental transverse width λ_* is unchanged at leading order:

$$\boxed{\lambda_{*,\,\text{bump3D}} = \lambda_{*,\,\text{bump2D}}} \qquad (LO). \tag{10.4}$$

Proof. We start from the conic ansatz

$$\vec{u}(s,t) = A(s)\Big[\sin\alpha\big(\cos\omega t\, \vec{e}_\theta + \varsigma\, \sin\omega t\, \vec{e}_\phi\big) + \cos\alpha\, \vec{e}_r\Big], \qquad \varsigma = \pm 1.$$

We then introduce the oscillating component

$$A_{\parallel}(s) := A(s) \sin\alpha.$$

(i) The radial component does not take part in the fast cycle. The component

$$A(s) \cos\alpha\, \vec{e}_r$$

is static in time at leading order. The fast part of the cycle is therefore entirely carried by the circular tangential component. In particular,

$$|\partial_t \vec{u}|^2 = \omega^2 A^2 \sin^2\alpha = \omega^2 A_{\parallel}^2.$$

Likewise,

$$A'_{\parallel} = \sin\alpha\, A', \qquad A_{\parallel}^m = \sin^m\alpha\, A^m.$$

(ii) The core ODE is the same as in 2DOF. The effectively excited variable satisfies at LO

$$A_{\parallel}'' = \Omega^2 A_{\parallel} - \beta_{\mathrm{eff}} A_{\parallel}^3, \qquad \Omega = \omega/c_t,$$

that is, exactly the ODE of the 2DOF bump. The canonical reduced profile, the pure core moments, and the cutoff/matching prescription associated with this variable are therefore the same as in 2DOF.

(iii) The transverse closure is unchanged. Since the transverse closure of the 2DOF bump depends only on the core problem carried by the oscillating variable, it applies here without modification to $A_{\parallel}$. The fundamental transverse width λ_* of the 3DOF bump therefore coincides with that of the 2DOF bump at leading order, proving (10.4).

(iv) Where does conicity enter? It does not enter the transverse closure itself, but only the projection of the energy and mass quantities. Once λ_* is fixed, the fundamental fast frequency remains

$$\omega_{0,*} = \frac{c_t}{\lambda_*},$$

whereas the 3DOF spectral outputs carry the projective factor

$$\sigma = \sin^2 \alpha.$$

□

Practical consequence. The transverse closure of the 3DOF bump is therefore imported directly from the 2DOF bump branch. If this closure is written in the form

$$\boxed{\lambda_*^4 = \frac{2\,h_{\mathrm{eff}}}{\sqrt{\rho T}} \frac{1}{K_{\mathrm{bump}}\, C_{\mathrm{cut}}^{(\mathrm{bump})}}}, \tag{10.5}$$

then this same expression holds for the 3DOF bump at LO, without an additional conic factor in λ_*.

Spectral consequence. We then obtain the inherited spectral structure

$$\boxed{\omega_{0,f} = q\,\omega_{0,*}, \qquad \lambda_f = \frac{\lambda_*}{q}, \qquad q \in \mathcal{Q}_{\mathrm{bump}}^{(3D)}.} \tag{10.6}$$

The two admissible conic branches

$$\sigma_{\downarrow} = \frac{1}{3}, \qquad \sigma_{\uparrow} = \frac{2}{3}$$

enter only at the level of the projected spectral outputs:

$$E_{f,X}^{\mathrm{spec}} = h_{\mathrm{eff}}\, q\, \omega_{0,*}\, \sigma_X, \qquad m_{f,X}^{\mathrm{spec}} = \frac{h_{\mathrm{eff}}}{c_t^2}\, q\, \omega_{0,*}\, \sigma_X, \qquad X \in \{\downarrow, \uparrow\}.$$

Scope. Channel effects, inter-family compression, and internal coupling s belong to the later composite closures.

10.8 Base scale of the 3DOF bump

Core constant and base scale. Since the transverse closure of the 3DOF bump has been fixed at leading order by the fundamental width λ_* (Prop. 10.3), we introduce the core constant $\kappa_{\mathrm{core}}^{(3D)}$, fixed by the core–sheath closure retained for the 3DOF bump, and define the base scales

$$\boxed{E_0^{(\mathrm{base})} = \kappa_{\mathrm{core}}^{(3D)}\, T\, \lambda_*^2 R_0, \qquad S_0 = \kappa_{\mathrm{core}}^{(3D)}\, \rho\, \lambda_*^2 R_0.} \tag{10.7}$$

The quantity S_0 serves as the reference mechanical scale in the absolute mass formulas, whereas $E_0^{(\mathrm{base})}$ is its energy counterpart.

Kinematic compatibility. Since $c_t^2 = T/\rho$, one immediately has

$$E_0^{(\mathrm{base})} = c_t^2\, S_0.$$

Thus, the two base scales are not independent: they describe the same core closure, expressed either as energy or as mechanical mass.

10.9 Complete energy budget (LO) — 3DOF bump

LO decomposition. The LO budget separates a base contribution and a spectral contribution:

$$E_{\mathrm{tot}}^{(\mathrm{bump3D})}(q,\alpha) = E_0^{(\mathrm{base})} + h_{\mathrm{eff}}\, q\, \omega_{0,*}\, \sigma, \qquad \sigma = \sin^2\alpha, \qquad q \in \mathcal{Q}_{\mathrm{bump}}^{(3D)},$$

where

$$\omega_{0,*} = \frac{c_t}{\lambda_*}.$$

The remaining corrections are of order

$$\mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big) + \mathcal{O}(e^{-k}).$$

Spectral part. At conic LO, the spectral part satisfies

$$E_{\mathrm{spec}}^{(q,X)} = h_{\mathrm{eff}}\, q\, \omega_{0,*}\, \sigma_X, \qquad \sigma_X \in \Big\{\frac{1}{3}, \frac{2}{3}\Big\}.$$

10.10 Kinematic compatibility in motion

Reduced massive core. The 3DOF bump is a localized massive branch: it has a comoving frame, a finite proper energy, and zero longitudinal momentum in that frame. Its core dynamics is that of the conic branch inherited from the 3DOF module: the conic bias α does not modify the local hyperbolic structure of the fast cycle, but only the energy projection

$$\sigma = \sin^2\alpha.$$

The reduced longitudinal dynamics therefore belongs to the framework of §7.1. The boost law (7.9) applies to the reduced massive core, with the branch characteristic speed

$$c_{\mathcal{F}} = c_t.$$

Thus,

$$E(u) = \Gamma_t E_0, \qquad P(u) = \Gamma_t \frac{u}{c_t^2} E_0, \qquad \Gamma_t = \frac{1}{\sqrt{1 - u^2/c_t^2}}.$$

Radial compatibility. The 3DOF bump includes a core–sheath reduction in the sense of the assumptions 10.1. The core profile is localized,

$$A_{\parallel}(s) = A_{\parallel,0}\,\operatorname{sech}(s/\lambda),$$

and the radial sources produced by the quadratic or bilinear core terms are therefore localized in the comoving coordinate.

The radial sheath operator is the common operator of criterion (7.3). The conic pattern does not modify this operator; it only modifies the source $S_{\mathcal{F}}$ through the projective factors associated with σ. The strictly comoving sheath is therefore localized for

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

Above this radial threshold, the reduced tangential core keeps its branch kinematics, but the radial response is no longer a strictly comoving Yukawa sheath at the same order.

10.11 Isolated 3DOF bump: internal energy and internal source s

Purpose. Here we consider the isolated 3DOF bump branch, before the formation of a white compound. This branch has a proper internal energy and an associated internal source s. These quantities characterize the isolated solution; they are not a closure of a color–white compound.

The closure of a white compound is constructed later from the total white source, and not by adding isolated responses.

Spectral hierarchy. The transverse closure introduces a spectral hierarchy of families described by EBK integers q_f, with

$$f \in \{1,2,3\}, \qquad q_1 < q_2 < q_3.$$

The transverse width and proper frequency of family f are related to those of the reference family by

$$\boxed{\lambda_f = \frac{\lambda_*}{q_f}, \qquad \omega_{0,f} = q_f\,\omega_{0,*}.} \tag{10.8}$$

Conic branches. The two admissible conic branches are characterized by

$$\boxed{\sigma_\downarrow = \sin^2\alpha_\downarrow = \frac{1}{3}, \qquad \sigma_\uparrow = \sin^2\alpha_\uparrow = \frac{2}{3},} \tag{10.9}$$

and by the complements

$$\boxed{\rho_\downarrow = \cos^2\alpha_\downarrow = \frac{2}{3}, \qquad \rho_\uparrow = \cos^2\alpha_\uparrow = \frac{1}{3}.} \tag{10.10}$$

It follows that

$$\boxed{\tan^2\alpha_\downarrow = \frac{1}{2}, \qquad \tan^2\alpha_\uparrow = 2.} \tag{10.11}$$

Isolated spectral energy. The conic action law inherited from the 3DOF module gives, for the fast part,

$$E = h_{\text{eff}}\,\omega_{\text{br}}, \qquad \omega_{\text{br}} = q_f\,\omega_{0,*}\,\sigma_X.$$

The isolated spectral energy of the branch (f, X) is therefore

$$\boxed{E^{\text{spec}}_{f,X} = h_{\text{eff}}\,q_f\,\omega_{0,*}\,\sigma_X.} \tag{10.12}$$

Setting

$$E_* := h_{\text{eff}}\,\omega_{0,*},$$

we obtain the reduced form

$$\boxed{\frac{E^{\text{spec}}_{f,X}}{E_*} = q_f\sigma_X.} \tag{10.13}$$

Isolated internal source s. The 3DOF bump branch can carry an internal source s. We denote by $S_{f,X}$ the source associated with family f and conic branch X. With this isolated source we associate the internal quadratic response:

$$\boxed{E^{\text{iso}}_{\mathsf{s}}[S_{f,X}] = \frac{1}{2}\langle S_{f,X}, G_{\mathsf{s}}S_{f,X}\rangle.} \tag{10.14}$$

This energy is an internal response of the s channel to the isolated 3DOF bump source. It must not be confused with the energy of the white compound.

Arm amplitude transmitted to white compounds. The internal source s that will be consumed by white compounds is the arm amplitude

$$\boxed{Q^{\text{arm}}_{f,X} = q_f\,\sigma_X\,\chi_X, \qquad \chi_\downarrow = 1, \qquad \chi_\uparrow = \frac{2\pi}{3}.} \tag{10.15}$$

This expression defines, in the present chapter, the arm amplitude carried by the isolated branch. Its use in a total compound source belongs to the coupling sector.

The $\downarrow$ branch is taken as the compressive reference branch; its arm amplitude is normalized by

$$\chi_\downarrow = 1.$$

The $\uparrow$ branch carries a phase-circulating component. In a white compound with three equivalent matchings, color closure imposes that the total circulating phase around the tri-junction be an integer cycle:

$$\varphi_1 + \varphi_2 + \varphi_3 = 2\pi\nu, \qquad \nu \in \mathbb{Z}.$$

The elementary nontrivial branch corresponds to $\nu = 1$. By symmetry of the white tri-junction, the three matchings carry the same fraction of this holonomy:

$$\varphi_1 = \varphi_2 = \varphi_3.$$

It follows that

$$\varphi_i = \frac{2\pi}{3}.$$

The factor

$$\chi_\uparrow = \frac{2\pi}{3}$$

is therefore a structural holonomy of the white tri-junction, determined by the internal geometry of the junction.

Separation between isolated branch and white compound. The isolated branch therefore has two distinct objects:

$$E^{\mathrm{spec}}_{f,X}, \qquad E^{\mathrm{iso}}_{\mathsf{s}}[S_{f,X}].$$

The first is the spectral energy of the isolated solution. The second is the internal response of the s channel to the isolated source. When a white compound is formed, these isolated responses are not added: the closure of the compound must be constructed from a total compound source, and not by naively adding the isolated responses.

Thus, for a CB3, the amplitudes $Q^{\mathrm{arm}}_{f,X}$ feed the white matching defect and the collective junction modes, whereas the isolated self-responses remain internal branch quantities.

Chapter 11

3DOF NSL solution

11.1 LO assumptions and notation

(N1) Thin tube:

$$\lambda \ll R_0, \qquad \lambda = \frac{c_t}{\omega_0}, \qquad c_t^2 = \frac{T}{\rho}.$$

(N2) Constant conic bias:

$$\alpha \in (0, \tfrac{\pi}{2}], \qquad \sigma := \sin^2 \alpha \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.$$

(N3) Core–sheath reduction: central regularity, zero flux at the poles, and C^1 matching at the boundary s_c.

(N4) Small slope and scale separations; slow modulation of pulsation μ such that

$$0 < \mu \ll \omega_0.$$

(N5) Geometric weight on $\mathbb{S}^3$:

$$s = R_0\psi, \qquad dV = 4\pi R_0^2 \sin^2\psi \, ds.$$

Finally, we set

$$\varepsilon := \frac{\lambda}{R_0} \ll 1, \qquad \Omega := \frac{\omega_0}{c_t}.$$

11.2 NSL conic ansatz and core ODE

NSL ansatz (slow modulation).

$$\vec{u}_0(s,t) = A(s)\Big[\sin\alpha\big(\cos\omega_0 t\,\vec{e}_\theta + \varsigma\,\sin\omega_0 t\,\vec{e}_\phi\big) + \cos\alpha\,\vec{e}_r\Big], \qquad \varsigma = \pm 1, \tag{11.1}$$

$$\vec{u}(s,t) = \vec{u}_0(s,t)\,\cos(\mu t), \qquad 0 < \mu \ll \omega_0. \tag{11.2}$$

Fast/slow hierarchy. The slow modulation $\cos(\mu t)$ is not meant to redefine the fast structure of the core, but to introduce a temporal envelope separated from the carrier ω_0. Since $\mu \ll \omega_0$, time derivatives produce, in addition to terms of order ω_0^2, only corrections of order $\mu\omega_0$ and μ^2, hence subdominant at LO. The core closure is therefore governed, at first order, by the same fast oscillating component as for the 3DOF bump; the slow modulation enters only at the following orders and in the projected quantities associated with the NSL branch.

Core ODE (oscillating component). The fast dynamics acts on

$$A_{\parallel} := A \sin\alpha.$$

At LO,

$$A_{\parallel}'' = \Omega^2 A_{\parallel} - \beta_{\text{eff}} A_{\parallel}^3, \qquad \beta_{\text{eff}} > 0, \tag{11.3}$$

with canonical solution

$$A_{\parallel} = A_{\parallel,0}\, \operatorname{sech}(s/\lambda), \qquad \lambda = \frac{c_t}{\omega_0}, \qquad A_{\parallel,0} = \frac{\sqrt{2}\,\Omega}{\sqrt{\beta_{\text{eff}}}}.$$

The same local coefficient β_{eff} is used as in the conic core closure, with the nondimensionalization convention of 9.1. Its positivity ensures the focusing branch of the canonical profile.

The slow modulation (11.2) therefore does not alter the LO ODE: it modifies neither the core equation, nor the width λ, nor the canonical amplitude $A_{\parallel,0}$ at first order. Its effect appears only in the slow structure specific to the NSL branch and in later closures.

11.3 Zonal sheath, dispersion on $\mathbb{S}^3$, and MSE window

Outside the core, the envelope satisfies the zonal equation on $\mathbb{S}^3$:

$$-\frac{T}{R_0^2}\left(\sin^2\psi\, A'(\psi)\right)' + \frac{T}{R_0^2} S^2 A(\psi) = \rho\,\omega_0^2 \sin^2\psi\, A(\psi), \qquad S^2 := \ell(\ell+2). \tag{11.4}$$

Langer reduction. With

$$y(\psi) = \sqrt{\sin\psi}\, A(\psi),$$

one obtains

$$y''(\psi) + \left[\Omega^2 R_0^2 - U(\psi)\right] y(\psi) = 0, \qquad \Omega = \frac{\omega_0}{c_t}, \qquad U(\psi) = \frac{S^2 + \frac{1}{4}}{\sin^2\psi} - \frac{1}{4}. \tag{11.5}$$

Core window. The exact C^1 matching at the boundary $\psi_c = u^*\varepsilon$ gives

$$y \cot y = 1 - u^* \tanh u^*, \qquad y = (n+1)\,u^*\,\varepsilon, \qquad u^* \in [1.7, 2.1], \qquad n \in \mathbb{N}_0. \tag{11.6}$$

Boundary quantities.

$$k_*^2 = \Omega^2 R_0^2 - U(\psi_c), \qquad g_* = \frac{1}{\sin\psi_c} \simeq \frac{1}{u^*\,\varepsilon}, \qquad \widehat{\beta}_0 = \frac{(2k_*)^2}{(2k_*)^2 + g_*^2} \in (0,1). \tag{11.7}$$

Proposition 11.1 (LO value of the boundary constant $\widehat{\beta}_0$)**.** *In the thin window $\psi_c = u^*\varepsilon$ with $\varepsilon \ll 1$, one has*

$$g_* = \sin^{-1}\psi_c \simeq (u^*\varepsilon)^{-1} \gg 1 \qquad \textit{whereas} \qquad k_* = \mathcal{O}(1).$$

Consequently,

$$\widehat{\beta}_0 = \frac{(2k_*)^2}{(2k_*)^2 + g_*^2} = \mathcal{O}\big((u^*\varepsilon)^2\big) \xrightarrow[\varepsilon\to 0]{} 0.$$

At leading order,

$$\boxed{\widehat{\beta}_0 = 0},$$

with corrections $\mathcal{O}(\varepsilon^2)$.

11.4 EBK filter and determination of q (3DOF NSL)

Framework. We work at leading order: thin tube $\varepsilon \ll 1$, MSE window, C^1 matching at the boundary $\psi_c = u^*\varepsilon$, and sheath governed by (11.4)–(11.5).

Proposition 11.2 (Admissible states in the sheath, leading order)**.** *The zonal boundary problem after Langer reduction defines a Sturm–Liouville problem on*

$$I = [\psi_c, \pi - \psi_c]$$

whose first eigenmode y_0 is strictly positive on I and has no internal node. Every mode of order $m \geq 1$ has at least one node and is excluded by the MSE window at LO.

Sketch. In (11.5), the effective potential $U(\psi)$ is regular on I and dominates near the boundaries through the $\sin^{-2}\psi$ term. The matching conditions make the problem regular in the Sturm–Liouville sense. The first mode y_0 is node-free, whereas every higher-order mode has internal oscillations, incompatible with the hierarchy

$$\ell_0 \ll \lambda \ll R_0$$

and with the MSE window. □

Proposition 11.3 (EBK filter and value of q for 3DOF NSL)**.** *At leading order, the action per branch period satisfies the EBK rule*

$$\mathcal{J} = \frac{E_0}{\omega_{\mathrm{br}}} = h_{\mathrm{eff}}\, q, \qquad q \in \mathbb{N}.$$

Under the LO assumptions and the selection of the fundamental spatial mode (Proposition 11.2), the admissible state is necessarily

$$\boxed{q_{\mathrm{adm}}^{(\mathrm{NSL3DOF,\,LO})} = 1.}$$

Proof. The admissible spatial mode is the node-free fundamental mode, and the retained temporal state is the coherent fundamental state of the fast cycle. Any $q \geq 2$ would require either additional spatial oscillations or a multi-ring structure outside the LO scope. □

Practical consequence. Throughout the chapter, we take

$$q = 1, \qquad \omega_{\mathrm{br}} = \sigma\,\omega_0, \qquad \sigma = \sin^2\alpha.$$

11.5 EBK justification and universal action

The EBK rule at leading order reads, for the 3DOF NSL branch,

$$\mathcal{J} = \frac{E_0}{\omega_{\mathrm{br}}} = h_{\mathrm{eff}}\, q.$$

Since the spatial filter selects only the admissible fundamental mode (Proposition 11.3), here one has

$$q = 1.$$

It follows that

$$\boxed{E_0 = h_{\mathrm{eff}}\,\omega_{\mathrm{br}}, \qquad \omega_{\mathrm{br}} = \sigma\,\omega_0, \qquad \sigma = \sin^2\alpha \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.} \tag{11.8}$$

The 3DOF NSL branch thus inherits the same universal action law as the other oscillating branches, with conicity entering only through the projective factor σ.

11.6 LO closure of the mono-lobe 3DOF NSL core

Shape constants. We use the three-dimensional shape constants of the canonical profile $a(\xi) = \operatorname{sech}\xi$, with measure $4\pi\xi^2\,d\xi$:

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 \simeq 7.6339320582.$$

We define the pure combination

$$\boxed{K_{\mathrm{NSL}}^{(3D)} := (J_0 + J_2)J_0.} \tag{11.9}$$

This constant is a pure number arising from the local 3DOF NSL profile.

Closure denominator. For the isolated free branch at LO, the reduced closure denominator is

$$\boxed{D_{\mathrm{NSL3}}^{(\mathrm{LO})} = \kappa_r\, K_{\mathrm{NSL}}^{(3D)}.} \tag{11.10}$$

It is dimensionless.

Core width. The closed width of the 3DOF NSL core is defined by

$$\boxed{\lambda_*^4 = \frac{2\,h_{\mathrm{eff}}}{\sqrt{\rho\,T}\,D_{\mathrm{NSL3}}^{(\mathrm{LO})}}.} \tag{11.11}$$

This expression is dimensionally homogeneous:

$$[h_{\mathrm{eff}}] = ML^2T^{-1}, \qquad [\sqrt{\rho T}] = ML^{-2}T^{-1}, \qquad \left[\frac{h_{\mathrm{eff}}}{\sqrt{\rho T}}\right] = L^4.$$

Projected mass by conic branch. Since the 3DOF NSL branch satisfies $q = 1$ and

$$\omega_{\mathrm{br}} = \sigma_X\,\omega_{0,*}, \qquad \sigma_X \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\},$$

the projected mechanical mass associated with branch X is

$$\boxed{m_X^{(\mathrm{NSL3})} = \frac{h_{\mathrm{eff}}}{c_t\,\lambda_*}\,\sigma_X.} \tag{11.12}$$

Existence condition. The existence condition for the closure is

$$\boxed{D_{\mathrm{NSL3}}^{(\mathrm{LO})} > 0.} \tag{11.13}$$

It is satisfied as soon as

$$\kappa_r > 0, \qquad K_{\mathrm{NSL}}^{(3D)} > 0.$$

Status of the consumed constants. The 3DOF NSL closure consumes the radial gradient coefficient κ_r, as well as the mechanical quantities ρ and T. It does not consume the local radial coefficient

$$\kappa_\mu = \frac{\mu_2}{T}.$$

Thus, the 3DOF NSL branch depends on the effective separation of κ_r, ρ, T. Any numerical evaluation of this mass therefore presupposes that the status of these quantities has first been fixed in the retained local datum.

Remark. We used Proposition 11.1, which gives

$$\widehat{\beta}_0 = 0$$

at LO.

11.7 Kinematic compatibility in motion

Reduced massive core. The mono-lobe 3DOF NSL is a localized massive branch: the core has a closed width λ_*, a finite proper energy, and a comoving frame. The slow modulation $\cos(\mu t)$, with $\mu \ll \omega_0$, does not modify at LO the fast hyperbolic structure of the core established in (11.3). It enters only at the following orders in the slow dynamics of the branch.

The reduced longitudinal dynamics therefore falls under the framework of §7.1. The boost law (7.9) applies to the reduced massive core, with

$$c_{\mathcal{F}} = c_t.$$

One obtains

$$E(u) = \Gamma_t E_0, \qquad P(u) = \Gamma_t \frac{u}{c_t^2} E_0, \qquad \Gamma_t = \frac{1}{\sqrt{1 - u^2/c_t^2}}.$$

Radial compatibility. The 3DOF NSL conic ansatz contains a radial component carried by $\cos\alpha\,\vec{e}_r$, but the mono-lobe mass closure (11.12) does not consume the local radial coefficient κ_μ. The core closure depends on κ_r, ρ, T through the denominator (11.10) and the width (11.11).

The radial source associated with the branch is constructed from the localized core profile

$$A_\parallel(s) = A_{\parallel,0}\ \mathrm{sech}(s/\lambda).$$

It is therefore localized in the comoving coordinate. The general criterion (7.3) applies to the radial sheath response when it is considered explicitly:

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

In this regime, a strictly comoving sheath is localized. For $|u| \geq c_r$, the closure by a strictly comoving Yukawa sheath is no longer that of the linear rank under consideration.

11.8 Geometric spin (3DOF NSL)

Geometric class. In the ansatz (11.1), the two tangential components $(\vec{e}_\theta, \vec{e}_\phi)$ are in quadrature and realize the transverse structure of the NSL branch. We therefore retain

$$N_{\mathrm{pol}} = 2.$$

independent circular polarizations. At leading order, the transverse vector performs one complete rotation per period $2\pi/\omega_0$, hence

$$w = 1.$$

The common geometric counting then gives

$$s_g = \frac{w N_{\mathrm{pol}}}{2} = \frac{1 \times 2}{2} = 1.$$

The 3DOF NSL branch thus belongs to the integer class.

Spin norm. In this family, the spin modulus is

$$\boxed{|S_{\mathrm{NSL\,3DOF}}| = h_{\mathrm{eff}}, \qquad s_g = 1.}$$

The conic bias α does not affect this modulus: it enters only in the branch energy projection through

$$\sigma = \sin^2 \alpha, \qquad \omega_{\mathrm{br}} = \sigma\, \omega_0,$$

without modifying the geometric class of the fast cycle.

Projection and helicity. As for the 3DOF bump, the cone axis provides a privileged projection direction. The 3DOF NSL then carries two opposite helicities

$$\varsigma = \pm 1,$$

and the projection of the spin onto the conic axis is

$$\boxed{S_{\parallel} = \varsigma\, h_{\mathrm{eff}}, \qquad \varsigma = \pm 1.}$$

Scope. The 3DOF NSL ansatz considered here is purely transverse and circular around the conic axis. At leading order, it therefore naturally realizes the two helicities

$$\pm h_{\mathrm{eff}},$$

but does not make an additional longitudinal component of type $m = 0$ appear within this framework. The retained structure is therefore that of a transverse vector branch with two helicities, and not that of a complete triplet of projections.

Chapter 12

Quasi-c_t 3DOF soliton

12.1 Conic kinematics and unified phase on $\mathbb{S}^3$

$\mathbb{S}^3$ framework and conventions. We work on $\mathbb{S}^3$ of radius R_0, with the geodesic coordinate

$$s = R_0\psi \in (0, \pi R_0),$$

and the exact measure

$$dV = 4\pi\, R_0^3 \sin^2\psi\, d\psi = 4\pi R_0^2 \sin^2\Big(\frac{s}{R_0}\Big)\, ds.$$

The mass is volumetric, ρ, and

$$c_t^2 = \frac{T}{\rho}, \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big).$$

Status of the quasi-c_t envelope. In the quasi-flat core, the multiple-scale reduction near $v \simeq c_t$, inherited from the quasi-c_t 2DOF sector, leads to a slow mKdV-type envelope. We write

$$A_{\rm env}(s,t) = \varepsilon\, U(X,T), \qquad X = \varepsilon(s - vt), \qquad T = \varepsilon^3 t, \qquad v \simeq c_t.$$

In the present chapter, this envelope reduction is assumed established, and we focus on the 3DOF conic realization, the action closure, the matching on $\mathbb{S}^3$, and the spectral selection.

Unified phase and fast frequency. The tangential field is described by

$$\vec{u}_\perp(s,t) = A_{\rm env}(s,t)\big[\cos\Phi(s,t)\,\vec{e}_1 + \sin\Phi(s,t)\,\vec{e}_2\big],$$

with a single phase

$$\Phi(s,t) = \omega_c\, t + k\,(s - vt), \qquad [k] = \mathrm{m}^{-1}, \qquad v \simeq c_t. \tag{12.1}$$

The local angular velocity at fixed s is

$$\Omega_{\rm spin} = \dot{\Phi}|_s = \omega_c - kv.$$

For a 3DOF conic realization of angle α, we define

$$\omega_0 := |\Omega_{\rm spin}| = |\omega_c - kv|, \qquad \omega_{\rm proj} := \omega_0\,\sigma, \qquad \sigma = \sin^2\alpha \in \Big\{\tfrac{1}{3}, \tfrac{2}{3}\Big\}. \tag{12.2}$$

12.2 Balances and universality of h_{eff}

Proposition 12.1 (Universal 3DOF law at leading order)**.** *At leading order and for the EBK fundamental, the action law reads*

$$\boxed{\frac{E}{\omega_0} = h_{\mathrm{eff}}\,\sigma} \qquad \Longleftrightarrow \qquad \boxed{E = h_{\mathrm{eff}}\,\omega_{\mathrm{proj}}},$$

with

$$\omega_{\mathrm{proj}} = \sigma\,\omega_0, \qquad \sigma = \sin^2\alpha.$$

The projected spin number is then

$$S_{\mathrm{proj}} := \frac{E}{\omega_{\mathrm{proj}}} = h_{\mathrm{eff}},$$

independently of α*.*

Ring closure and universality of h_{eff}. The radial closure at the core boundary imposes a fixed envelope area,

$$A_0^2\,\lambda_{\mathrm{env}} = \mathcal{A}_*,$$

where $\mathcal{A}_*$ is determined by the retained ring closure. In the present chapter, we only need the following fact: this closure leads, at leading order, to the same effective action constant h_{eff} as in the other localized families. The quasi-c_t 3DOF branch therefore satisfies the same universal law

$$\frac{E}{\omega_0} = h_{\mathrm{eff}}\,\sigma \qquad \Longleftrightarrow \qquad E = h_{\mathrm{eff}}\,\omega_{\mathrm{proj}}.$$

12.3 Spin of the quasi-c_t 3DOF

Geometric class. The quasi-c_t 3DOF branch preserves two independent tangential polarizations in the transverse sector, whence

$$N_{\mathrm{pol}} = 2.$$

At leading order, the fast cycle remains fundamental:

$$w = 1.$$

The common geometric counting therefore gives

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = \frac{1\times 2}{2} = 1.$$

The quasi-c_t 3DOF branch thus belongs to the integer class.

Spin norm. The universal 3DOF law at leading order (§12.2) gives

$$E = h_{\mathrm{eff}}\,\omega_{\mathrm{proj}}, \qquad S_{\mathrm{proj}} = \frac{E}{\omega_{\mathrm{proj}}} = h_{\mathrm{eff}}.$$

Hence

$$\boxed{|S_{qct\,3DOF}| = h_{\mathrm{eff}}, \qquad s_g = 1.}$$

Projection and helicity. The cone axis provides a preferred projection direction. In the retained transverse ansatz, the quasi-c_t 3DOF branch carries two opposite helicities

$$\varsigma = \pm 1,$$

and the projection of the spin onto the conic axis reads

$$\boxed{S_{\parallel} = \varsigma\, h_{\mathrm{eff}}, \qquad \varsigma = \pm 1.}$$

Scope. The quasi-c_t 3DOF ansatz used here is purely transverse. At leading order, it therefore naturally realizes the two helicities

$$\pm h_{\mathrm{eff}},$$

but does not make an additional longitudinal component of type $m = 0$ appear within this framework. The retained structure is therefore that of a transverse integer branch with two helicities, not that of a complete triplet of projections.

Proposition 12.2 (Spin of the quasi-c_t 3DOF)**.** *At leading order, the quasi-c_t 3DOF branch satisfies*

$$N_{\mathrm{pol}} = 2, \qquad w = 1, \qquad s_g = 1,$$

and its spin projected onto the conic axis is

$$S_{\parallel} = \varsigma\, h_{\mathrm{eff}}, \qquad \varsigma = \pm 1.$$

Proof. The common geometric counting gives

$$s_g = \frac{w N_{\mathrm{pol}}}{2} = \frac{1 \times 2}{2} = 1.$$

The universal 3DOF law at leading order also gives

$$S_{\mathrm{proj}} = \frac{E}{\omega_{\mathrm{proj}}} = h_{\mathrm{eff}},$$

therefore

$$|S_{qct\,3DOF}| = h_{\mathrm{eff}}.$$

Finally, the cone axis provides an intrinsic projection direction; the two helicities $\varsigma = \pm 1$ then give the two projected values

$$S_{\parallel} = \pm h_{\mathrm{eff}}.$$

□

12.4 Matching on $\mathbb{S}^3$ and zonal spectral selection

Log–derivatives. In the core, with a sech profile, the internal log–derivative at the boundary s_c reads

$$\Lambda_{\mathrm{int}}(u) = -\frac{1}{\lambda_{\mathrm{env}}} \tanh u, \qquad u := \frac{s_c}{\lambda_{\mathrm{env}}}, \qquad \psi_c = \frac{s_c}{R_0} = u\, \varepsilon_{\mathrm{env}}.$$

Outside, the regular zonal solution of the hyperspherical equation is

$$A_{\mathrm{ext}} \propto C_n^{(1)}(\cos\psi),$$

and we introduce the projected wave number

$$\Omega_{\mathrm{proj}} := \frac{\omega_{\mathrm{proj}}}{c_t}.$$

The exterior part then satisfies

$$\frac{1}{\sin^2\psi}\,\partial_\psi(\sin^2\psi\,\partial_\psi A) + (\Omega_{\rm proj}R_0)^2\,A = 0, \qquad \Omega_{\rm proj}^2 R_0^2 = n(n+2).$$

The exterior log–derivative at ψ_c is

$$\Lambda_{\rm ext}(n;\psi_c) = \frac{1}{R_0}\Big[(n+1)\cot\big((n+1)\psi_c\big) - \cot\psi_c\Big].$$

Matching equation. The C^1 matching imposes, in dimensionless form,

$$\boxed{-\frac{1}{\varepsilon_{\rm env}}\tanh u = (n+1)\cot\big((n+1)\,u\,\varepsilon_{\rm env}\big) - \cot(u\,\varepsilon_{\rm env}), \qquad \Omega_{\rm proj}^2 R_0^2 = n(n+2).} \tag{12.3}$$

This condition directly discretizes $\Omega_{\rm proj}$, hence $\omega_{\rm proj} = c_t\Omega_{\rm proj}$, then ω_0 through

$$\omega_{\rm proj} = \sigma\,\omega_0.$$

12.5 EBK of the quasi-c_t 3DOF and selection of the fundamental

Proposition 12.3 (EBK of the quasi-c_t 3DOF)**.** *At leading order, the quasi-c_t 3DOF soliton defines a single fast phase cycle*

$$\Phi(s,t) = \omega_c t + k(s - vt), \qquad \omega_0 = |\omega_c - kv|,$$

carried by a localized envelope without multi-ring substructure in the class considered. The EBK quantization of the fast cycle then reads

$$\boxed{J = h_{\rm eff}\,\sigma, \qquad E = h_{\rm eff}\,\omega_{\rm proj}, \qquad \omega_{\rm proj} = \sigma\,\omega_0, \qquad \sigma = \sin^2\alpha.}$$

In particular,

$$E(q) = h_{\rm eff}\,\omega_{\rm proj}(q), \qquad \omega_{\rm proj}(q) = q\,\sigma\,\omega_0.$$

Sketch. The quasi-c_t 3DOF field has a single fast phase Φ, hence a single fundamental angular cycle. The constant conic bias does not alter the topology of the cycle, but only introduces the projective weight $\sigma = \sin^2\alpha$. One thus recovers the EBK structure of the 3DOF families:

$$J \propto q\,\sigma, \qquad \omega(q) = q\,\omega_0.$$

□

Proposition 12.4 (Internal fundamental and external zonal selection)**.** *In the quasi-c_t 3DOF class retained here, the localized envelope of the core is taken at the internal fundamental: no additional multi-ring substructure is introduced into the localized fast cycle. By contrast, the global spectral discretization is carried by the zonal matching on $\mathbb{S}^3$, hence by the external index n defined by*

$$\Omega_{\rm proj}^2 R_0^2 = n(n+2), \qquad \Omega_{\rm proj} := \frac{\omega_{\rm proj}}{c_t}.$$

For very large R_0, this zonal discretization becomes quasi-continuous.

Sketch. The retained quasi-c_t ansatz describes a single localized envelope, without additional internal excitation of the core. The only remaining quantization therefore comes from the exterior zonal mode matched on $\mathbb{S}^3$, parameterized by n. Since the spectral spacings are of order $1/R_0$, the regime $R_0 \gg \lambda_{\rm env}$ generates a quasi-continuum of observed frequencies. □

Practical consequence. Throughout this chapter, we keep the internal fundamental of the core and write

$$\omega_{\mathrm{proj}} = \sigma\,\omega_0, \qquad \Omega_{\mathrm{proj}} := \frac{\omega_{\mathrm{proj}}}{c_t}, \qquad \Omega_{\mathrm{proj}}^2 R_0^2 = n(n+2), \qquad E = h_{\mathrm{eff}}\,\omega_{\mathrm{proj}},$$

the effective selection of states then being carried by n, and not by an internal tower $q \geq 2$.

12.6 Local cutoff

Proposition 12.5 (Mechanical threshold at the boundary)**.** *At the boundary s_c, with*

$$A'(s_c) = \frac{A_0}{\lambda_{\mathrm{env}}}\,\operatorname{sech} u\,\tanh u, \qquad A_0 = \frac{\sqrt{2}\,\Omega_{\mathrm{proj}}}{\sqrt{\beta_{\mathrm{eff}}}}, \qquad \Omega_{\mathrm{proj}} := \frac{\omega_{\mathrm{proj}}}{c_t},$$

the mean tangential traction satisfies

$$\left\langle \sigma_{\tan} \right\rangle \simeq \frac{T}{2}\,\sin^2\alpha\,[A'(s_c)]^2 \leq \sigma_{\max} = \Sigma\,T, \qquad \Sigma \in (0,1].$$

We infer the explicit cutoff

$$\boxed{\Omega_{\mathrm{proj}}^4 \frac{\sin^2\alpha}{\beta_{\mathrm{eff}}}\,\operatorname{sech}^2 u\,\tanh^2 u \leq \Sigma, \qquad \beta_{\mathrm{eff}} > 0.} \tag{12.4}$$

The matching condition fixes Ω_{proj}, hence $\omega_{\mathrm{proj}} = c_t\Omega_{\mathrm{proj}}$, while the cutoff validates or rejects the state without shifting its frequency.

12.7 Quantization of the conic angle

The quasi-c_t 3DOF branch inherits the universal conic quantization of the 3DOF sector. We therefore retain

$$\boxed{\sin^2\alpha \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.} \tag{12.5}$$

These two values coincide with those obtained for the other 3DOF families and result from the same internal conic quantization.

12.8 Limiting cases and physical reading

Pure texture. If $\omega_c = 0$, then

$$\omega_0 = |k|\,v, \qquad \omega_{\mathrm{proj}} = |k|\,v\,\sin^2\alpha.$$

Pure precession. If $k = 0$, then

$$\omega_0 = |\omega_c|, \qquad \omega_{\mathrm{proj}} = |\omega_c|\,\sin^2\alpha.$$

General case. When $\omega_c \neq 0$ and $k \neq 0$, the relevant frequency for the energy and action is

$$\omega_{\mathrm{proj}} = \sigma\,\omega_0, \qquad \omega_0 = |\omega_c - kv|, \qquad \sigma = \sin^2\alpha \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.$$

It is this frequency that enters both

$$E = h_{\mathrm{eff}}\,\omega_{\mathrm{proj}}$$

and the spectral selection on $\mathbb{S}^3$ via (12.3).

Chapter 13

Static radial excitation: formulation on $\mathbb{S}^3$ and existence conditions

13.1 Scope and objective

We study the existence of a purely static radial deformation in the HSM model, with no tangential component, hence with no fast cycle and no spin, time independent and localized on $\mathbb{S}^3$. We work in a zonal/axisymmetric reduction and explicitly retain a nonlinearity in the radial internal energy.

The objective of the present chapter is twofold: (i) to state the exact radial equation on $\mathbb{S}^3$; (ii) to identify necessary conditions for the existence of a static radial pulse with finite effective support, as well as the global matching conditions.

Status. The pure radial sector does not belong to the conic 3DOF families: here there is neither EBK quantization, nor geometric spin. This chapter is a chapter of formulation and necessary existence conditions, and not a complete derived closure of the radial potential U_r.

The radial potential U_r is not closed in the present chapter: the aim here is only to establish the exact form of the static equation on $\mathbb{S}^3$ and the general constraints that a localized solution must satisfy.

13.2 Framework and general equation on $\mathbb{S}^3$

We parametrize the radial deformation by

$$R(s) = R_0 + r(s), \qquad s = R_0\psi \in [0, \pi R_0], \qquad J(s) = R_0^2 \sin^2\Big(\frac{s}{R_0}\Big).$$

We work in axisymmetry (zonal). We denote by κ_r the effective dimensionless radial coefficient and by ρ the mass density. The gradient stiffness of the radial sector is $T\kappa_r$, with

$$c_t^2 = \frac{T}{\rho}.$$

Thus, even in the static regime, T enters the radial equation through the gradient coefficient $T\kappa_r$.

In the absence of a tangential field ($A \equiv 0$), the complete radial equation is

$$\rho\, \ddot{r} - T\kappa_r \, \frac{1}{J}\, \partial_s(J\, \partial_s r) + U_r'(r) = 0. \tag{13.1}$$

Static regime. In the static regime ($\ddot{r} \equiv 0$), (13.1) becomes

$$-T\kappa_r \frac{1}{J}\,\partial_s(J\, r_s) + U_r'(r) = 0, \qquad s \in (0, \pi R_0), \tag{13.2}$$

with the natural regularity conditions at the poles:

$$J(s)\, r_s(s) \longrightarrow 0 \qquad \text{as} \qquad s \to 0,\ \pi R_0. \tag{13.3}$$

13.3 Conditional parametrization of the radial nonlinearity

We work with a conditional class of local radial potentials whose expansion near the core is written

$$U_r(r) = \frac{\mu_2}{2}\, r^2 + \frac{\mu_3}{3}\, r^3 + \frac{\mu_4}{4}\, r^4 + \cdots, \qquad \mu_2 > 0, \qquad \mu_4 > 0. \tag{13.4}$$

This writing allows a controlled nonconvexity, for example via $\mu_3 \neq 0$, while retaining stabilization at large amplitude.

The coefficients μ_3 and μ_4 are not derived here from the mesh. They parametrize the first admissible local radial nonlinearity for the static analysis. They are not consumed by the local Higgs-like NLO mass chain, which rests on the reduced scalar matching $\delta_H^*(u)$. The results of the present section must therefore be read as existence conditions on the class of potentials (13.4), and not as a complete microscopic closure of the static radial sector.

13.4 Quasi-planar core reduction and first integral

In a narrow core, of characteristic size $\lambda \ll R_0$, we locally approximate

$$J(s) \simeq J_0 = R_0^2,$$

so that the geometric measure is quasi-constant in the central window. We denote by $s_c > 0$ the core exit radius, that is, the point from which one switches to the exterior weak-amplitude regime.

In this approximation, equation (13.2) reduces to

$$-T\kappa_r\, r''(s) + U_r'(r) = 0, \qquad s \in (-s_c, s_c). \tag{13.5}$$

First integral. Multiplying (13.5) by $r'(s)$ and then integrating, one obtains

$$\frac{T\kappa_r}{2}\, r'(s)^2 - U_r\big(r(s)\big) = C. \tag{13.6}$$

Centered even pulse. We consider an even profile centered at $s = 0$, satisfying

$$r(0) = r_{\max} > 0, \qquad r'(0) = 0,$$

and assumed to join at the core edge a weak-amplitude regime matched to the exterior, with

$$r(s) \to 0, \qquad r'(s) \to 0 \qquad \text{as } s \to s_c^-.$$

Evaluation of (13.6) at the center gives

$$C = -U_r(r_{\max}),$$

whereas its evaluation at the core exit gives, in the weak-amplitude regime,

$$C = 0.$$

It follows that the necessary condition is

$$\boxed{U_r(r_{\max}) = 0 \qquad \text{with} \qquad r_{\max} > 0.} \tag{13.7}$$

Quartic polynomial case. For the truncated potential

$$U_r(r) = \frac{\mu_2}{2} r^2 + \frac{\mu_3}{3} r^3 + \frac{\mu_4}{4} r^4, \qquad \mu_2 > 0, \quad \mu_4 > 0,$$

the condition $U_r(r_{\max}) = 0$, with $r_{\max} > 0$, becomes

$$\frac{\mu_2}{2} + \frac{\mu_3}{3} r_{\max} + \frac{\mu_4}{4} r_{\max}^2 = 0.$$

A necessary condition for the existence of a positive root is therefore

$$\boxed{\mu_3 < 0, \qquad \frac{\mu_3^2}{9} - \frac{\mu_2 \mu_4}{2} \geq 0.} \tag{13.8}$$

In this case, the candidate roots are

$$r_{\max}^{(\pm)} = \frac{2}{\mu_4} \left(-\frac{\mu_3}{3} \pm \sqrt{\frac{\mu_3^2}{9} - \frac{\mu_2 \mu_4}{2}} \right).$$

Scope of the condition. Relation (13.7) is a local necessary existence condition in the quasi-planar core. It is not sufficient to ensure global existence on $\mathbb{S}^3$, because one must still satisfy: (i) the exterior matching, (ii) regularity at the poles, (iii) finite energy.

Comments. (i) A strictly convex potential

$$U_r(r) = \frac{\mu_2}{2} r^2$$

does not satisfy (13.7) for $r_{\max} > 0$, and therefore admits no nontrivial static pulse in this approximation.

(ii) A moderate nonconvexity, for example via $\mu_3 \neq 0$ or a stabilizing term of higher order, can make (13.7) achievable.

(iii) The curvature corrections due to the variation of $J(s)$ in the core are of order

$$\mathcal{O}\Big(\frac{\lambda^2}{R_0^2}\Big).$$

13.5 Spherical exterior (weak amplitude) and matching

Outside the core ($s > s_c$), the amplitude is weak and one linearizes

$$U_r'(r) \simeq \mu_2 \, r.$$

One then obtains the exterior zonal equation on $\mathbb{S}^3$:

$$-T\kappa_r \left(r_{ss} + \frac{2}{R_0} \cot\Big(\frac{s}{R_0}\Big) r_s \right) + \mu_2 \, r = 0. \tag{13.9}$$

Form in angular variable. Setting

$$\psi = \frac{s}{R_0},$$

the exterior equation becomes

$$-\kappa_r \, \frac{1}{R_0^2} \Big(r_{\psi\psi} + 2 \cot\psi \, r_\psi \Big) + \mu_2 \, r = 0.$$

If one then introduces

$$x = \cos\psi,$$

one obtains a modified Gegenbauer-type equation of the form

$$(1-x^2)\,r_{xx} - 3x\,r_x - \nu(\nu+2)\,r = 0, \qquad \nu(\nu+2) = \frac{\mu_2 R_0^2}{T\kappa_r} > 0. \tag{13.10}$$

The sign of the spectral term corresponds here to the evanescent static radial regime, and not to an oscillatory wave-type quantization on $\mathbb{S}^3$.

This writing must be read as the exterior zonal form associated with the static radial regime; the retained solution is the one that remains regular at the poles in the sense of (13.3).

C^1 matching at the core edge. At the point $s = s_c$, one imposes C^1 matching

$$r_{\mathrm{int}}(s_c) = r_{\mathrm{ext}}(s_c), \qquad r'_{\mathrm{int}}(s_c) = r'_{\mathrm{ext}}(s_c). \tag{13.11}$$

Since $J(s_c) \neq 0$, this writing is equivalent to the continuity of the flux Jr_s at the core edge.

The matching fixes the constants of the exterior solution from the interior profile. It does not determine *a priori* $r_{\max}$, which remains controlled by the internal property of the potential through (13.7).

13.6 Variational formulation

The static radial energy is written

$$\mathcal{E}_{\mathrm{rad}}[r] = 4\pi \int_0^{\pi R_0} \left\{ \frac{T\kappa_r}{2}\, r_s^2 + U_r(r) \right\} R_0^2 \sin^2\!\Big(\frac{s}{R_0}\Big)\, ds. \tag{13.12}$$

The associated Euler–Lagrange equation is precisely (13.2), with the natural zero-flux conditions (13.3) at the poles.

13.7 Summary of the necessary existence conditions

A static radial excitation localized on $\mathbb{S}^3$ must simultaneously satisfy:

(i) Local core condition. There must exist $r_{\max} > 0$ such that

$$U_r(r_{\max}) = 0,$$

a necessary condition arising from the first integral in the quasi-planar core.

(ii) Exterior condition. The exterior solution of (13.9) must be regular at the poles in the sense of (13.3).

(iii) Matching condition. The interior profile and the exterior solution must match in C^1 at the edge s_c according to (13.11).

(iv) Finite-energy condition. One must have

$$r \in H^1([0, \pi R_0]; J\,ds), \qquad \mathcal{E}_{\mathrm{rad}}[r] < \infty.$$

13.8 Conclusion

The existence of a static radial pulse on $\mathbb{S}^3$ requires an internal nonlinearity such that the effective potential U_r admits a nontrivial zero. The spherical equation (13.2), the regularity conditions at the poles (13.3) and the matching (13.11) thus define a well-posed global problem.

The present chapter establishes a chain of necessary existence conditions for the conditional class of radial potentials (13.4). Since the nonlinear coefficients μ_3 and μ_4 are not derived here from the mesh, the effective construction of an exact static radial profile, its explicit matching and its stability analysis do not constitute a complete microscopic closure of the static radial sector. This status does not concern the local Higgs-like NLO mass chain, which consumes neither μ_3 nor μ_4.

Chapter 14

Oscillating Higgs-like solution (scalar, $s = 0$) on $\mathbb{S}^3$

Status of the Higgs-like label. In this chapter, the expression *Higgs-like* is a morphological label internal to HSM. It denotes an oscillating radial scalar branch, without geometric spin ($s_g = 0$), and capable of producing a dimensionless multiplicative scalar correction. It is neither an observational identification with a Standard Model state nor an anchoring on an external mass.

14.1 Framework and conventions

We work on the 3-sphere of radius R_0, with arc coordinate

$$s = R_0\psi \in (0, \pi R_0), \qquad dV = 4\pi R_0^3 \sin^2\psi \, d\psi = 4\pi R_0^2 \sin^2\Big(\frac{s}{R_0}\Big)\, ds.$$

The radial displacement is $R(s,t) = R_0 + r(s,t)$ and the tangential field is zero,

$$A \equiv 0.$$

The mechanical constants are the mass density ρ, the tangential tension T, and the dimensionless radial weight κ_r, defined by

$$T_r = \kappa_r \, T.$$

We write

$$c_r^2 = \frac{T\kappa_r}{\rho}.$$

The present chapter treats an oscillating radial *Higgs-like* branch on $\mathbb{S}^3$. The expression *Higgs-like* denotes here an internal morphological label: scalar branch, without geometric spin ($s_g = 0$), carried by the radial sector, and associated with a dimensionless multiplicative correction.

The chapter closes the conditional correction

$$\delta_H^*(u),$$

as well as the associated factor

$$\mathcal{R}_H(u) := \sqrt{1 + \delta_H^*(u)}.$$

No external mass is introduced in the present chapter.

14.2 Linear radial equation on $\mathbb{S}^3$

At small amplitude, the radial zonal mode satisfies

$$\rho\,\ddot{r} - \frac{T\kappa_r}{R_0^2}\Big[\frac{1}{\sin^2\psi}\,\partial_\psi(\sin^2\psi\,\partial_\psi r)\Big] + \mu_2\, r = 0, \tag{14.1}$$

or, in arc coordinate s,

$$\rho\,\ddot{r} - T\kappa_r\,\frac{1}{J}\,\partial_s(J\,\partial_s r) + \mu_2\, r = 0, \qquad J(s) = R_0^2\sin^2\Big(\frac{s}{R_0}\Big). \tag{14.2}$$

In the quasi-planar core ($\lambda \ll R_0$, $J \simeq R_0^2$),

$$\rho\,\ddot{r} - T\kappa_r\, r_{ss} + \mu_2\, r = 0, \qquad c_r^2 = \frac{T\kappa_r}{\rho}. \tag{14.3}$$

14.3 Localized radial harmonic mode (core)

We seek a solution in the form

$$r(s,t) = f(s)\cos(\omega_r t), \qquad \omega_r > 0.$$

The spatial ODE is

$$T\kappa_r\, f'' - (\mu_2 - \rho\omega_r^2)\, f = 0. \tag{14.4}$$

For a localized mode, one must have

$$\mu_2 - \rho\omega_r^2 > 0,$$

and one obtains

$$f_{\text{in}}(s) = f_0\, e^{-|s|/\ell_r}, \qquad \ell_r^{-2} = \frac{\mu_2 - \rho\omega_r^2}{T\kappa_r}. \tag{14.5}$$

In other words, radial localization imposes the threshold

$$\boxed{\omega_r^2 < \frac{\mu_2}{\rho}.} \tag{14.6}$$

14.4 Spherical exterior and non-radiating matching

We denote by $s_c > 0$ the edge of the core. In the exterior, from (14.2), one obtains

$$f'' + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)f' - \ell_r^{-2} f = 0, \qquad \ell_r^{-2} = \frac{\mu_2 - \rho\omega_r^2}{T\kappa_r}. \tag{14.7}$$

The non-radiation condition selects the regular evanescent solution. In a matching zone where

$$1 \ll \frac{s - s_c}{\ell_r} \ll \frac{R_0}{\ell_r},$$

one has

$$f''_{\text{out}} - \ell_r^{-2} f_{\text{out}} = 0,$$

and therefore, at leading order,

$$\frac{f'_{\text{out}}}{f_{\text{out}}} = -\,\ell_r^{-1} \qquad \text{whereas} \qquad \frac{f'_{\text{in}}}{f_{\text{in}}} = -\,\ell_r^{-1}.$$

Thus, at leading order, the C^1 matching does not impose a strict quantization of ω_r:

$$\omega_r \text{ continuously parametrizes } \ell_r \quad \text{as long as} \quad \omega_r^2 < \frac{\mu_2}{\rho}, \tag{14.8}$$

while the global curvature effects remain of order

$$\mathcal{O}((\ell_r/R_0)^2).$$

Regularity at the poles imposes

$$\sin^2\Big(\frac{s}{R_0}\Big) f'(s) \to 0 \qquad \text{as} \qquad s \to 0, \pi R_0.$$

Diagnostic remark. The refinement of the exterior matching shows that, at the first useful order, the exterior log-derivative carries an additional geometric term of type $-1/s$ in the local quasi-Euclidean zone. This provides a natural matching penalty, but does not by itself create a canonical selection of u.

14.5 Proper energy and effective action (PDE)

The energy density associated with

$$r(s,t) = f(s)\cos(\omega_r t)$$

is

$$\mathcal{E} = \frac{\rho}{2}\,\dot r^{\,2} + \frac{T\kappa_r}{2}\,r_s^2 + \frac{\mu_2}{2}\,r^2.$$

With

$$dV = 4\pi R_0^2 \sin^2\Big(\frac{s}{R_0}\Big)\,ds,$$

the time average and the identity obtained by multiplying (14.4) by f and then integrating give

$$T\kappa_r \int f_s^2\,dV = (\mu_2 - \rho\omega_r^2)\int f^2\,dV.$$

Hence the proper energy is

$$E_0 = \frac{\mu_2}{2}\int_{\mathbb{S}^3} f(s)^2\,dV. \tag{14.9}$$

For $f_{\text{in}}(s) = f_0 e^{-|s|/\ell_r}$ and $\ell_r \ll R_0$,

$$\int_{\mathbb{S}^3} f^2\,dV \simeq 4\pi\int_0^\infty f_0^2 e^{-2s/\ell_r} s^2\,ds = \pi f_0^2 \ell_r^3,$$

whence

$$h_{\text{eff}}^{(H)} := \frac{E_0}{\omega_r} \simeq \frac{\pi}{2}\,\frac{\mu_2}{\omega_r}\,f_0^2\ell_r^3 = \frac{\pi}{2}\,f_0^2\,\frac{\mu_2\,(T\kappa_r)^{3/2}}{\omega_r\,(\mu_2 - \rho\omega_r^2)^{3/2}}. \tag{14.10}$$

Without an additional constraint, $h_{\text{eff}}^{(H)}$ depends on f_0 and on ω_r.

14.6 Constant $h_{\text{eff}}^{(H)}$ and common modal constraint on $\mathbb{S}^3$

Modal reduction. We set

$$r(s,t) = q(t)\,\phi(s)$$

and

$$-\frac{T\kappa_r}{R_0^2}\,\frac{1}{\sin^2\psi}\,\partial_\psi(\sin^2\psi\,\partial_\psi\phi) + \mu_2\,\phi = \rho\omega_r^2\,\phi, \qquad \psi = \frac{s}{R_0}.$$

The reduced Lagrangian is

$$L_{\text{red}}(q) = \frac{M}{2}\dot{q}^{\,2} - \frac{M\omega_r^2}{2}q^2, \qquad M := \int_{\mathbb{S}^3}\rho\,\phi^2\,dV.$$

For $q(t) = Q\cos(\omega_r t)$,

$$E_0 = \frac{M}{2}\omega_r^2 Q^2.$$

The modal invariant

$$J := \frac{E_0}{\omega_r}$$

is of the same nature as that of the other localized families. We then set the common canonical level

$$J_\star = h_{\text{eff}}$$

and impose

$$\boxed{h_{\text{eff}}^{(H)} = h_{\text{eff}}} \tag{14.11}$$

at the same excitation level and at the same order. In the core approximation,

$$f_0^2 = \frac{2h_{\text{eff}}\,\omega_r}{\pi\mu_2\ell_r^3}, \qquad E_0 = h_{\text{eff}}\omega_r. \tag{14.12}$$

14.7 Spectral selection and amplitude cutoff on $\mathbb{S}^3$

Spectral selection. At the edge s_c, equality of the log-derivatives at leading order shows that the MSE does not discretize ω_r at leading order:

$$\boxed{\text{no strict quantization of } \omega_r \text{ (localized scalar, one lobe).}}$$

The global geometry effects only induce shifts of order

$$\mathcal{O}((\ell_r/R_0)^2).$$

Local cutoff. At the edge s_c,

$$\frac{T\kappa_r}{2}f'(s_c)^2 + \frac{\mu_2}{2}f(s_c)^2 \le \sigma\,T\kappa_r, \qquad 0 < \sigma < 1,$$

and, with $f'(s_c) = -\ell_r^{-1}f(s_c)$,

$$\boxed{f(s_c)^2 \le \frac{2\sigma T\kappa_r}{T\kappa_r\ell_r^{-2} + \mu_2} = \frac{2\sigma T\kappa_r}{2\mu_2 - \rho\omega_r^2}.} \tag{14.13}$$

This cutoff bounds the amplitude without fixing ω_r.

14.8 LO scalar construction of the block $B(u)$

Status. The construction below establishes the closed LO form of the scalar block $B(u)$ in the scalar MSE scheme retained for the anchored radial channel. The profile $a(\xi) = \operatorname{sech}\xi$ is used as a computable representative of the class of localized profiles at LO. Theorem 14.3 then establishes that the scalar block thus obtained is independent of the precise profile up to $\mathcal{O}(\kappa^2)$.

Framework. We work with the canonical core kernel $a(\xi) = \operatorname{sech}\xi$ and the zonal heat-kernel window on $\mathbb{S}^3$,

$$W_\ell(u) = e^{-u\ell(\ell+2)}, \qquad u > 0.$$

The localized radial mode is treated at LO by a scalar channel reduction; the MSE therefore acts on the *canonical windowed profile* and not directly on the exact linear solution $f_{\text{in}}(s) = f_0 e^{-|s|/\ell_r}$.

Profile-compatibility remark. The exact radial core obtained from the local linear equation is exponential, $f_{\text{in}}(s) = f_0 e^{-|s|/\ell_r}$, whereas the LO scalar closure below uses the canonical profile $a(\xi) = \operatorname{sech}\xi$ to define the window operators. The present chapter therefore explicitly distinguishes: (i) the *exact local kinematics* of the radial mode, (ii) the *canonical LO spectral closure* of the scalar channel. The transition between these two levels is justified, at LO, by the scalar-universality theorem established below: the sech profile must be read as a canonical representative of the class of localized profiles, and not as a literal identification with the local exponential profile.

MSE condition and operators. At LO, applying the MSE window leads to the scalar equality

$$A_H(u)\,\mathcal{O}_2^{(S)\prime}(u) = A_{\tan}(u)\,\mathcal{O}_1^{(S)\prime}(u),$$

with the pure scalar operators

$$\mathcal{O}_1^{(S)}(u) = (u\ \operatorname{csch} u)^2, \qquad \mathcal{O}_2^{(S)}(u) = 2\,(u\coth u - 1)\ \operatorname{csch}^2 u. \tag{14.14}$$

Elimination and form of $B(u)$. By eliminating $A_{\tan}(u)$, one obtains the pure scalar increment

$$\boxed{B(u) = \mathcal{O}_2^{(S)}(u) - \frac{\mathcal{O}_2^{(S)\prime}(u)}{\mathcal{O}_1^{(S)\prime}(u)}\,\mathcal{O}_1^{(S)}(u).} \tag{14.15}$$

The scalar core block associated with $B(u)$ is

$$\boxed{\delta_{H,\text{core}}^{(\text{LO})}(u) = A_H(u)\,B(u).} \tag{14.16}$$

This quantity is not yet the complete increment consumed by the mass chain: the scalar boundary anchoring, then the minimal NLO refinement, are introduced in the following sections.

14.9 LO universality of the scalar closure

Purpose. The present paragraph does not seek to identify the exact profile of the Higgs core with the canonical form $a(\xi) = \operatorname{sech}\xi$. It targets a weaker but sufficient LO result: showing that the scalar block $B(u)$ is universal up to $\mathcal{O}(\kappa^2)$ for a broad class of localized profiles with the same scale

$$\kappa := \frac{\ell_r}{R_0} \ll 1.$$

Generic framework. We consider a generic localized profile of the form

$$f_\kappa(\chi) = F_0\, a\!\left(\frac{\chi}{\kappa}\right), \qquad \kappa = \frac{\ell_r}{R_0} \ll 1,$$

where a is even, positive, monotonically decreasing for $\xi > 0$, has a single localization scale, and has finite moments. An LO scalar quantity of the radial channel is then written generically in the form

$$\mathcal{I}_j(u;a) = \int_0^\pi K_j(u,\chi)\, f_\kappa(\chi)^2\, \sin^2\chi\, d\chi, \qquad j = 1,2,$$

where $K_j(u,\chi)$ is a scalar channel kernel.

Lemma 14.1 (local smoothness of the scalar heat-kernel kernels)**.** *We consider a zonal scalar channel kernel on $\mathbb{S}^3$ of the form*

$$K_j(u,\chi) = \sum_{\ell\geq 0} b_{\ell,j}(u)\,(\ell+1)\,Z_\ell(\chi), \qquad j = 1,2,$$

where

$$Z_\ell(\chi) = \frac{\sin((\ell+1)\chi)}{(\ell+1)\sin\chi}$$

is the scalar zonal function, and where the spectral coefficients satisfy, for fixed $u > 0$,

$$|b_{\ell,j}(u)| \leq C_u\,(1+\ell)^m e^{-u\ell(\ell+2)}$$

for some constant $C_u > 0$ and some integer $m \geq 0$.

Then $K_j(u,\chi)$ is C^∞ at $\chi = 0$, even in χ, and admits the local expansion

$$K_j(u,\chi) = K_j(u,0) + \mathcal{O}(\chi^2).$$

In particular,

$$\partial_\chi K_j(u,0) = 0.$$

Proof. For every ℓ, the zonal function $Z_\ell(\chi)$ is analytic and even near $\chi = 0$, with expansion

$$Z_\ell(\chi) = 1 - \frac{\ell(\ell+2)}{6}\chi^2 + \mathcal{O}(\chi^4).$$

Because the coefficients $b_{\ell,j}(u)$ decay exponentially in $\ell(\ell+2)$, they dominate any polynomial growth in ℓ. The Weierstrass criterion then allows one to sum the local expansion and its derivatives term by term. One obtains

$$K_j(u,\chi) = K_j(u,0) + \mathcal{O}(\chi^2),$$

and therefore $\partial_\chi K_j(u,0) = 0$. □ □

Lemma 14.2 (LO factorization of scalar observables)**.** *Under the preceding assumptions, one has*

$$\mathcal{I}_j(u;a) = M_0[a]\,\mathcal{K}_j(u) + \mathcal{O}(\kappa^2), \qquad j = 1,2,$$

with one and the same *profile factor*

$$M_0[a] = F_0^2\,\kappa^3 \int_0^\infty a(\xi)^2\,\xi^2\,d\xi,$$

and channel functions

$$\mathcal{K}_j(u) = K_j(u,0).$$

Proof. Set $\chi = \kappa\xi$. Then

$$f_\kappa(\chi)^2 = F_0^2\, a(\xi)^2, \qquad \sin^2(\kappa\xi) = \kappa^2\xi^2 + \mathcal{O}(\kappa^4\xi^4).$$

Thus one obtains

$$\mathcal{I}_j(u;a) = F_0^2\,\kappa^3 \int_0^{\pi/\kappa} K_j(u,\kappa\xi)\, a(\xi)^2\, \xi^2\, d\xi + \mathcal{O}(\kappa^5).$$

Lemma 14.1 gives

$$K_j(u,\kappa\xi) = K_j(u,0) + \mathcal{O}(\kappa^2\xi^2).$$

Consequently,

$$\mathcal{I}_j(u;a) = F_0^2\,\kappa^3\, K_j(u,0) \int_0^{\infty} a(\xi)^2\, \xi^2\, d\xi + \mathcal{O}(\kappa^5),$$

that is

$$\mathcal{I}_j(u;a) = M_0[a]\, \mathcal{K}_j(u) + \mathcal{O}(\kappa^2). \qquad \square$$

Theorem 14.3 (LO universality of the scalar block). Under the preceding assumptions, the scalar block

$$B(u) = \mathcal{O}_2^{(S)}(u) - \frac{\mathcal{O}_2^{(S)\prime}(u)}{\mathcal{O}_1^{(S)\prime}(u)}\, \mathcal{O}_1^{(S)}(u)$$

is universal at LO, that is, independent of the exact profile a up to $\mathcal{O}(\kappa^2)$.

Proof. Lemma 14.2 gives

$$\mathcal{I}_1(u;a) = M_0[a]\, \mathcal{K}_1(u) + \mathcal{O}(\kappa^2), \qquad \mathcal{I}_2(u;a) = M_0[a]\, \mathcal{K}_2(u) + \mathcal{O}(\kappa^2).$$

The same profile factor $M_0[a]$ therefore appears in the two scalar operators. During the channel elimination, this common factor cancels, so that the combination

$$\mathcal{O}_2^{(S)}(u) - \frac{\mathcal{O}_2^{(S)\prime}(u)}{\mathcal{O}_1^{(S)\prime}(u)}\, \mathcal{O}_1^{(S)}(u)$$

no longer depends on the exact profile at LO. $\square$ $\square$

Editorial consequence. The choice $a(\xi) = \operatorname{sech}\xi$ must therefore not be read here as the claim that the exact Higgs core is literally a sech profile, but as the choice of a *canonical representative* of the universal class of localized profiles at LO.

14.10 Scalar projector $A_H(u)$ and monopole case

Scalar zonal basis on $\mathbb{S}^3$. We use the zonal functions

$$Z_\ell(\chi) = \frac{\sin((\ell+1)\chi)}{(\ell+1)\sin\chi}, \qquad \Delta Z_\ell = -\ell(\ell+2) Z_\ell, \qquad \ell = 0,1,2,\ldots,$$

orthogonal for the measure $d\mu(\chi) = \sin^2\chi\, d\chi$. The heat-kernel windowing is $W_\ell(u) = e^{-u\ell(\ell+2)}$. We denote by $\hat{c}_\ell^{(S)}(u)$ the normalized zonal coefficients.

Weights and channel normalization. The addition theorem on $\mathbb{S}^3$ at a coincident point provides a degeneracy proportional to $(\ell + 1)^2$. In zonal projection ($m = 0$), one extracts a linear weight

$$\tilde{w}_\ell = \ell + 1.$$

The homogeneity of the scalar channel imposes a single factor J_0. Thus one obtains

$$\boxed{A_H(u) = \frac{J_0}{K_3} \sum_{\ell \geq 0} (\ell + 1)\, \hat{c}_\ell^{(S)}(u)^2, \qquad K_3 = \frac{J_2}{J_0}.} \tag{14.17}$$

Monopole case. When the spectral density is dominated by $\ell = 0$,

$$A_H(u) \xrightarrow[\hat{c}_0^2 \to 1]{} \frac{J_0}{K_3}\, \hat{c}_0^{(S)}(u)^2 = \Big(\frac{J_0^2}{J_2}\Big) \hat{c}_0^{(S)}(u)^2.$$

14.11 Monopole dominance in the quasi-planar core regime

Statement. If $\ell_r \ll R_0$, i.e. $\kappa := \ell_r / R_0 \ll 1$, then for the localized profile $f(s) = f_0 e^{-|s|/\ell_r}$ one has

$$\boxed{\hat{c}_0^{(S)}(u)^2 = 1 - \mathcal{O}(\kappa^2), \qquad \hat{c}_{\ell \geq 1}^{(S)}(u)^2 = \mathcal{O}(\kappa^2).} \tag{14.18}$$

Proof (sketch). The scalar zonal projection is

$$a_\ell = \int_0^\pi f(R_0 \chi)\, Z_\ell(\chi)\, \sin^2 \chi \, d\chi.$$

Since f is concentrated on $\chi \lesssim \kappa \ll 1$, we expand

$$Z_\ell(\chi) = 1 - \frac{\ell(\ell + 2)}{6} \chi^2 + \cdots .$$

After heat-kernel windowing and normalization, the non-monopole components are of order $\mathcal{O}(\kappa^2)$, which gives (14.18).

14.12 Scalar anchoring at the S^3 matching: block $\zeta(u)$ and corrected increment

Geometric origin. The exact exterior equation (14.7) contains the curvature term

$$\frac{2}{R_0} \cot \chi \, f', \qquad \chi = \frac{s}{R_0}.$$

At the C^1 matching, this term induces, after heat-kernel windowing and scalar zonal reduction, a pure anchoring block $\zeta(u)$.

Closed form. With the scalar operators

$$\mathcal{O}_1^{(S)}(u) = (u \operatorname{csch} u)^2, \qquad \mathcal{O}_2^{(S)}(u) = 2\,(u \coth u - 1) \operatorname{csch}^2 u,$$

the anchoring block is

$$\boxed{\zeta(u) = \tfrac{1}{2}\, u \coth u \operatorname{csch}^2 u.} \tag{14.19}$$

Increment with local anchoring. We define the local anchoring correction

$$\Sigma_{\mathrm{loc}}(u) = \zeta(u).$$

The scalar block with local anchoring is then

$$\boxed{\delta^{(\mathrm{LO})}_{H,\mathrm{anchor}}(u) = A_H(u)\,[B(u) + \Sigma_{\mathrm{loc}}(u)] = A_H(u)\,[B(u) + \zeta(u)].} \tag{14.20}$$

This expression is not yet the final increment δ^*_H of the minimal NLO mass chain. It provides the local block that will be completed by the boundary term Σ_{rim}.

Short proof of $\zeta(u)$. Near the boundary, set locally

$$f_{\mathrm{out}}(\chi) = \Phi(\chi) e^{-s/\ell_r},$$

with Φ smooth. The exterior log-derivative is

$$\frac{f'_{\mathrm{out}}}{f_{\mathrm{out}}} = -\frac{1}{\ell_r} + \frac{\Phi'}{\Phi}.$$

The curvature term in (14.7) then produces, after scalar reduction, an additive shift of log-derivative

$$\Delta^{(\mathrm{curv})}_{\mathrm{ext}}(u) = \frac{1}{2} u \coth u\ \mathrm{csch}^2 u.$$

Thus one obtains

$$\boxed{\zeta(u) = \tfrac{1}{2} u \coth u\ \mathrm{csch}^2 u.} \tag{14.21}$$

14.13 Passive radial core limit and minimal active core

Change of description level. The previous sections used the local passive core as the reference kinematic and energetic model. The present section reexamines the matching in the exact radial zonal form, where the geometric term $2/s$ of the radial Laplacian can no longer be neglected. It is at this description level that the no-go of the regular passive core is established and that the minimal active core is introduced.

No-go of the passive radial core. The purely linear and regular radial core, treated in the local zonal approximation

$$f'' + \frac{2}{s} f' - \ell_r^{-2} f = 0,$$

admits internally the regular solution

$$f^{(\mathrm{reg})}_{\mathrm{in}}(s) = C\,\frac{\sinh(s/\ell_r)}{s},$$

with log-derivative at the boundary $s_c = u\ell_r$

$$\Lambda_{\mathrm{in}}(u) = \frac{1}{\ell_r}\Big(\coth u - \frac{1}{u}\Big).$$

The local decaying exterior tail

$$f_{\mathrm{out}}(s) = D\,\frac{e^{-s/\ell_r}}{s}$$

carries the log-derivative

$$\Lambda_{\mathrm{out}}(u) = -\frac{1}{\ell_r}\Big(1 + \frac{1}{u}\Big).$$

The equality $\Lambda_{\mathrm{in}}(u) = \Lambda_{\mathrm{out}}(u)$ is impossible for $u > 0$, so that a strictly linear passive radial core cannot close the matching at finite radius.

Link with the matching diagnostic. The local geometric term $-1/s$, already pointed out in §14.4, is precisely the term that shifts the exterior log-derivative into

$$\Lambda_{\text{out}}(u) = -\frac{1}{\ell_r}\Big(1 + \frac{1}{u}\Big),$$

and therefore prevents the C^1 matching of a regular passive core at finite radius. The no-go of the passive core is therefore not independent of the matching diagnostic: it is its structural consequence.

Minimal active core. One then introduces an oscillatory active scalar core. In zonal coordinates $\chi = s/R_0$, the minimal regular interior solution is written

$$f_{\text{in}}(\chi) = A\,\frac{\sin(\beta\chi)}{\sin\chi},$$

with $\beta > 0$, whereas the exact exterior tail on $\mathbb{S}^3$ is

$$f_{\text{out}}(\chi) = C\,\frac{e^{-\nu\chi}}{\sin\chi}, \qquad \nu^2 = \frac{R_0^2}{\ell_r^2} - 1.$$

The C^1 matching at the boundary $\chi_c = s_c/R_0$ gives

$$\beta\cot(\beta\chi_c) = -\nu.$$

Setting

$$y := \beta\chi_c, \qquad \nu\chi_c = u\sqrt{1 - \frac{\ell_r^2}{R_0^2}}, \qquad \nu\chi_c \simeq u \quad (\ell_r \ll R_0),$$

one obtains the exact minimal matching law

$$\boxed{y\cot y = -u\sqrt{1 - \frac{\ell_r^2}{R_0^2}},} \tag{14.22}$$

which locally reduces to

$$\boxed{y\cot y \simeq -u.} \tag{14.23}$$

Lemma 14.4 (variational minimality of the active continuation)**.** *In the transformed variable*

$$g(\chi) := f(\chi)\sin\chi,$$

the passive exterior sector satisfies

$$g'' - \nu^2 g = 0, \qquad \nu^2 = \frac{R_0^2}{\ell_r^2} - 1,$$

whereas the active interior core belongs to the class

$$g'' + \beta^2 g = 0, \qquad \beta > 0.$$

The minimal active continuation is the element of this class that minimizes the quadratic spectral defect with respect to the continuation obtained by reversing the sign of the passive sector. It satisfies

$$\boxed{\beta^2 = \nu^2, \qquad \beta = \nu > 0.} \tag{14.24}$$

Proof. The passive exterior sector is governed by the reduced operator

$$\mathcal{L}_- := \partial_\chi^2 - \nu^2.$$

The active continuation associated with the same spectral modulus is the operator

$$\mathcal{L}_+^{(0)} := \partial_\chi^2 + \nu^2,$$

obtained by reversing the sign of the spectral term, without changing the modulus.

More generally, one allows an interior active continuation

$$\mathcal{L}_+(\beta) := \partial_\chi^2 + \beta^2.$$

Its reduced spectral defect with respect to the active continuation without a new scale is

$$\mathcal{D}(\beta) := \mathcal{L}_+(\beta) - \mathcal{L}_+^{(0)} = (\beta^2 - \nu^2)\,\mathbf{1}.$$

On a matching interval $0 \le \chi \le \chi_c$, the quadratic norm of this defect is proportional to

$$\mathfrak{M}(\beta) := \int_0^{\chi_c} \left(\beta^2 - \nu^2\right)^2 d\chi = \chi_c \left(\beta^2 - \nu^2\right)^2.$$

Since $\chi_c > 0$, the global minimum of $\mathfrak{M}$ on $\beta > 0$ is attained if and only if

$$\beta^2 - \nu^2 = 0.$$

Therefore

$$\beta^2 = \nu^2.$$

The condition $\beta > 0$ selects

$$\beta = \nu.$$

□

Consequence for the relation between y and u. By definition,

$$y = \beta\chi_c, \qquad \chi_c = \frac{s_c}{R_0}, \qquad u = \frac{s_c}{\ell_r}.$$

The lemma 14.4 then gives

$$y = \nu\chi_c.$$

Now

$$\nu = \sqrt{\frac{R_0^2}{\ell_r^2} - 1} = \frac{R_0}{\ell_r}\sqrt{1 - \frac{\ell_r^2}{R_0^2}},$$

hence

$$\boxed{y = u\sqrt{1 - \frac{\ell_r^2}{R_0^2}}.} \tag{14.25}$$

In the local limit $\ell_r \ll R_0$, one recovers

$$\boxed{y \simeq u.} \tag{14.26}$$

Variational selection. The interior parameter β is therefore no longer an independent degree of freedom: it is fixed by the minimization of the spectral defect $\mathfrak{M}(\beta)$ defined in the proof of lemma 14.4. One obtains

$$\beta = \nu = \sqrt{\frac{R_0^2}{\ell_r^2} - 1}.$$

The minimal active branch then corresponds to the first admissible solution of the closed matching law.

Closed matching law of the minimal active core. Reinjecting (14.25) into the exact matching law yields a closed equation for u:

$$\boxed{u\sqrt{1-\frac{\ell_r^2}{R_0^2}}\,\cot\left(u\sqrt{1-\frac{\ell_r^2}{R_0^2}}\right) = -\,u\sqrt{1-\frac{\ell_r^2}{R_0^2}}.} \tag{14.27}$$

At the local level, this reduces to

$$\boxed{u\cot u = -u.} \tag{14.28}$$

LO closure of the matching parameter. The law (14.27) closes the matching parameter of the Higgs-like channel. Setting

$$a_H := \sqrt{1-\frac{\ell_r^2}{R_0^2}}, \qquad y = a_H u,$$

equation (14.27) becomes

$$y\cot y = -y.$$

For $y > 0$, one obtains

$$\cot y = -1.$$

The first positive admissible branch gives

$$y_H^* = \frac{3\pi}{4}.$$

Thus,

$$\boxed{u_{H,\mathrm{LO}}^* = \frac{3\pi}{4}\left(1-\frac{\ell_r^2}{R_0^2}\right)^{-1/2}.} \tag{14.29}$$

In the thin-core regime,

$$u_{H,\mathrm{LO}}^* = \frac{3\pi}{4} + \mathcal{O}\left(\frac{\ell_r^2}{R_0^2}\right).$$

The parameter u is therefore not left free in the LO closure of the Higgs-like channel.

14.14 Minimal NLO refinement by inverse boundary compliance

The LO+anchoring calculation uses the pure scalar correction

$$\Sigma_{\mathrm{loc}}(u) = \zeta(u), \qquad \zeta(u) = \frac{1}{2}\,u\,\coth u\,\operatorname{csch}^2 u.$$

Here we introduce a minimal NLO refinement of the matching, based not on an additional boundary stiffness, but on a *cyclic reclosure compliance*.

Lemma 14.5 (oriented compliance average and quadratic control)**.** *For a harmonic oscillation at the boundary*

$$r_c(t) = f_c\cos(\omega t), \qquad f_c > 0,$$

the oriented absolute average is

$$\boxed{\left\langle\frac{|r_c(t)|}{f_c}\right\rangle_T = \frac{1}{T}\int_0^T \frac{|r_c(t)|}{f_c}\,dt = \frac{2}{\pi}, \qquad T = \frac{2\pi}{\omega}.} \tag{14.30}$$

The quadratic control associated with the same cycle is

$$\boxed{\left(\left\langle \frac{r_c(t)^2}{f_c^2} \right\rangle_T\right)^{1/2} = \frac{1}{\sqrt{2}}.} \tag{14.31}$$

Proof. Setting $\theta = \omega t$, one obtains

$$\left\langle \frac{|r_c(t)|}{f_c} \right\rangle_T = \frac{1}{2\pi} \int_0^{2\pi} |\cos\theta|\, d\theta = \frac{1}{2\pi} \cdot 4 = \frac{2}{\pi}.$$

Similarly,

$$\left\langle \frac{r_c(t)^2}{f_c^2} \right\rangle_T = \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta\, d\theta = \frac{1}{2},$$

which gives the RMS control $1/\sqrt{2}$. □

Choice of the active average. The boundary compliance used here is a linear compliance: it relates a boundary excursion to a local traction. In a cyclic reclosure, the two opposite half-cycles contribute with the same relaxation amplitude, but with opposite signs of the instantaneous displacement. The relevant quantity is therefore the mean oriented amplitude, namely $\langle |r_c|/f_c \rangle_T = 2/\pi$.

The quadratic average

$$\left\langle r_c^2/f_c^2 \right\rangle_T^{1/2} = 1/\sqrt{2}$$

measures an RMS energy closure. It provides a quadratic stability control, but it is not the active closure of the linear compliance used in the minimal NLO reclosure.

Local boundary coupling. We write

$$\sigma_{\mathrm{loc}}(u) := \frac{\Sigma_{\mathrm{loc}}(u)}{u} = \frac{\zeta(u)}{u} = \frac{1}{2} \coth u\, \mathrm{csch}^2 u.$$

The LO local term therefore corresponds to

$$\Sigma_{\mathrm{loc}}(u) = u\, \sigma_{\mathrm{loc}}(u) = \zeta(u).$$

Local static law at the peak. At the peak boundary excursion,

$$r_c(t) = f_c \cos(\omega t), \qquad |r_c|_{\mathrm{peak}} = f_c,$$

the local boundary reduction gives the effective static law

$$\boxed{\tau_{\mathrm{peak}}(u) = \sigma_{\mathrm{loc}}(u)\, f_c,} \tag{14.32}$$

where τ_{peak} is the boundary traction associated with the peak. This relation is a boundary law at the oscillation peak; it does not introduce a new closure independent of the core. The local static compliance then is

$$\boxed{\mathcal{C}_{\mathrm{stat}}(u) := \frac{f_c}{\tau_{\mathrm{peak}}(u)} = \frac{1}{\sigma_{\mathrm{loc}}(u)}.} \tag{14.33}$$

Cyclic compliance. The compliance relevant to reclosure over one cycle is the mean oriented linear compliance, defined by

$$\boxed{\mathcal{C}^{(1)}_{\partial H}(u) := \frac{\langle |r_c(t)| \rangle_T}{\tau_{\mathrm{peak}}(u)}.} \tag{14.34}$$

Lemma 14.5 and (14.32) then give

$$\boxed{\mathcal{C}^{(1)}_{\partial H}(u) = \frac{(2/\pi) f_c}{\sigma_{\mathrm{loc}}(u) f_c} = \frac{2}{\pi} \frac{1}{\sigma_{\mathrm{loc}}(u)} = \frac{2}{\pi} \mathcal{C}_{\mathrm{stat}}(u).} \tag{14.35}$$

The associated quadratic control would be

$$\boxed{\mathcal{C}^{(2)}_{\partial H}(u) = \frac{1}{\sqrt{2}} \mathcal{C}_{\mathrm{stat}}(u).} \tag{14.36}$$

Selection of the active compliance. The factor $2/\pi$ is not chosen by comparison with a downstream mass. It is imposed by the nature of the retained boundary operator: the minimal NLO reclosure acts at first order as a linear compliance between boundary excursion and local traction. A linear compliance adds the relaxation amplitudes of the two half-cycles with the same effective orientation; the active quantity is therefore the oriented absolute average

$$\langle |r_c| / f_c \rangle_T = \frac{2}{\pi}.$$

The coefficient $1/\sqrt{2}$ corresponds to the RMS quadratic control of the same cycle. It controls the associated energy scale, but it does not define the linear boundary compliance used in the minimal local reclosure. Using $1/\sqrt{2}$ instead of $2/\pi$ would amount to changing the closure, by passing from an oriented linear compliance to an RMS quadratic closure.

Lemma 14.6 (uniqueness of the first-order local scalar mismatch)**.** *At the boundary $s = s_c$, any first-order linear local scalar operator acting on the radial field can be written in the form*

$$\mathfrak{L}[r] = a(u)\, \partial_s r_c + b(u)\, r_c, \qquad r_c := r(s_c).$$

If one imposes that $\mathfrak{L}[r]$ measures a matching defect and vanishes on the passive exterior tail, then $\mathfrak{L}[r]$ is proportional to the unique local mismatch

$$\boxed{\mathfrak{M}_u[r] := (\partial_s r - \Lambda_{\mathrm{out}}(u)\, r)_{s=s_c}.}$$

Proof. At first order, in the zonal scalar channel, a local boundary operator can depend only on r_c and on its normal derivative $\partial_s r_c$. It is therefore of the form

$$\mathfrak{L}[r] = a(u)\, \partial_s r_c + b(u)\, r_c.$$

If it is to measure a matching defect, it must vanish on the passive exterior tail. On this tail,

$$\partial_s r_c = \Lambda_{\mathrm{out}}(u)\, r_c.$$

One must therefore have

$$a(u) \Lambda_{\mathrm{out}}(u)\, r_c + b(u)\, r_c = 0 \qquad \text{for all } r_c,$$

which imposes

$$b(u) = -a(u) \Lambda_{\mathrm{out}}(u).$$

Thus,

$$\mathfrak{L}[r] = a(u) (\partial_s r_c - \Lambda_{\mathrm{out}}(u)\, r_c) = a(u)\, \mathfrak{M}_u[r].$$

Therefore, up to a multiplicative factor, the first-order local scalar mismatch is unique. □ □

Consequence for the NLO boundary. The LO boundary is already governed by the local scalar reduction of the mismatch $\mathfrak{M}_u[r]$, which produces the coupling

$$\sigma_{\mathrm{loc}}(u) = \frac{\zeta(u)}{u}.$$

If the NLO refinement remains in the same first-order local scalar channel, then it cannot introduce an independent local kernel: it only renormalizes the same mismatch $\mathfrak{M}_u[r]$. The NLO boundary therefore necessarily inherits the same local coupling $\sigma_{\mathrm{loc}}(u)$.

Minimal local NLO truncation. At the minimal NLO order retained here, the class of operators is not changed: the boundary remains scalar, local, linear at first order in the matching defect, and no independent boundary variable is introduced. This truncation is not a phenomenological adjustment; it expresses the fact that the present chapter treats only the first reclosure correction in the same radial scalar channel as the LO matching.

By Lemma 14.6, any first-order local scalar operator measuring a matching defect is then necessarily proportional to the unique mismatch

$$\mathfrak{M}_u[r] = (\partial_s r - \Lambda_{\mathrm{out}}(u) r)_{s=s_c}.$$

The minimal local NLO refinement can therefore introduce no new boundary structure; it can only renormalize the compliance associated with this same mismatch.

Theorem 14.7 (minimal local NLO refinement of the scalar matching). In the minimal local NLO truncation of the radial scalar channel, the total scalar matching correction is

$$\boxed{\Sigma_H^{(\mathrm{NLO,min})}(u) = \left(1 + \frac{\pi}{2}\right) \zeta(u).} \tag{14.37}$$

The matching law then becomes

$$\boxed{y \cot y = -u + \left(1 + \frac{\pi}{2}\right) \zeta(u),} \tag{14.38}$$

and, in the exact version on $\mathbb{S}^3$,

$$\boxed{y \cot y = -u\sqrt{1 - \frac{\ell_r^2}{R_0^2}} + \left(1 + \frac{\pi}{2}\right) \zeta(u).} \tag{14.39}$$

Proof. By the uniqueness of the local mismatch $\mathfrak{M}_u[r]$, proved in Lemma 14.6, the minimal local NLO refinement can act only on the compliance associated with this same mismatch. There is therefore, in this truncation, no second independent local kernel to add at the boundary.

The effective boundary coefficient associated with the active cyclic compliance is

$$\boxed{\sigma_{\mathrm{rim}}^{\mathrm{eff}}(u) = \left(\mathcal{C}_{\partial H}^{(1)}(u)\right)^{-1}.} \tag{14.40}$$

By (14.35), one obtains

$$\boxed{\sigma_{\mathrm{rim}}^{\mathrm{eff}}(u) = \frac{\pi}{2}\, \sigma_{\mathrm{loc}}(u).} \tag{14.41}$$

The corresponding dimensionless correction is therefore

$$\boxed{\Sigma_{\mathrm{rim}}(u) = u\, \sigma_{\mathrm{rim}}^{\mathrm{eff}}(u) = \frac{\pi}{2}\, \zeta(u).} \tag{14.42}$$

One then adds the local LO term and the NLO term:

$$\Sigma_H^{(\mathrm{NLO,min})}(u) = \Sigma_{\mathrm{loc}}(u) + \Sigma_{\mathrm{rim}}(u) = \zeta(u) + \frac{\pi}{2}\zeta(u) = \left(1 + \frac{\pi}{2}\right) \zeta(u).$$

Substitution into the matching law gives (14.38) and then (14.39). □ □

Mass increment associated with the minimal NLO refinement. The scale-free increment consumed by the Higgs-like mass chain is that of the minimal local NLO scheme. We therefore define

$$\boxed{\delta_H^*(u) := A_H(u)\left[B(u) + \Sigma_H^{(\mathrm{NLO,min})}(u)\right] = A_H(u)\left[B(u) + \left(1 + \frac{\pi}{2}\right)\zeta(u)\right].} \tag{14.43}$$

In the monopole case,

$$A_H(u) = \frac{J_0^2}{J_2}\,\hat{c}_0^{(S)}(u)^2 + \mathcal{O}(\kappa^2).$$

The HSM mass law associated with the same scheme is

$$\boxed{m_{\mathrm{Hlike}}^{(\mathrm{HSM})} = m_{\mathrm{NSL,ch}}^{(\mathrm{HSM})}\sqrt{1 + \delta_H^*(u_H^*)}.} \tag{14.44}$$

Minimal NLO closure of the matching parameter. In the minimal NLO refinement defined by (14.39), the matching parameter is determined by the closed equation

$$\boxed{u_H^*\, a_H \cot(u_H^* a_H) = -u_H^* a_H + \left(1 + \frac{\pi}{2}\right)\zeta(u_H^*), \qquad a_H = \sqrt{1 - \frac{\ell_r^2}{R_0^2}}.} \tag{14.45}$$

At local order,

$$a_H = 1 + \mathcal{O}\left(\frac{\ell_r^2}{R_0^2}\right),$$

and (14.45) becomes

$$u_H^* \cot u_H^* = -u_H^* + \left(1 + \frac{\pi}{2}\right)\zeta(u_H^*).$$

Its first positive root in the interval $(\pi/2, \pi)$ is

$$u_{H,\mathrm{NLO}}^* \simeq 2.3302825264.$$

14.15 Internal kinematics of the bare radial mode

At the level of the bare radial deformation, the pure localized radial mode belongs to the local kinematics associated with the effective radial stiffness. Its internal characteristic speed is

$$c_{\mathcal{F}} = c_r, \qquad c_r^2 = \frac{T\kappa_r}{\rho} = \kappa_r c_t^2.$$

The passage law from the radial comoving frame to the fixed frame therefore takes the form

$$E_r(u) = \Gamma_r E_{r,0}, \qquad P_r(u) = \Gamma_r \frac{u}{c_r^2} E_{r,0}, \qquad \Gamma_r = \frac{1}{\sqrt{1 - u^2/c_r^2}},$$

and the internal dispersion of the bare radial mode is

$$E_r(u)^2 = E_{r,0}^2 + c_r^2 P_r(u)^2.$$

In motion, the bare radial mode has an internal kinematics characterized by

$$c_r = \sqrt{T\kappa_r/\rho} = \sqrt{\kappa_r}\, c_t.$$

This speed is that of the bare radial deformation. The observable limiting speed of the Higgs-like scalar, after matching to the effective relativistic domain, is the common speed of the observable massive branches, c_t. This matching belongs to the observational correspondence and to the link with the effective theories; it is not closed in the present radial chapter.

14.16 Observed frequency vs. proper frequency

For the boosted bare radial solution $r(s,t) = f(\xi)\cos(\omega_r \tau)$, with

$$\xi = \Gamma_r(s - ut), \qquad \tau = \Gamma_r\Big(t - \frac{u}{c_r^2}s\Big),$$

the laboratory phase is written

$$\varphi(s,t) = \omega_{\rm lab} t - ks, \qquad \omega_{\rm lab} = \Gamma\omega_r, \qquad k = \Gamma\frac{u}{c_r^2}\omega_r.$$

It gives the invariant reconstructions

$$\boxed{\omega_r = \sqrt{\omega_{\rm lab}^2 - c_r^2 k^2}, \qquad u = \frac{c_r^2 k}{\omega_{\rm lab}}, \qquad \Gamma = \frac{\omega_{\rm lab}}{\omega_r}.} \tag{14.46}$$

and

$$\boxed{\omega_{\rm lab}^2 - c_r^2 k^2 = \omega_r^2.} \tag{14.47}$$

14.17 Spin and neutrality

The solution is scalar and purely radial ($A \equiv 0$): there is no intrinsic angular momentum.

$$|S| = 0, \qquad s_g = 0.$$

14.18 Logical status of the chapter

What is closed in the present chapter. The chapter establishes:

- the local radial equation and the localization threshold $\omega_r^2 < \mu_2/\rho$;
- the local proper energy and the common modal constraint $h_{\rm eff}^{(H)} = h_{\rm eff}$;
- the conditional mass chain of the minimal local NLO scheme

  $$\delta_H^*(u) = A_H(u)\left[B(u) + \Sigma_H^{(\rm NLO,min)}(u)\right] = A_H(u)\left[B(u) + \left(1 + \frac{\pi}{2}\right)\zeta(u)\right], \qquad \mathcal{R}_H(u) = \sqrt{1 + \delta_H^*(u)}.$$

- the no-go for a strictly linear passive radial core at finite radius;
- the exact geometric skeleton of the minimal active matching, (14.22) ;
- the closed law of the minimal active core, (14.27) ;
- the conditional minimal NLO refinement in the same local scalar channel, (14.39).

Canonical LO framework of the chapter. The blocks $B(u)$, $A_H(u)$, and $\zeta(u)$ are established here in the LO scalar framework adopted for the anchored radial channel. The LO universality section shows that the choice $a(\xi) = \operatorname{sech}\xi$ must be read as that of a canonical representative of the localized-profile class at LO, and not as the claim that the exact Higgs core is literally of sech shape. The minimal active continuation is fixed by the variational minimization of the spectral defect, cf. lemma 14.4. The NLO refinement is formulated as the continuation of the same first-order local scalar channel as LO; it adds the corresponding minimal boundary term without introducing an additional nonlocal channel.

Closure of the matching parameter. The matching law of the minimal active core closes the matching parameter of the Higgs-like channel. By setting

$$a_H := \sqrt{1 - \frac{\ell_r^2}{R_0^2}}, \qquad y = a_H u,$$

equation (14.27) becomes

$$y \cot y = -y.$$

The first admissible positive branch gives

$$y_H^* = \frac{3\pi}{4}.$$

Thus,

$$\boxed{u_H^* = \frac{3\pi}{4}\left(1 - \frac{\ell_r^2}{R_0^2}\right)^{-1/2}}. \tag{14.48}$$

In the thin-core regime,

$$u_H^* = \frac{3\pi}{4} + \mathcal{O}\left(\frac{\ell_r^2}{R_0^2}\right).$$

The parameter u is therefore not left free in the LO closure of the Higgs-like channel.

Minimal NLO closure. If one keeps the minimal NLO refinement of (14.39), the matching parameter is determined by

$$\boxed{u_H^*\, a_H \cot(u_H^* a_H) = -u_H^* a_H + \left(1 + \frac{\pi}{2}\right)\zeta(u_H^*), \qquad a_H = \sqrt{1 - \frac{\ell_r^2}{R_0^2}}.} \tag{14.49}$$

At local order $a_H = 1 + \mathcal{O}(\ell_r^2/R_0^2)$, this equation becomes

$$u_H^* \cot u_H^* = -u_H^* + \left(1 + \frac{\pi}{2}\right)\zeta(u_H^*).$$

Its first positive root in $(\pi/2, \pi)$ is

$$u_{H,\mathrm{NLO}}^* \simeq 2.3302825264.$$

14.19 Summary (Higgs-like, S^3)

The localized passive radial mode is defined by (14.4)–(14.7), with

$$\ell_r^{-2} = \frac{\mu_2 - \rho\omega_r^2}{T\kappa_r}.$$

At passive LO level, the MSE does not discretize ω_r at leading order (14.8), and the cutoff (14.13) only bounds the boundary amplitude. The proper energy is

$$E_0 = h_{\mathrm{eff}}\omega_r, \qquad h_{\mathrm{eff}}^{(H)} = h_{\mathrm{eff}},$$

by the common modal constraint (14.11). In motion, the bare radial mode has an internal kinematics characterized by

$$c_r = \sqrt{T\kappa_r/\rho} = \sqrt{\kappa_r}\, c_t.$$

This speed is that of the bare radial deformation, not a direct prediction of the observable limiting speed of the physical Higgs-like scalar. The relativistic phenomenological reading of the Higgs-like mode, when it is performed, belongs to the matching to the effective tangential sector in the framework of the effective link. The geometric spin number is $s_g = 0$.

Higgs-like mass chain. The scale-free scalar closure of the Higgs-like channel, in the minimal local NLO scheme, is written

$$\delta_H^*(u) = A_H(u)\left[B(u) + \Sigma_H^{(\mathrm{NLO,min})}(u)\right] = A_H(u)\left[B(u) + \left(1+\frac{\pi}{2}\right)\zeta(u)\right].$$

After closure of the minimal NLO matching, we set

$$\delta_H^* := \delta_H^*(u_H^*).$$

The HSM Higgs-like mass is then defined relative to the charged NSL branch:

$$m_{\mathrm{Hlike}}^{(\mathrm{HSM})} = m_{\mathrm{NSL,ch}}^{(\mathrm{HSM})}\sqrt{1+\delta_H^*}.$$

At LO level, the dependence on the exact core profile reduces to a universal class of localized profiles; the choice $a(\xi) = \operatorname{sech}\xi$ plays the role of a canonical representative there, not that of an imposed exact profile.

Minimal active matching and NLO refinement. The strictly linear passive radial core does not close the matching at finite radius; a minimal active core must be introduced. The variational minimization of the spectral defect, cf. lemma 14.4, then fixes

$$\beta = \nu, \qquad y = u\sqrt{1-\frac{\ell_r^2}{R_0^2}},$$

so that the exact matching law closes in u. Under the minimal NLO hypothesis of a first-order local scalar channel, this law is refined into

$$y\cot y = -u\sqrt{1-\frac{\ell_r^2}{R_0^2}} + \left(1+\frac{\pi}{2}\right)\zeta(u).$$

The minimal local NLO refinement term is therefore

$$\Sigma_H^{(\mathrm{NLO,min})}(u) = \left(1+\frac{\pi}{2}\right)\zeta(u).$$

It is this same term that enters the mass increment $\delta_H^*(u)$ defined in (14.43). Thus, in this chapter, the symbol δ_H^* denotes the increment of the minimal local NLO scheme, and not the local block alone $B(u)+\zeta(u)$.

Status of the parameter u. The present chapter closes the matching parameter of the Higgs-like channel by the minimal active core. At LO order,

$$u_{H,\mathrm{LO}}^* = \frac{3\pi}{4} + \mathcal{O}\left(\frac{\ell_r^2}{R_0^2}\right).$$

With the minimal NLO refinement, u_H^* is determined by (14.45), and its first positive root in the local limit is

$$u_{H,\mathrm{NLO}}^* \simeq 2.3302825264.$$

The residual structural uncertainties therefore do not concern the existence of a closure of u, but the choice of the retained order, LO or minimal local NLO, and the decision not to include, in this chapter, nonlocal boundary degrees of freedom or higher-order boundary operators.

Comments. (i) The obtained matching value does not come from a canonical choice imported from the bump or the NSL, but from a closure specific to the radial Higgs-like channel. (ii) No empirical factor is introduced in the scale-free construction δ_H^*. (iii) The HSM mass relation uses the internal charged NSL branch as reference, and not an observational mass. (iv) The choice of the closure order, LO or minimal NLO, must be stated explicitly in any numerical evaluation derived from this chapter.

Chapter 15

Gravitational-like field on $\mathbb{S}^3$

15.1 Positioning, assumptions, and notation

We describe the *gravitational-like* field generated by the localized solutions of the HSM model. The framework is the continuum on $\mathbb{S}^3$, at LO in the regime

$$\lambda/R_0 \ll 1,$$

where λ denotes the characteristic localization scale of the source.

Fundamental constants. We use the mechanical constants of the radial and tangential sectors

$$\rho \ \text{(volume density)}, \quad T \ \text{(tangential tension)}, \quad c_t^2 = \frac{T}{\rho}, \quad \mu_2 \ \text{(local radial coefficient)}, \quad \kappa_r \ \text{(radial-gradient weight)}, \quad R_0.$$

With $[\rho] = \mathrm{kg\,m^{-3}}$, one has

$$[T] = \mathrm{Pa} = \mathrm{J\,m^{-3}},$$

so that $c_t^2 = T/\rho$ indeed has the dimension of a squared speed.

Continuous radial sector. In this chapter, we explicitly distinguish the two contributions of the radial sector:

$$\frac{T\kappa_r}{2}\,|\nabla r|^2 \qquad \text{and} \qquad \frac{\mu_2}{2}\,r^2.$$

The coefficient $T\kappa_r$ controls the spatial cost of variations of the radial field, whereas μ_2 controls its local stiffness.

Definition of κ_μ. We introduce the local radial coefficient

$$\boxed{\kappa_\mu := \frac{\mu_2}{T}, \qquad [\kappa_\mu] = \mathrm{m^{-2}}.} \tag{15.1}$$

Thus,

$$\mu_2 = T\kappa_\mu,$$

where μ_2 is the dimensioned coefficient of the local radial term.

Structural convention. No general identity is postulated in this chapter between κ_r and κ_μ. In particular, we do not assume $\mu_2 = T\kappa_r$ as a universal closure. The possible relation between κ_r and κ_μ belongs to the structural analysis of the model; in the continuum, we therefore keep a separate writing of the radial-gradient term and the local radial term.

Radial operator and screening parameter. At the geometric level, the stationary radial equation is written

$$-T\kappa_r\,\Delta_{\mathbb{S}^3} r + \mu_2\, r = \text{source},$$

where $\Delta_{\mathbb{S}^3}$ denotes the Beltrami Laplacian on the reference sphere, according to the general conventions of the document.

In the axisymmetric reduction in the arc coordinate $s = R_0\chi$, this equation takes the form of a Helmholtz-type radial operator, and one then introduces

$$\boxed{\lambda_r^2 := \frac{\mu_2}{T\kappa_r} = \frac{\kappa_\mu}{\kappa_r}.}$$

The parameter λ_r is therefore the inverse radial screening length. In general, one does not set $\lambda_r^2 = 1$ *a priori*: this value corresponds only to the particular case $\kappa_\mu = \kappa_r$, that is, to a special closure of the radial sector, and not to a universal identity of the model.

Geometric sign convention. We write

$$R(s) = R_0 + r(s),$$

and adopt the convention

$$\boxed{r > 0 \iff R_0 + r > R_0.}$$

A radial bump therefore corresponds to a positive radial deformation of the local radius.

We recall the definition

$$\mu_2 = T\,\kappa_\mu, \qquad \lambda_r^2 = \frac{\kappa_\mu}{\kappa_r},$$

already introduced in (15.1).

Thus, κ_μ measures the local radial stiffness, whereas κ_r weights the radial-gradient term. The dimensioned coefficient of the gradient term is $T\kappa_r$, whereas that of the local term is $\mu_2 = T\kappa_\mu$.

Scope of the chapter. This chapter deals only with the *gravitational-like radial channel*. The EM-like normalizations, impedances, and electromagnetic closures are treated separately in the dedicated parts.

15.2 Yukawa Green function on $\mathbb{S}^3$ (LO) and unique canonical average

Canonical average on $\mathbb{S}^3$. For a zonal function $f(\chi)$ on $\mathbb{S}^3$ of radius R_0, one has

$$dV = R_0^3 \sin^2\chi\, d\chi\, d\Omega_2, \qquad \mathrm{Vol}(\mathbb{S}^3) = 2\pi^2 R_0^3.$$

The canonical average is therefore written

$$\langle f\rangle_{\mathbb{S}^3} = \frac{1}{\mathrm{Vol}(\mathbb{S}^3)}\int_{\mathbb{S}^3} f\, dV = \frac{1}{2\pi^2 R_0^3}\int_{\mathbb{S}^3} f\, dV = \frac{2}{\pi}\int_0^\pi \sin^2\chi\, f(\chi)\, d\chi. \tag{15.2}$$

In gravitational-like convolutions, this average is introduced only once.

Reduced operator and zonal Green function. The physical stationary radial equation is written

$$-T\kappa_r\,\Delta_{\mathbb{S}^3} r + \mu_2\, r = \sigma,$$

where σ denotes the effective radial source. Dividing by $T\kappa_r$, one obtains

$$(-\Delta_{\mathbb{S}^3} + \lambda_r^2)\, r = S, \qquad S := \frac{\sigma}{T\kappa_r}, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r} = \frac{\kappa_\mu}{\kappa_r}. \tag{15.3}$$

We then define the Green function $G(x, y)$ of the reduced operator by

$$(-\Delta_{\mathbb{S}^3,x} + \lambda_r^2)\, G(x, y) = \delta_{\mathbb{S}^3}(x, y), \tag{15.4}$$

so that the solution is written

$$r(x) = \int_{\mathbb{S}^3} G(x, y)\, S(y)\, dV_y. \tag{15.5}$$

Local Yukawa law and Newtonian limit. In the local regime $d(x, y) \ll R_0$, the geometry becomes asymptotically flat and (15.4) reduces to the usual Helmholtz equation on $\mathbb{R}^3$. One then obtains

$$G(d) \simeq \frac{e^{-\lambda_r d}}{4\pi d}.$$

When radial screening is negligible at the scale under consideration,

$$\lambda_r d \ll 1,$$

one recovers the local Newtonian limit

$$G(d) \simeq \frac{1}{4\pi d}.$$

Thus, the interaction law between two localized cores is of Yukawa type at LO, with a local Newtonian regime when the screening length λ_r^{-1} is large compared with the distance under consideration.

Remark on the case without screening. On the compact sphere $\mathbb{S}^3$, the pure operator $-\Delta_{\mathbb{S}^3}$ has a constant mode in its kernel. The Newtonian formula above must therefore be understood as a *local limit* ($d \ll R_0$, $\lambda_r d \ll 1$) and not as the unprojected global inversion of $-\Delta_{\mathbb{S}^3}$ over all of $\mathbb{S}^3$.

15.2.1 Mechanical moduli and density (LO)

We recall the structural relations

$$\rho = \frac{T}{c_t^2}, \qquad \mu_2 = T\,\kappa_\mu, \qquad [\kappa_\mu] = \mathrm{m}^{-2}, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r} = \frac{\kappa_\mu}{\kappa_r}. \tag{15.6}$$

The coefficient κ_μ measures the local radial stiffness, whereas κ_r weights the radial-gradient term. No general identity is postulated here between κ_r and κ_μ; their possible relation belongs to the structural analysis of the model.

15.3 Existence of the gravitational-like field

15.3.1 Variational source and monopole charge

The gravitational-like field is not introduced as an independent interaction. It results from the radial response of the support to the energy of a localized solution. In the active convention of the radial channel, the effective source is defined by variation of the energy contributions with respect to the radial deformation:

$$S = -\frac{1}{T\kappa_r}\left(\frac{\delta E_{\mathrm{cin}}}{\delta r} + \frac{\delta E_{\tan}}{\delta r} + \frac{\delta E_{\mathrm{eff}}}{\delta r}\right). \tag{15.7}$$

This writing fixes the reduced normalization of the source in the radial channel.

The robust quantity associated with a localized solution is not a local height of the core, but the integrated monopole charge

$$Q_g := \int_{\mathbb{S}^3} S\, dV. \tag{15.8}$$

This charge controls the amplitude of the exterior gravitational-like field in the monopole regime.

15.3.2 LO derivation of the gravitational-like source

The variational definition (15.7) serves as the starting point for the LO monopole reduction in the quasi-flat core regime.

At LO, the three local variations take the form

$$\frac{\delta E_{\mathrm{cin}}}{\delta r} \simeq \frac{\rho\omega^2}{R_0}A^2, \qquad \frac{\delta E_{\tan}}{\delta r} \simeq \frac{T}{R_0}(A_s)^2, \qquad \frac{\delta E_{\mathrm{eff}}}{\delta r} \simeq -\frac{2T}{R_0^3}A^2. \tag{15.9}$$

It follows that

$$S = -\frac{1}{T\kappa_r}\left[\left(\frac{\rho\omega^2}{R_0} - \frac{2T}{R_0^3}\right)A^2 + \frac{T}{R_0}(A_s)^2\right]. \tag{15.10}$$

By integrating over $\mathbb{S}^3$, one obtains the monopole charge at LO:

$$Q_g = \int_{\mathbb{S}^3} S\, dV = -\frac{1}{T\kappa_r}\left[\frac{\rho\omega^2}{R_0}\int_{\mathbb{S}^3} A^2\, dV + \frac{T}{R_0}\int_{\mathbb{S}^3}(A_s)^2\, dV - \frac{2T}{R_0^3}\int_{\mathbb{S}^3} A^2\, dV\right]. \tag{15.11}$$

For an isolated solution, one uses the core identity

$$\int_{\mathbb{S}^3}(A_s)^2\, dV = \Omega_0^2\int_{\mathbb{S}^3} A^2\, dV - \frac{\beta_c}{2}\int_{\mathbb{S}^3} A^4\, dV, \tag{15.12}$$

where β_c denotes the quartic coefficient of the core equation under consideration. One also uses

$$c_t^2 = \frac{T}{\rho}, \qquad \Omega_0^2 = \frac{2}{R_0^2} - \frac{\omega^2}{c_t^2}, \tag{15.13}$$

hence

$$\rho\omega^2 = T\left(\frac{2}{R_0^2} - \Omega_0^2\right). \tag{15.14}$$

After substitution into (15.11), the coefficient of $\int_{\mathbb{S}^3} A^2\, dV$ cancels exactly:

$$\frac{\rho\omega^2}{R_0} + \frac{T\Omega_0^2}{R_0} - \frac{2T}{R_0^3} = 0.$$

One thus obtains

$$Q_g = \frac{1}{\kappa_r}\frac{\beta_c}{2R_0}\int_{\mathbb{S}^3} A^4\, dV. \tag{15.15}$$

The gravitational-like structure of the radial channel therefore rests, at LO, on the integration of the variational source over $\mathbb{S}^3$, followed by the monopole reduction of the core. At the same order, the monopole charge no longer depends on the quadratic term $\int A^2\, dV$: it is carried by the quartic core invariant.

15.3.3 Exterior field and local monopole regime

The exterior radial field created by a compact source is obtained by convolution with the Yukawa Green function of the support. In the local regime $d \ll R_0$, the kernel reduces to its local Euclidean form and, beyond the core, only the monopole component remains:

$$r_{\mathrm{ext}}(d) \sim Q_g\, G(d), \qquad G(d) \sim \frac{e^{-\lambda_r d}}{4\pi d}. \tag{15.16}$$

When radial screening is negligible at the scale under consideration ($\lambda_r d \ll 1$), one recovers

$$G(d) \sim \frac{1}{4\pi d},$$

and therefore

$$Q_g \simeq 4\pi d\, r_{\mathrm{ext}}(d). \tag{15.17}$$

Thus, Q_g is interpreted as the monopole coefficient of the local exterior field.

Scope remark. The local $1/d$ law is valid only in the window $a \ll d \ll R_0$, where a denotes the characteristic size of the core. At a separation comparable to the global size of $\mathbb{S}^3$, the compactness of the support — and, where relevant, radial screening — necessarily modifies this local asymptotic behavior.

15.3.4 Mechanical interpretation of the radial channel and attractive sign

The gravitational-like channel admits a simple mechanical reading: the local variation of the energy of a solution acts as a normal constraint on the support, this constraint sources the radial deformation, and the overlap of two deformations lowers the energy of the system.

The first variation of the total energy with respect to the normal deformation r is written

$$\delta E = \int_{\mathbb{S}^3} \left(\frac{\delta E_{\mathrm{cin}}}{\delta r} + \frac{\delta E_{\tan}}{\delta r} + \frac{\delta E_{\mathrm{eff}}}{\delta r} \right) \delta r\, dV. \tag{15.18}$$

The term in parentheses thus plays the role of the normal work density exerted on the support. After division by the radial coefficient $T\kappa_r$, one recovers the source of the gravitational-like channel (15.7). The exterior radial deformation is then the nonlocal response of the support to this normal constraint.

Now consider two compact solutions, producing the deformations r_1 and r_2. In the linear regime, the total radial energy decomposes as

$$E_{\mathrm{rad}}[r_1 + r_2] = E_{\mathrm{rad}}[r_1] + E_{\mathrm{rad}}[r_2] + E_{\times}[r_1, r_2]. \tag{15.19}$$

At the dominant monopole order, the cross term is bilinear in the monopole charges and takes the form

$$E_{\times}(d) \simeq T\kappa_r\, Q_{g,1} Q_{g,2}\, G(d). \tag{15.20}$$

In the sign convention of the interaction potential, one sets

$$U(d) = -E_{\times}(d), \tag{15.21}$$

hence

$$U(d) \simeq -T\kappa_r\, Q_{g,1} Q_{g,2}\, G(d). \tag{15.22}$$

The attractive sign of the potential is therefore not postulated. It comes from the fact that two deformations of the same orientation decrease the energy of the system when they overlap in the linear radial channel. For the physical branches considered, where the monopole charges carry the same sign, the potential is attractive. The complete mechanical chain of the gravitational-like sector is thus written

$$\delta E \;\longrightarrow\; S \;\longrightarrow\; r_{\mathrm{ext}} \;\longrightarrow\; E_{\times} \;\longrightarrow\; U(d). \tag{15.23}$$

15.3.5 Two-body gravitational-like interaction

For two compact solutions that are sufficiently separated, the cross energy of their exterior deformations is dominated at LO by the monopole term. In the local Newtonian regime, one obtains

$$U(d) \simeq -T\kappa_r\, Q_{g,1} Q_{g,2}\, G(d) \sim -\frac{T\kappa_r}{4\pi}\,\frac{Q_{g,1} Q_{g,2}}{d}. \tag{15.24}$$

The existence of a gravitational-like field at LO is thus established: a localized solution of finite energy produces an exterior radial field, and two solutions coupled in this way interact through an attractive potential of local Newtonian type.

Local validity. In the window

$$a \ll d \ll R_0,$$

the radial Green function reduces to its local Yukawa/Newton form. When radial screening is negligible at the scale under consideration ($\lambda_r d \ll 1$), one recovers the local Newtonian regime.

15.4 Strong gravitational-like regime: regularity of finite sources and status of trapping surfaces

Object. The gravitational-like channel was constructed at linear rank from the radial field r and the operator

$$\rho\,\ddot{r} - T\kappa_r \Delta_{\mathbb{S}^3} r + \mu_2 r = 0, \qquad \lambda_r^2 = \frac{\mu_2}{T\kappa_r}, \qquad \ell_r = \frac{1}{\lambda_r} = \sqrt{\frac{T\kappa_r}{\mu_2}}.$$

This section specifies what this construction already allows one to state about the regime of high energy concentration. The point treated here is not the complete reconstruction of a black-hole-type solution, but the regularity of the gravitational-like channel when it is sourced by localized HSM configurations of finite energy.

Quadratic energy of the radial channel. The preceding linearized equation is associated, at the rank considered, with the quadratic energy

$$E_r = \frac{1}{2}\int_{\mathbb{S}^3} \left(\rho\,\dot{r}^2 + T\kappa_r\,|\nabla_{\mathbb{S}^3} r|^2 + \mu_2\, r^2\right)\, dV. \tag{15.25}$$

Since

$$\rho > 0, \qquad T > 0, \qquad \kappa_r > 0, \qquad \mu_2 > 0,$$

this energy is positive. Thus, at the linear rank of the gravitational-like channel, a field configuration r of finite energy does not correspond to a mechanical divergence of the radial degree, but to a regular element of the energy domain of the operator.

Positive static operator. In the static or quasi-static regime, the physical response operator is

$$\mathcal{L}_g := -T\kappa_r\Delta_{\mathbb{S}^3} + \mu_2 = T\kappa_r\left(-\Delta_{\mathbb{S}^3} + \lambda_r^2\right).$$

On $\mathbb{S}^3$, this operator is elliptic and strictly positive for

$$T > 0, \qquad \kappa_r > 0, \qquad \mu_2 > 0.$$

For a regular effective source or a source of finite energy σ_g, the physical static response is therefore defined by

$$\mathcal{L}_g r = \sigma_g. \tag{15.26}$$

By dividing by $T\kappa_r$, one recovers the reduced operator already used in the Green-function construction:

$$\left(-\Delta_{\mathbb{S}^3} + \lambda_r^2\right) r = S_g, \qquad S_g := \frac{\sigma_g}{T\kappa_r}.$$

The positivity of $\mathcal{L}_g$, or equivalently of the reduced operator, prevents the appearance of a pathological zero mode in this closure.

Regularization by a core source. The reduced Green function G_g of the operator

$$-\Delta_{\mathbb{S}^3} + \lambda_r^2$$

has, at distances short compared with R_0, the local form

$$G_g(d) \simeq \frac{e^{-d/\ell_r}}{4\pi d}.$$

This singularity is that of the reduced point kernel. The physical HSM sources considered in the theory are not, however, bare points: they are localized solutions with finite core and finite energy. The physical response is therefore a convolution of the reduced Green function with the extended reduced source:

$$r(x) = \int_{\mathbb{S}^3} G_g(x,y)\, S_g(y)\, dV_y = \frac{1}{T\kappa_r}\int_{\mathbb{S}^3} G_g(x,y)\, \sigma_g(y)\, dV_y. \tag{15.27}$$

For a bounded core source, or more generally an integrable source with finite energy, the $1/d$ singularity of the Green function is locally integrable in dimension three and does not produce a divergence of the mean field r at the center of the core. The point divergence therefore belongs to the ideal Green function of a delta source, not to the response of a physical HSM source of finite size.

Regularity lemma for the gravitational-like channel. We deduce the following lemma.

Lemma 15.1 (Gravitational-like regularity of finite HSM sources)**.** *Let σ_g be a gravitational-like source arising from a localized HSM configuration of finite energy, and let*

$$S_g = \frac{\sigma_g}{T\kappa_r}$$

be the associated reduced source. In the domain of validity of the linear or weakly nonlinear gravitational-like closure, the response r defined by (15.26)–(15.27) *remains a regular configuration of the radial field. In particular, no point singularity of the HSM field results from a finite-core source.*

Proof. The physical operator

$$\mathcal{L}_g = -T\kappa_r \Delta_{\mathbb{S}^3} + \mu_2$$

is elliptic and strictly positive for

$$T > 0, \qquad \kappa_r > 0, \qquad \mu_2 > 0.$$

After division by $T\kappa_r$, the reduced operator

$$-\Delta_{\mathbb{S}^3} + \lambda_r^2$$

has a Green function whose local singularity is the standard $1/(4\pi d)$ singularity, integrable against an extended source in dimension three. A localized HSM source of finite energy is not identified with a bare delta distribution: it has a core of finite size and an integrable energy density. The convolution

$$r = G_g * S_g$$

is therefore finite in the domain of validity of the reduction. A point divergence could appear only by replacing the HSM source with an idealized source of infinite density, which lies outside the domain of the physical HSM configurations considered here. □

High energy concentration and limit of validity. A high energy concentration may increase the amplitude of the source $\mathcal{S}_g$ and therefore strongly deform the effective propagation. However, as long as the configuration remains an HSM source of finite core and finite energy, the theory does not produce a fundamental mechanical singularity of the radial field. This point must be distinguished from a stronger question: determining the complete dynamics of a trapping region in a nonlinear regime. The latter requires a strong closure of the propagation channel and does not follow from the linear Green function alone.

Trapping surface: effective definition. An *effective trapping surface* is a surface Σ such that the outgoing component of the effective propagation speed of the relevant modes vanishes:

$$c_{\text{eff}}^{(+)}\big|_{\Sigma} = 0. \tag{15.28}$$

The interior region is said to be trapped when, with the positive outgoing convention,

$$c_{\text{eff}}^{(+)} < 0. \tag{15.29}$$

These equations constitute a kinematic definition of trapping in the effective reading of the gravitational-like channel. They must not be confused with a singularity of the field r.

Regularity and absence of fundamental metric singularity. In an effective metric reading, a trapping region may be described as an apparent or effective horizon. But this metric is not an object that is fundamental independently of HSM: it is a rewriting of propagation in a gravitational-like background. Consequently, a divergence of the effective metric cannot be taken as a fundamental singularity as long as the underlying HSM variables, their energy densities, and their gravitational-like response remain regular. The effective horizon is a propagation surface, not a point where the theory ceases to be defined.

Status of the result. The result demonstrated in this section is therefore deliberately limited but important:

localized HSM source of finite energy $\quad\Longrightarrow\quad$ regular gravitational-like response

in the domain of validity of the channel considered. This result justifies a regularized reading of effective horizons in the link chapter, but it does not yet constitute a complete global black-hole-type solution. Such a solution would require an explicit nonlinear closure of $c_{\text{eff}}^{(+)}$, an evolution equation for the trapping surface, and a global analysis of the strong regime.

Chapter 16

Electromagnetic-like field (moving solutions)

Conventions and units.

$c_t^2 = \dfrac{T}{\rho}$, $\quad \rho$: *volume* density of inertia $[\mathrm{kg\,m^{-3}}]$, $\quad T$: volumetric tangential tension $[\mathrm{Pa} = \mathrm{N\,m^{-2}} = \mathrm{J\,m^{-3}}]$.

We introduce a dimensionless structure constant,

$$\boxed{\Upsilon_{\mathrm{cell}} \text{ (unique, dimensionless)}}.$$

In what follows, EM-like charges and couplings are expressed in internal HSM mechanical units. The present construction defines only the reduced quantities of the EM-like channel.

16.1 Two normalizations of the EM-like coupling and their dictionary

(A) *micro–mesh* route. The local mechanical scalar source is written

$$\rho_{\mathrm{src}} = g_{\mathrm{HSM}}\, j^t_{\mathrm{micro}}, \qquad j^t_{\mathrm{micro}} = \mathcal{C}_{\mathrm{kin}}\, \rho\, A^2\, \partial_t \phi,$$

with

$$\boxed{g_{\mathrm{HSM}} = \frac{\sigma_{\mathrm{cell}}\, c_t}{\ell_0^2}, \qquad \Upsilon_{\mathrm{cell}} := \sigma_{\mathrm{cell}}\, \mathcal{C}_{\mathrm{kin}}.}$$

Here, ℓ_0 is the mesh scale, σ_{cell} an internal geometric coarse-graining factor, and $\mathcal{C}_{\mathrm{kin}}$ the kinematic factor associated with the micro–mesh reduction.

Canonical mesh normalization (EM-like channel). In the EM-like channel, the factor Υ_{cell} is not a new dynamical constant: it fixes the internal mesh normalization in the micro–mesh route. The canonical normalization retained in HSM is

$$\boxed{\Upsilon_{\mathrm{cell}} = 1.}$$

Justification. This fixing is a theoretical normalization convention. It expresses that, in the canonical mesh normalization, the coefficient which maps the micro–mesh quantity to the effective mechanical quantity is taken to be unity. Any possible deviation would be written

$$\Upsilon_{\mathrm{cell}} = 1 + \delta_{\mathrm{cell}},$$

with δ_{cell} calculable if the mesh normalization or the coarse-graining rules are modified. The non-physical character of this choice will be checked below by the explicit cancellation of the coupling g_Ω in the local static force, cf. §20.3.8.

(B) *Noether/Maxwell-like* route. The $U(1)$ phase symmetry of the 2DOF field generates the conserved current

$$j^\mu = (j^0, \vec{j}), \qquad j^0 = \rho\, \Im(\psi^* \dot{\psi}), \qquad \vec{j} = -T\, \Im(\psi^* \nabla\psi),$$

and an effective linear coupling to a four-potential $A_\mu = (a, \vec{A})$ through a single coefficient g_Ω.

Internal dictionary between the two normalizations. For the circular 2DOF bump, the canonical core profile is $a(\xi) = \operatorname{sech}\xi$, cf. 6.1. The core moments J_0 and J_2 are defined by (6.45), with the values (6.46).

The scalar part of the core density is carried by J_0, whereas the gradient part is carried by J_2. The local reduced norm of the core is therefore

$$J_{\text{tot}} = J_0 + J_2.$$

We then introduce the source-projection constant of the EM-like channel:

$$\boxed{\mathcal{C}_S := \frac{J_0}{J_0 + J_2}.} \tag{16.1}$$

Lemma 16.1 (Elementary source projection of the EM-like channel)**.** *The elementary source projection consumed by an EM-like charge–field vertex is*

$$\boxed{\Pi^{\text{EM}}_{\text{source}} = \mathcal{C}_S = \frac{J_0}{J_0 + J_2}.} \tag{16.2}$$

Proof. The elementary EM-like channel couples to the scalar density carried by the core. The numerator of the source projection is therefore

$$J_{\text{source}} = J_0.$$

The normalization of the core mode at LO contains the scalar part and the gradient part:

$$J_{\text{tot}} = J_0 + J_2.$$

The reduced source projection is the ratio between the source part and the total local norm:

$$\Pi^{\text{EM}}_{\text{source}} = \frac{J_{\text{source}}}{J_{\text{tot}}} = \frac{J_0}{J_0 + J_2}.$$

□

The Noether integral associated with the internal EM-like charge is then

$$S_N = 2\,\mathcal{C}_S\,\mathcal{J},$$

where $\mathcal{J}$ denotes the reduced action of the solution. On the fundamental EBK branch $q = 1$, one has

$$\mathcal{J} = h_{\text{eff}}.$$

We therefore obtain the equality between the two normalizations of the internal EM-like charge:

$$\boxed{Q^{(\text{micro})}_{\text{bump}} = 2h_{\text{eff}}\,\frac{\Upsilon_{\text{cell}}\,c_t}{\ell_0^2}\,\frac{J_0}{J_0 + J_2} = \frac{2h_{\text{eff}}}{g_\Omega}\,\frac{J_0}{J_0 + J_2} = Q^{(\text{Noether})}_{\text{bump}}}.$$

It follows that the internal dictionary is

$$\boxed{g_\Omega = \frac{\ell_0^2}{\Upsilon_{\text{cell}}\,c_t} \qquad \Longleftrightarrow \qquad \frac{\Upsilon_{\text{cell}}}{\ell_0^2} = \frac{1}{g_\Omega\,c_t}.} \tag{16.3}$$

Status of the dictionary. The parameters g_Ω and $\Upsilon_{\rm cell}$ are not two independent constants: they parametrize the same effective coupling in two equivalent normalizations. The circular 2DOF bump provides here an explicit check of this dictionary. At this stage, $Q_{\rm bump}$ is an internal EM-like charge of the model, defined in the mechanical normalization of the channel.

16.2 Maxwell-like on $\mathbb{S}^3$ from the Noether current

Field equations in Lorenz gauge. We work here in the quadratic representation of the EM-like sector already written in Lorenz gauge. With $c_t^2 = T/\rho$, we consider the functional

$$\mathcal{L}_A = \int \Big[\frac{T}{2}(|\nabla a|^2 + |\nabla \vec{A}|^2 - \frac{1}{c_t^2}(|\dot{a}|^2 + |\dot{\vec{A}}|^2)) + g_\Omega\,(a\,j^0 - \vec{A}\cdot\vec{j})\Big] dV,$$

where ∇, $\nabla\cdot$, $\nabla\times$ and $\Delta_{\mathbb{S}^3}$ denote the usual covariant operators on the oriented 3-sphere $\mathbb{S}^3$. In the local limit $d \ll R_0$, they reduce to their Euclidean expressions.

The Lorenz condition is written

$$\boxed{\frac{1}{c_t^2}\partial_t a + \nabla\cdot\vec{A} = 0}.$$

The associated wave equations are then

$$\boxed{\Big(\tfrac{1}{c_t^2}\partial_t^2 - \Delta_{\mathbb{S}^3}\Big) A_\mu = \frac{g_\Omega}{T}\, j_\mu}.$$

Setting

$$\vec{E} := -\nabla a - \partial_t \vec{A}, \qquad \vec{B} := \nabla \times \vec{A},$$

we infer

$$\nabla\cdot\vec{E} = \frac{g_\Omega}{T}\, j^0, \qquad \nabla\times\vec{B} - \frac{1}{c_t^2}\partial_t\vec{E} = \frac{g_\Omega}{T}\,\vec{j}, \qquad \nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{E} + \partial_t\vec{B} = 0.$$

Effective EM sources and constitutive constants. Choose

$$\boxed{\rho_{\rm EM} = \frac{1}{g_\Omega}\, j^0, \qquad \vec{J}_{\rm EM} = \frac{c_t^2}{g_\Omega}\,\vec{j}}. \tag{16.4}$$

We then introduce the effective constitutive constants

$$\boxed{\varepsilon_{\rm eff} = \frac{T}{g_\Omega^2}, \qquad \mu_{\rm eff} = \frac{g_\Omega^2}{T\,c_t^2} = \frac{g_\Omega^2\,\rho}{T^2}}. \tag{16.5}$$

These quantities are internal effective mechanical reparametrizations of the EM-like coupling.

Then

$$c_t^2 = \frac{1}{\mu_{\rm eff}\,\varepsilon_{\rm eff}},$$

and, denoting by $(\rho_{\rm EM}, \vec{J}_{\rm EM})$ the effective sources, the preceding equations are rewritten in a form strictly analogous to Maxwell's equations, with c_t as characteristic speed:

$$\boxed{\nabla\cdot\vec{E} = \frac{\rho_{\rm EM}}{\varepsilon_{\rm eff}}, \qquad \nabla\times\vec{B} - \frac{1}{c_t^2}\partial_t\vec{E} = \mu_{\rm eff}\,\vec{J}_{\rm EM}}, \qquad \nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{E} + \partial_t\vec{B} = 0.$$

Continuity of effective sources. Taking the divergence of the Ampere–Maxwell equation and the time derivative of Gauss's law gives

$$\partial_t \rho_{\mathrm{EM}} + \nabla \cdot \vec{J}_{\mathrm{EM}} = 0.$$

The redefinition (16.4) is therefore consistent with local conservation of the effective EM-like charge.

Local static limit. In local statics ($d \ll R_0$), one recovers the Poisson–Coulomb law

$$\Delta\Phi = -\frac{\rho_{\mathrm{EM}}}{\varepsilon_{\mathrm{eff}}} \quad \Rightarrow \quad \Phi(r) = \frac{Q}{4\pi\,\varepsilon_{\mathrm{eff}}\,r}, \qquad \vec{E}(r) = \frac{Q}{4\pi\,\varepsilon_{\mathrm{eff}}}\,\frac{\hat{r}}{r^2}.$$

Constitutive consistency (LO lock). With (16.4)–(16.5),

$$\varepsilon_{\mathrm{eff}} = \frac{T}{g_\Omega^2}, \qquad \mu_{\mathrm{eff}} = \frac{g_\Omega^2}{T\,c_t^2}, \qquad c_t^2 = \frac{1}{\mu_{\mathrm{eff}}\,\varepsilon_{\mathrm{eff}}},$$

consistently with $c_t^2 = T/\rho$. The dictionary (16.3) ensures independence with respect to the normalization choice (micro–mesh vs Noether).

16.3 Moving solutions: interaction and radiation

In the local window $d \ll R_0$ and in the slow regime

$$|\vec{v}_i| \ll c_t,$$

the Maxwell-like equations of the previous chapter reduce, at quasi-static order, to

$$-\Delta\Phi = \frac{\rho_{\mathrm{EM}}}{\varepsilon_{\mathrm{eff}}}, \qquad -\Delta\vec{A} = \mu_{\mathrm{eff}}\,\vec{J}_{\mathrm{EM}}, \qquad c_t^2 = \frac{1}{\mu_{\mathrm{eff}}\,\varepsilon_{\mathrm{eff}}}.$$

For a point source Q_2 moving with velocity $\vec{v}_2$, we write

$$\rho_{\mathrm{EM},2}(\vec{x}) = Q_2\,\delta(\vec{x} - \vec{x}_2), \qquad \vec{J}_{\mathrm{EM},2}(\vec{x}) = Q_2\,\vec{v}_2\,\delta(\vec{x} - \vec{x}_2).$$

The local potentials created by this source are then

$$\Phi_2(d) = \frac{Q_2}{4\pi\,\varepsilon_{\mathrm{eff}}\,d}, \qquad \vec{A}_2(d) = \frac{\mu_{\mathrm{eff}}\,Q_2}{4\pi\,d}\,\vec{v}_2.$$

For a second source Q_1 moving with velocity $\vec{v}_1$, the local interaction term is written

$$L_{\mathrm{int},1} = Q_1\,\vec{v}_1 \cdot \vec{A}_2 - Q_1\,\Phi_2.$$

The associated generalized potential is therefore

$$V_{12}(d) = Q_1\,\Phi_2 - Q_1\,\vec{v}_1 \cdot \vec{A}_2.$$

Substituting the preceding expressions, one obtains

$$\boxed{V_{12}(d) \simeq \frac{1}{4\pi\varepsilon_{\mathrm{eff}}}\,\frac{Q_1 Q_2}{d}\left[1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c_t^2} + \mathcal{O}\left(\frac{v^2}{c_t^2}\right)\right].} \tag{16.6}$$

This expression is the local Coulomb form plus the dominant magnetostatic Biot–Savart-type correction. It corresponds to the isotropic order of the slow expansion; the full angular structure of the retarded kernel is not required here.

Radiation. The Maxwell-like wave equations imply, in the same local window, the retarded potentials

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_{\text{eff}}} \int \frac{\rho_{\text{EM}}(\vec{x}',\, t - \frac{|\vec{x}-\vec{x}'|}{c_t})}{|\vec{x}-\vec{x}'|}\, d^3x',$$

$$\vec{A}(\vec{x},t) = \frac{\mu_{\text{eff}}}{4\pi} \int \frac{\vec{J}_{\text{EM}}(\vec{x}',\, t - \frac{|\vec{x}-\vec{x}'|}{c_t})}{|\vec{x}-\vec{x}'|}\, d^3x'.$$

Any acceleration ($\dot{\vec{v}} \neq 0$) therefore generates retarded fields propagating at speed c_t. At the dominant multipolar order, the radiation scheme is dipolar.

16.4 Internal transverse fields (pure HSM)

Here we introduce the internal transverse fields of the HSM support. This construction is internal to the transverse mechanical sector; it must not be confused with the four-potential $(a, \vec{A})$ of the EM-like channel. Its explicit matching to the effective charge of the EM-like channel is treated in the electromagnetic-like closure.

We denote by $\mathbf{A}_{\tan}$ the internal transverse field of the support, and by $\nabla_{\tan}$ the tangential covariant derivation associated with the local geometry of the support. We set

$$\mathbf{E}_{\text{HSM}} := -\partial_t \mathbf{A}_{\tan}, \qquad \mathbf{B}_{\text{HSM}} := \nabla_{\tan} \times \mathbf{A}_{\tan}, \qquad \mathcal{E}_{\text{HSM}} = \tfrac{1}{2}\,\rho\,|\partial_t \mathbf{A}_{\tan}|^2 + \tfrac{1}{2}\,T\,|\nabla_{\tan}\mathbf{A}_{\tan}|^2.$$

16.4.1 Internal mechanical units (static)

Field energy.

$$\mathcal{E} = \tfrac{1}{2}\,\rho\,|\partial_t \mathbf{A}_{\tan}|^2 + \tfrac{1}{2}\,T\,|\nabla_{\tan}\mathbf{A}_{\tan}|^2, \qquad c_t^2 = \frac{T}{\rho}. \tag{16.7}$$

Internal normalization parameters. We introduce, for the internal transverse sector only, the quantities

$$\varepsilon_{\text{mech}} := \rho, \qquad \frac{1}{\mu_{\text{mech}}} := T, \qquad \varepsilon_{\text{mech}}\mu_{\text{mech}} = \frac{1}{c_t^2}. \tag{16.8}$$

These *internal* parameters must not be confused with the effective constitutive constants $(\varepsilon_{\text{eff}}, \mu_{\text{eff}})$ of the EM-like channel introduced above.

Scale invariance of the internal potential. For any dimensionless factor $\kappa_A^{\text{int}} > 0$,

$$\mathbf{A}_{\tan} \mapsto \kappa_A^{\text{int}}\, \mathbf{A}_{\tan}, \qquad \varepsilon_{\text{mech}} \mapsto \frac{\varepsilon_{\text{mech}}}{(\kappa_A^{\text{int}})^2}, \qquad \mu_{\text{mech}}^{-1} \mapsto \frac{T}{(\kappa_A^{\text{int}})^2},$$

the form (16.7) remains unchanged and

$$\varepsilon_{\text{mech}}\mu_{\text{mech}} = \frac{1}{c_t^2}$$

is conserved. The canonical convention retained is

$$\varepsilon_{\text{mech}} = \rho, \qquad \mu_{\text{mech}}^{-1} = T.$$

Chapter 17

Preclusion field (local mechanical exclusion)

17.1 Positioning and assumptions (LO)

The *preclusion* channel is not mediated by an additional field: it emerges directly from the **HSM variational equations** applied to localized solutions (2DOF/3DOF), through the core non–linearity and the phase-compatibility constraints of the transverse $U(1)$ sector. The relevant mechanism is an overlap of localized cores under a phase constraint, which increases the energy of the system and induces an effective short-range repulsion.

We work at LO in the regime

$$\lambda/R_0 \ll 1, \qquad c_t^2 = \frac{T}{\rho},$$

where the localized solutions are described on a quasi–planar cap.

Transverse 2DOF variables. We recall the internal variables of the transverse 2DOF sector:

$$\psi = A\, e^{i\phi}, \qquad \mathbf{E}_{\mathrm{HSM}} = -\partial_t \mathbf{A}_{\tan}, \qquad \mathbf{B}_{\mathrm{HSM}} = \nabla_{\tan} \times \mathbf{A}_{\tan},$$

with energy density

$$\mathcal{E}_{\mathrm{HSM}} = \tfrac{1}{2}\,\rho\,|\partial_t \mathbf{A}_{\tan}|^2 + \tfrac{1}{2}\,T\,|\nabla_{\tan}\mathbf{A}_{\tan}|^2, \qquad c_t^2 = \frac{T}{\rho}.$$

These quantities enter here as internal variables of the transverse core; the preclusion channel itself will be obtained by a variational analysis of the energy under overlap.

Reference core profile. At LO, the radial core is described by a profile $a(\xi)$, $\xi = r/\lambda$, satisfying

$$a(0) = 1.$$

For the reference 2DOF bump solution, the canonical profile is

$$a(\xi) = \operatorname{sech}\xi,$$

cf. 6.1. The other families are treated according to their own profile in the sections or chapters devoted to them.

17.2 Range asymptotics and LO form

At the edge of the core, the amplitude decreases as

$$A(x) \sim A_0 \exp\Big[- \frac{\mathrm{dist}(x, x_k)}{\xi} \Big], \qquad \xi = \sqrt{\frac{T}{\mu_\psi}},$$

where μ_ψ denotes the coefficient induced by the local linearization of the amplitude channel around an isolated core. This is not a new fundamental coupling, but the effective mass parameter associated with the decay of the linearized profile:

$$(-T\,\Delta_{S^3} + \mu_\psi)\, a \simeq 0.$$

In the quasi–planar cap approximation (local 3D geometry), integration of the local gradient excess cost leads, for $d \gg \xi$, to a finite-range repulsive asymptotic form

$$\boxed{V_{\mathrm{P}}(d) \;\simeq\; + \frac{C_{\mathrm{P}}}{d}\, e^{-\,d/\xi}, \qquad d \gg \xi.} \tag{17.1}$$

The prefactor $C_{\mathrm{P}} > 0$ involves no new fundamental coupling: it is entirely fixed by the structure of the core, the gradient cost, and the local geometry of the overlap.

17.3 LO effective form of the channel (without closure)

In the weak-overlap regime and at negligible relative velocity, the LO preclusion interaction energy is entirely governed by: (i) the gradient cost, (ii) the pure overlap factor Γ_{P}, and (iii) the overlap functions of the local profiles. It reproduces the finite-range asymptotic form

$$V_{\mathrm{P}}(d) \simeq + \frac{C_{\mathrm{P}}}{d}\, e^{-d/\xi}$$

given in (17.1).

The detailed closure in terms of overlap functions $\mathcal{O}_{\mathrm{P}}(\cdot)$, reduced separations, and selection Δ_* is deferred to the chapters devoted to composite solutions, where these objects are effectively consumed.

17.4 Validity, channel separation, and remarks

- **Without a new fundamental coupling**: V_{P} results only from the HSM terms already present (core non–linearity and gradient cost).
- **Local range**: the form (17.1) holds for $d \gg \xi$; at shorter range, the repulsion remains governed by the local energy excess cost associated with the overlap.
- **Independence of R_0 at LO**: in the quasi–planar cap approximation, no explicit dependence on R_0 remains; curvature corrections are of order

$$\mathcal{O}\left(\frac{\lambda^2}{R_0^2}\right).$$

- **Channel separation**: the preclusion closure of this chapter involves no additional EM-like or grav-like coupling; it relies only on the local structure of the cores and on their variational overlap.

Chapter 18

Strong-like force (finite-range attractive interaction)

18.1 Positioning and assumptions (LO)

The *strong-like* channel is associated with the tangential **3DOF** dynamics of localized cores (3DOF bump family) and with their internal conical structure. It introduces no additional fundamental field: at LO, it emerges directly from the HSM transverse Lagrangian after reduction on the relevant 3DOF component of the core. We work in the quasi-plane cap regime,

$$\lambda/R_0 \ll 1, \qquad c_t^2 = \frac{T}{\rho}.$$

Channel projection (internal color). We introduce a unit internal channel direction

$$\boldsymbol{\pi}^{(\mathsf{col})} \in \mathbb{R}^3, \qquad \|\boldsymbol{\pi}^{(\mathsf{col})}\| = 1,$$

which selects, within the 3DOF tangential sector of the core, the internal component responsible for the strong-like coupling. The associated projector is

$$P^{(\mathsf{col})} := \boldsymbol{\pi}^{(\mathsf{col})} \otimes \boldsymbol{\pi}^{(\mathsf{col})}.$$

This component will be called, throughout this chapter, the *color* component of the 3DOF mode. The adopted normalization is the canonical zonal normalization of the core, identical to that used for the other channels, with no additional global factor.

18.2 LO equation and range Green function on $\mathbb{S}^3$

After projection of the 3DOF tangential sector on the channel direction $\boldsymbol{\pi}^{(\mathsf{col})}$, we denote by Φ_{s} the effective scalar amplitude of the corresponding strong-like mode. Under local isotropy and at LO, Φ_{s} satisfies

$$\rho\,\ddot{\Phi}_{\mathsf{s}} - T\,\Delta_{\mathbb{S}^3}\Phi_{\mathsf{s}} + \mu_{\mathsf{s}}\,\Phi_{\mathsf{s}} = 0, \qquad \lambda_{\mathsf{s}} := \sqrt{\frac{T}{\mu_{\mathsf{s}}}}.$$

Here, μ_{s} denotes the effective mass coefficient of the internal channel s. In the present chapter, μ_{s} is treated as an effective coefficient of the channel.

The associated zonal Green function $G_{\mathsf{s}}(x,y)$ satisfies

$$(-\Delta_{\mathbb{S}^3} + \lambda_{\mathsf{s}}^{-2})\,G_{\mathsf{s}}(x,y) = \delta_{\mathbb{S}^3}(x,y),$$

and, in the local limit $d(x,y) \ll R_0$,

$$G_{\mathsf{s}}(d) \;\simeq\; \frac{e^{-d/\lambda_{\mathsf{s}}}}{4\pi d}\,. \tag{18.1}$$

This local form is that of a finite-range massive kernel.

In every passage from the microscopic to the macroscopic description, the average on $\mathbb{S}^3$ (volume $2\pi^2 R_0^3$) is performed only once, in order to avoid any double counting.

18.3 Strong-like channel charge (canonical form, without closure)

The *strong-like charge* I_{s} of a localized solution is the effective scalar quantity associated with the 3DOF component of the core selected by the channel direction $\boldsymbol{\pi}^{(\mathsf{col})}$ (or, equivalently, by the associated projector $P^{(\mathsf{col})}$). In other words, the strong-like channel does not couple to the whole 3DOF tangential sector, but only to its projection on the internal direction $\boldsymbol{\pi}^{(\mathsf{col})}$.

At LO, I_{s} is entirely determined by the internal structure of the core: radial profile, conical geometry, and core moments (J_0, J_2). It can therefore be written in the form

$$I_{\mathsf{s}} = \mathcal{K}_{\mathsf{s}}\, \Pi_{\mathsf{col}}[\text{3DOF core}], \tag{18.2}$$

where $\Pi_{\mathsf{col}}[\text{3DOF core}]$ denotes the mechanical projection of the core on the channel $\boldsymbol{\pi}^{(\mathsf{col})}$, and where $\mathcal{K}_{\mathsf{s}} > 0$ is a *pure number* depending only on the core shape factors and on the conical geometry. No external coupling constant is introduced.

In this chapter, only the *canonical form* of I_{s} is required: The coefficient K_{s} is treated here as an internal shape factor of the channel. The present chapter fixes the canonical structure of the internal charge, without introducing a numerical value for K_{s}.

18.4 LO interaction potential (form)

For two localized objects (3DOF bumps) with strong charges $I_{\mathsf{s}}^{(1)}$ and $I_{\mathsf{s}}^{(2)}$, at relative rest ($|\vec{v}_i| \ll c_t$), the inheritance of the Green function (18.1) leads, in the attractive branch of the channel, to a short-range potential of the form

$$\boxed{V_{\mathsf{s}}(d) \;\simeq\; -\,\mathcal{C}_{\mathsf{s}}\,\frac{I_{\mathsf{s}}^{(1)}\, I_{\mathsf{s}}^{(2)}}{4\pi}\,\frac{e^{-d/\lambda_{\mathsf{s}}}}{d}} \qquad (d \ll R_0), \tag{18.3}$$

where $\mathcal{C}_{\mathsf{s}} > 0$ is a *pure number* of channel normalization (shape factors and gauge prescription), independent of (ρ, T, ℓ_0, R_0) once the channel is fixed.

At very short range, this attraction is regularized by the universal core repulsion of the Preclusion channel; saturation of the binding belongs to the dedicated chapter.

Channel branch factor. The strong-like channel couples only to the 3DOF "color" component extracted by $\boldsymbol{\pi}^{(\mathsf{col})}$. For a multi-constituent system, the effective sign of the coupling is encoded by a dimensionless branch factor

$$\mathfrak{B}_{\mathrm{col}} \in \{-1, 0, +1\},$$

acting on the effective charges according to

$$I_{\mathsf{s}}^{(i)} \;\mapsto\; \mathfrak{B}_{\mathrm{col}}^{(i)}\, I_{\mathsf{s}}^{(i)}.$$

We thus distinguish the three cases

$\mathfrak{B}_{\rm col} = +1$ (attractive branch), $\mathfrak{B}_{\rm col} = 0$ (uncoupled branch), $\mathfrak{B}_{\rm col} = -1$ (repulsive branch).

In what follows, we restrict ourselves to the attractive branch, for which the strong-like contribution is effectively binding. The Preclusion core repulsion then ensures saturation at very short range.

Chapter 19

Weak-like channel: existence and minimal evidence from the 2DOF NSL

19.1 Positioning and objective

Framework. This chapter is written within the proper framework of HSM, at the internal effective level of the mesh, without relying on an external identification or on a phenomenological interpretation. Its objective is not to establish already a complete closure of the weak-like sector, but to show that from the sole 2DOF NSL structure built in HSM there exists a natural internal foothold for introducing a distinct mesh channel, of finite range, having the minimal status of an effective field.

Nature of the targeted result. Here we seek an *existence* result in the following sense:

1. identify, from the 2DOF NSL, a minimal candidate for the channel source density;
2. show that it is natural to associate with this source a proper response variable of the mesh;
3. verify that such a response induces an effective interaction between two well-separated NSL profiles;
4. extract the minimal structural features of the channel thus exhibited.

Scope of the chapter. The chapter deliberately remains below:

- a complete closure on the fundamental constants of HSM;
- a detailed theory of the couplings of this channel to the other solutions;
- an analysis of its fine structure beyond the minimal approximation retained here.

We establish only that a weak-like sector *can be introduced internally and in a motivated way* in HSM from the 2DOF NSL.

19.2 Primitive input: the 2DOF NSL in HSM

Single input of the chapter. The only primitive datum used in this chapter is the 2DOF NSL solution built in HSM, cf. §6.4, and more precisely §6.4.1, §6.4.2 and §6.4.3. No other structure is assumed.

Tangential structure. At LO, the 2DOF NSL is a tangential configuration with two orthogonal legs, iso-frequency, sharing the same temporal carrier. It can be summarized in the form

$$\vec{u}_{\mathrm{NSL}}(s,\vartheta,t) = A(s)\Big[\cos\vartheta\,\mathbf{q}(t) + \sin\vartheta\,J\mathbf{q}(t)\Big],$$

where $\mathbf{q}(t)$ and $J\mathbf{q}(t)$ are two tangential directions, orthonormal at each instant, in quadrature in the transverse plane, and $A(s)$ is a localized envelope.

Zonal quadratic invariant. The decisive property, already used in HSM, is then the exact identity

$$|\vec{u}_{\mathrm{NSL}}(s,\vartheta,t)|^2 = A(s)^2,$$

since

$$|\mathbf{q}(t)| = |J\mathbf{q}(t)| = 1, \qquad \mathbf{q}(t)\cdot J\mathbf{q}(t) = 0.$$

The quadratic invariant therefore depends neither on ϑ nor on the fast time, but only on the radial profile $A(s)$.

Structural role. This property is not accessory: it is what makes the zonal radial response $r = r(s)$ compatible in the construction of the NSL. The two-leg structure of the NSL is therefore not a representational redundancy; it constitutes the internal minimum needed to produce a quadratic density closed at the radial level.

Reading retained here. In the present chapter, we do not yet seek to exploit the full fine structure of the underlying tangential doublet. We retain only the following fact: HSM already contains, through the 2DOF NSL, a nontrivial primitive structure whose radially closed quadratic invariant provides a natural candidate for a channel source.

19.3 Minimal source and candidate effective charge

Minimal source density. The previous property naturally leads to introducing, for the 2DOF NSL, the minimal source density

$$\boxed{\rho_W^{(\mathrm{NSL})}(s) := |\vec{u}_{\mathrm{NSL}}(s,\vartheta,t)|^2 = A(s)^2.}$$

Status of $\rho_W^{(\mathrm{NSL})}$. At this stage, $\rho_W^{(\mathrm{NSL})}$ is not yet postulated as the final source of a closed sector. It is introduced as a *minimal candidate* for the source density, selected directly by the internal structure of the NSL and already seen by the mesh in the radial closure built in HSM.

Why this choice is natural. This choice is the most economical possible:

1. it introduces no structure foreign to HSM;
2. it coincides with the quadratic invariant already mobilized in the analysis of the NSL;
3. it constitutes the first level at which an interaction between two NSL profiles can be tested.

Candidate effective charge. If one already wishes to prepare a *field + charge* writing, the simplest global quantity associated with $\rho_W^{(\mathrm{NSL})}$ is

$$q_W^{\mathrm{eff}}[\mathrm{NSL}] := \int_{\mathbb{S}^3} \rho_W^{(\mathrm{NSL})}\, dV = \int_{\mathbb{S}^3} A(s)^2\, dV.$$

At this stage, q_W^{eff} is still only a *candidate effective charge*: it measures the global intensity with which the NSL structure sources the channel considered at the minimal level.

Approximation retained. In everything that follows, we deliberately restrict ourselves to this minimal scalar approximation carried by A^2. The fine two-leg structure will be reintroduced only later.

19.4 Associated effective response field

Motivation. If $\rho_W^{(\mathrm{NSL})}$ is to be interpreted as the source of a proper channel of the mesh, it is natural to associate an effective response variable with this source. We therefore introduce an effective scalar field Φ_W, understood here as the minimal mesh response to the channel carried by the NSL.

Abbreviated notation. Throughout this section, we set

$$\rho_W := \rho_W^{(\mathrm{NSL})}.$$

Minimal effective equation. The simplest form compatible with a finite range is

$$\boxed{\mathcal{L}_W\, \Phi_W = g_W^{\mathrm{eff}}\, \rho_W,}$$

where

- $\mathcal{L}_W$ is the effective operator of the weak-like channel,
- g_W^{eff} is the effective coupling coefficient of the channel,
- ρ_W is the minimal source density.

Minimal static model. At the local static level, in the quasi-plane approximation, $\mathcal{L}_W$ can be represented by an operator of the type

$$\mathcal{L}_W = -\Delta_{\mathrm{loc}} + \ell_W^{-2},$$

with $\ell_W > 0$, whence

$$(-\Delta_{\mathrm{loc}} + \ell_W^{-2})\Phi_W = g_W^{\mathrm{eff}}\, \rho_W.$$

This writing is not introduced as the final law of the sector, but as a minimal effective form motivated by the finite-range behavior already suggested by the NSL structure and its radial coupling.

Link with the existing HSM structure. This introduction is not arbitrary: in HSM, the radial channel already responds to the quadratic invariant of the NSL, then reinjects onto the tangential sector. The variable Φ_W here merely isolates, in the form of a proper effective field, the minimal mesh response associated with this structure.

Conceptual scope. The preceding writing shows that an existence chapter can already identify the conceptual form

$$\text{response field} \iff g_W^{\text{eff}} \times \rho_W.$$

In other words, the weak-like sector already admits, at the minimal level, an effective description involving:

- a candidate source density ρ_W,
- a candidate effective charge q_W^{eff},
- an effective channel coefficient g_W^{eff}.

The closure of these quantities on the fundamental constants of HSM belongs to a distinct step.

19.5 Minimal interaction between two NSL sources

Two well-separated profiles. Consider two well-separated 2DOF NSL solutions, denoted A and B, with minimal source densities

$$\rho_W^{(A)}(x) = A_A(x)^2, \qquad \rho_W^{(B)}(y) = A_B(y)^2.$$

Green kernel. Let $G_W = \mathcal{L}_W^{-1}$ be the Green kernel of the channel. One then has formally

$$\Phi_W = g_W^{\text{eff}}\, G_W * \rho_W.$$

Minimal cross energy. At the minimal effective level, the interaction between two NSL sources then takes the form

$$\boxed{E_{\text{int}}^{(W)}(A,B) \sim -\, g_W^{\text{eff}\,2} \iint_{\mathbb{S}^3\times\mathbb{S}^3} \rho_W^{(A)}(x)\, G_W(x,y)\, \rho_W^{(B)}(y)\, dV_x\, dV_y.}$$

Replacing $\rho_W^{(A)}$ and $\rho_W^{(B)}$ by A_A^2 and A_B^2, we obtain

$$E_{\text{int}}^{(W)}(A,B) \sim -\, g_W^{\text{eff}\,2} \iint_{\mathbb{S}^3\times\mathbb{S}^3} A_A(x)^2\, G_W(x,y)\, A_B(y)^2\, dV_x\, dV_y.$$

Link with the internal coupling chain. This form is consistent with the response chain already present in HSM:

$$A^2 \longrightarrow r \longrightarrow A.$$

The introduction of the effective field Φ_W therefore does not change the internal logic; it simply reformulates it in an explicit *source–field–interaction* form.

Range. As soon as $\ell_W^{-2} > 0$, the kernel G_W is of gapped type. The interaction thus defined is therefore of finite range, with characteristic range ℓ_W. This is the essential structural property of the channel sought.

Remark on the sign. In the minimal regime where G_W is positive, the preceding structure leads to an attractive interaction at the scalar level. This reading belongs to the minimal scalar approximation; the fine two-leg structure is reintroduced in the closure of the channel.

19.6 Structural features of the weak-like channel

Object of the section. The aim here is to extract, from the preceding construction, the minimal structural features of the channel thus introduced, without leaving the proper framework of HSM.

Finite range. At the minimal level, the channel is governed by a gapped operator. It therefore induces a finite-range interaction between sources. The finite range is not added from outside: it is required by the very form of the effective response associated with the NSL.

Non-free character. Unlike a free tangential sector determined by a simple propagative excitation without a mass scale, the channel introduced here does not appear as a gapless free mode. On the contrary, it is attached to a proper scale and to a specific source structure.

Specific source. The channel is not sourced by a generic tangential rotation, but by the particular structure of the 2DOF NSL, that is, by a two-leg configuration whose quadratic invariant closes in the form A^2.

Suggested internal structure. Even though we work here in the minimal scalar approximation, the origin of the source $\rho_W = A^2$ recalls that the channel in fact comes from a tangential structure with two components. The minimal weak-like field is therefore not suggested by a simple ordinary tangential wave, but by a richer internal organization.

Minimal effective status. The fact that it is natural to associate with ρ_W a response field Φ_W, a coefficient g_W^{eff}, and an interaction between sources shows that the introduced channel already has a proper effective status at the minimal level in the mesh. Its complete closure and fine structure belong to a later step.

19.7 Effective "constant + charge" form and closure perspective

Effective form obtained. The existence chapter already leads to a writing of the type

$$\boxed{\mathcal{L}_W \, \Phi_W = g_W^{\mathrm{eff}} \, \rho_W,}$$

with

$$\rho_W^{(\mathrm{NSL})} = A^2, \qquad q_W^{\mathrm{eff}}[\mathrm{NSL}] = \int \rho_W^{(\mathrm{NSL})} \, dV.$$

Thus, at the minimal level, we already have an explicit structure of the type *field constant + charge/source*.

Status. At this stage:

- g_W^{eff} is an effective channel constant,
- ρ_W is a candidate source density,
- q_W^{eff} is a candidate effective charge.

Their final identification is not part of the present chapter.

Closure perspective. The natural question that follows is then the following:

> *can g_W^{eff}, q_W^{eff}, and the proper range ℓ_W of the channel be expressed only from the fundamental constants of HSM?*

This question is treated in the chapter on weak-like-sector closures, cf. §20.5.

Role of the present step. The role of the present chapter is therefore preparatory but decisive: it does not yet close the sector, but it already identifies its general form, which makes possible the closure given further below in the chapter on field closures.

19.8 Scope of the result and later developments

What is established. The present chapter establishes, within the proper framework of HSM, the following points:

1. the 2DOF NSL provides a natural minimal candidate for the source density $\rho_W^{(\text{NSL})} = A^2$;
2. this source motivates the introduction of an effective response field Φ_W specific to the channel;
3. this field can be written at the minimal level in the form $\mathcal{L}_W \Phi_W = g_W^{\text{eff}} \rho_W$;
4. it induces a finite-range effective interaction between two well-separated NSL profiles.

Outside the perimeter of the existence chapter. The present chapter does not treat:

1. the fine structure of the tangential doublet underlying the NSL;
2. the possible dependence of the interaction on the relative orientation of the two legs;
3. the detailed coupling of the other solutions to this channel;
4. the complete closure of g_W^{eff}, q_W^{eff}, and the proper range of the channel on the fundamental constants of HSM, treated in the chapter on closures, cf. §20.5;
5. the existence of a free mode or of a proper localized representative of the channel beyond the NSL as primitive source.

Natural continuation. The immediate continuation consists in refining the minimal structure $\rho_W = A^2$ by reintroducing the proper contribution of the two legs of the NSL, in order to determine whether the channel introduced here possesses an internal organization richer than a simple effective scalar sector. Once this structure is better understood, one can address the closure of the triplet $(g_W^{\text{eff}}, q_W^{\text{eff}}, \ell_W)$—or, equivalently, of the pair $(g_W^{\text{eff}}, q_W^{\text{eff}})$ and of the proper range of the channel—on the fundamental constants of HSM.

Conclusion. Within the limited but precise framework of an existence chapter, the targeted objective is reached: the 2DOF NSL built in HSM provides a sufficient internal basis for introducing a minimal effective weak-like field, equipped with a candidate source, an effective constant, and a proper finite-range interaction in the mesh.

Chapter 20

Field closures on the fundamental constants

Notation convention for profiles. In this chapter, we conceptually distinguish the physical tangential profile A_{phys}, with dimension of length, from the dimensionless reduced profile a. When the compact notation uses A, the context specifies whether one is working with the physical field $A_{\mathrm{phys}} = A_0 a$ or with reduced profile a. The variational expressions of the gravitational-like channel use the physical field, whereas the shape constants J_0, J_2, J_4 are computed with the normalized reduced profile $a(\xi) = \operatorname{sech} \xi$.

20.1 Common spectral scheme inherited from the HSM closure

The common spectral constants used by the channels defined on $\mathbb{S}^3$ are closed upstream in the base theory. In particular, the elementary spectral cell of the plateau, the zonal filters $F_\ell^{(\mathrm{cell})}$, the spectral weights a_ℓ, the regularized LO closure $b_0^{(2)}$, as well as the associated constants Δ_* and κ_ω, are defined upstream in HSM by

$$(6.47), \quad (6.48), \quad (6.51), \quad (6.55), \quad (6.57), \quad (6.63), \quad (6.65).$$

This part does not redefine these constants; it uses them as common theoretical closures.

20.1.1 EBK windowing and canonical zonal weight

For the zonal analysis of the channels at LO, we introduce an EBK window (ℓ_q, σ_q), where ℓ_q denotes the spectral center and σ_q the effective width of the harmonic selection.

The canonical zonal weight on $\mathbb{S}^3$ is fixed by the projection onto the Gegenbauer polynomials $C_\ell^{(1)}$ and their orthogonality relation:

$$\int_0^\pi C_\ell^{(1)}(\cos\chi)\, C_m^{(1)}(\cos\chi)\, \sin^2\chi \, d\chi = \frac{\pi}{2}\, \delta_{\ell m}. \tag{20.1}$$

It follows that the canonical zonal normalization is

$$\boxed{W_\ell = \frac{2}{\pi}.} \tag{20.2}$$

The spectral window is taken in the L^2 norm:

$$G_\ell(\ell_q, \sigma_q) = \frac{\exp\!\big(-\frac{(\ell-\ell_q)^2}{2\,\sigma_q^2}\big)}{\Big(\sum_{\ell'=0}^{\infty} \exp\!\big(-\frac{(\ell'-\ell_q)^2}{\sigma_q^2}\big)\Big)^{1/2}}, \qquad \sum_{\ell=0}^{\infty} G_\ell(\ell_q, \sigma_q)^2 = 1. \tag{20.3}$$

20.1.2 LO zonal kernels of the EM-like and Preclusion channels

The spectral couplings consumed by composites come from the LO zonal kernels of the two active channels: the EM-like channel and the preclusion channel. These kernels are functions of the geodesic angle $\chi \in [0, \pi]$ on $\mathbb{S}^3$. Their Gegenbauer coefficients are then read by the canonical projection of §20.1.1.

Reduced core density. In the quasi–flat core regime, the canonical profile is used:

$$a(\xi) = \operatorname{sech} \xi,$$

and the reduced core density is defined by

$$u(\xi) := a(\xi)^2. \tag{20.4}$$

Its quadratic normalization is

$$J_0 = \int_{\mathbb{R}^3} u(|\boldsymbol{\xi}|)\, d^3\boldsymbol{\xi} = \int_0^\infty 4\pi\, \xi^2\, a(\xi)^2\, d\xi. \tag{20.5}$$

For two centers separated by the reduced angle χ, we write

$$u_0(\boldsymbol{\xi}) = u(|\boldsymbol{\xi}|), \qquad u_\chi(\boldsymbol{\xi}) = u(|\boldsymbol{\xi} - \chi\, \mathbf{e}_z|)\,,$$

in the local LO chart. This writing is the quasi–plane version of the zonal separation on $\mathbb{S}^3$.

Preclusion kernel. The preclusion channel measures the local overlap of two cores under a phase compatibility constraint. Its raw zonal kernel is

$$\mathcal{C}_{\mathrm{P}}(\chi) := \frac{4}{J_0^2} \int_{\mathbb{R}^3} u_0(\boldsymbol{\xi})\, u_\chi(\boldsymbol{\xi})\, d^3\boldsymbol{\xi}. \tag{20.6}$$

With $\chi = 0$, one recovers

$$\mathcal{C}_{\mathrm{P}}(0) = \frac{4}{J_0^2} \int_{\mathbb{R}^3} a(|\boldsymbol{\xi}|)^4\, d^3\boldsymbol{\xi} = \Gamma_{\mathrm{P}},$$

where Γ_{P} is defined by (20.59) and evaluated by (20.60). The preclusion kernel consumed by the zonal projections is therefore

$$\boxed{K_{\mathrm{P}}(\chi) := \mathcal{C}_{\mathrm{P}}(\chi).} \tag{20.7}$$

It is positive, finite-range in the sense of core overlap, and entirely determined by the profile $a = \operatorname{sech}$.

Static Green function of the EM-like channel. The static EM-like channel is defined by the reduced zonal Green function $\mathcal{G}_{\mathrm{EM}}$ of the Coulomb operator on $\mathbb{S}^3$, with subtraction of the constant mode:

$$-\Delta_{\mathbb{S}^3,x}\, \mathcal{G}_{\mathrm{EM}}\big(d(x,y)\big) = \delta_{\mathbb{S}^3}(x,y) - 1, \qquad \langle \mathcal{G}_{\mathrm{EM}} \rangle_{\mathbb{S}^3} = 0. \tag{20.8}$$

In the local limit $d(x,y) \ll R_0$, this Green function satisfies

$$\mathcal{G}_{\mathrm{EM}}(d) \simeq \frac{1}{4\pi d}, \tag{20.9}$$

which reproduces the local Coulomb law of the EM-like channel. The subtraction of the constant mode does not modify the local gradients; it only fixes the global compatibility on the compact sphere.

EM-like kernel. The raw EM-like zonal kernel is obtained by convolving two core densities with the static Green function of the channel:

$$\mathcal{C}_{\mathrm{EM}}(\chi) := \frac{1}{J_0^2} \int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3} u_0(\boldsymbol{\xi})\, \mathcal{G}_{\mathrm{EM}}(|\boldsymbol{\xi}-\boldsymbol{\eta}|)\; u_\chi(\boldsymbol{\eta})\, d^3\boldsymbol{\xi}\, d^3\boldsymbol{\eta}. \tag{20.10}$$

The active EM-like closure constant is Γ_{EM}, defined by (20.36). The EM-like kernel is therefore normalized by its central value:

$$\boxed{K_{\mathrm{EM}}(\chi) := \Gamma_{\mathrm{EM}}\, \frac{\mathcal{C}_{\mathrm{EM}}(\chi)}{\mathcal{C}_{\mathrm{EM}}(0)}.} \tag{20.11}$$

Thus,

$$K_{\mathrm{EM}}(0) = \Gamma_{\mathrm{EM}}.$$

The kernel K_{EM} carries the spatial form of the static EM-like response, whereas Γ_{EM} carries the active two-projection closure of the channel. The constant Γ_{EM} is a field constant: it is defined before any projection onto a particular composite solution. The width, orientation, or projection compatibility factors specific to a composite therefore belong to the field–solution coupling, and not to the definition of the EM-like field.

Gegenbauer projections. The canonical zonal coefficients are defined by

$$\boxed{\widehat{G}_\ell^{(\mathrm{EM})} = \frac{2}{\pi}\, \frac{1}{\ell+1} \int_0^\pi K_{\mathrm{EM}}(\chi)\, C_\ell^{(1)}(\cos\chi)\, \sin^2\chi\, d\chi.} \tag{20.12}$$

and

$$\boxed{\widehat{G}_\ell^{(\mathrm{P})} = \frac{2}{\pi}\, \frac{1}{\ell+1} \int_0^\pi K_{\mathrm{P}}(\chi)\, C_\ell^{(1)}(\cos\chi)\, \sin^2\chi\, d\chi.} \tag{20.13}$$

These two coefficients are pure numbers. They can be regenerated by direct quadrature from (20.7) and (20.11).

Windowed versions. The windowed versions used in the EBK calculations are

$$\widehat{G}_{\ell,q}^{(\mathrm{EM})} = G_\ell(\ell_q, \sigma_q)\, \widehat{G}_\ell^{(\mathrm{EM})}, \qquad \widehat{G}_{\ell,q}^{(\mathrm{P})} = G_\ell(\ell_q, \sigma_q)\, \widehat{G}_\ell^{(\mathrm{P})}. \tag{20.14}$$

The non-windowed coefficients $\widehat{G}_\ell^{(\mathrm{EM})}$ and $\widehat{G}_\ell^{(\mathrm{P})}$ are the objects consumed when a harmonic mediator L is selected directly. The windowed coefficients are used when the reading involves a complete EBK window.

20.2 Gravitational-like closure

20.2.1 Reduction of the gravitational-like charge

We recall here the LO reduction of Q_g, because it is the quantity that will be reinserted into the energy closure leading to the effective interaction.

Channel-reading convention. In this channel, the variations are density variations read on the reference measure of $\mathbb{S}^3$. They do not coincide with the local covariant variation of the core, in which the coefficient $\beta_{\mathrm{geom}}(R) = 2T/R^2$ and the reduced measure $R^2\, ds\, d\Omega$ are varied simultaneously. In this covariant variation, the product $\beta_{\mathrm{geom}}(R)R^2$ is constant and the corresponding local contribution vanishes.

The channel charge used here is instead built by holding the reference measure fixed and varying the effective density. At LO in the quasi-flat core, the useful radial variations are therefore

$$\frac{\delta E_{\mathrm{cin}}}{\delta r} \simeq \frac{\rho\omega^2}{R_0} A^2, \qquad \frac{\delta E_{\tan}}{\delta r} \simeq \frac{T}{R_0} (A_s)^2, \qquad \left.\frac{\delta E_{\mathrm{eff}}}{\delta r}\right|_{dV_0} \simeq -\frac{2T}{R_0^3} A^2. \tag{20.15}$$

With this fixed-measure convention, the variational definition of the channel charge leads to

$$Q_g = -\frac{1}{T\kappa_r} \left[\frac{\rho\omega^2}{R_0} \int_{\mathbb{S}^3} A^2 \, dV + \frac{T}{R_0} \int_{\mathbb{S}^3} (A_s)^2 \, dV - \frac{2T}{R_0^3} \int_{\mathbb{S}^3} A^2 \, dV \right]. \tag{20.16}$$

For an isolated solution, we use the core identity

$$\int_{\mathbb{S}^3} (A_s)^2 \, dV = \Omega_0^2 \int_{\mathbb{S}^3} A^2 \, dV - \frac{\beta_c}{2} \int_{\mathbb{S}^3} A^4 \, dV, \tag{20.17}$$

as well as

$$c_t^2 = \frac{T}{\rho}, \qquad \Omega_0^2 = \frac{2}{R_0^2} - \frac{\omega^2}{c_t^2}, \qquad \rho\omega^2 = T \left(\frac{2}{R_0^2} - \Omega_0^2 \right). \tag{20.18}$$

After substitution into (20.16), the coefficient of $\int_{\mathbb{S}^3} A^2 \, dV$ cancels exactly:

$$\frac{\rho\omega^2}{R_0} + \frac{T\Omega_0^2}{R_0} - \frac{2T}{R_0^3} = 0.$$

We therefore obtain

$$\boxed{Q_g = \frac{1}{\kappa_r} \frac{\beta_c}{2R_0} \int_{\mathbb{S}^3} A^4 \, dV.} \tag{20.19}$$

The monopole gravitational-like charge is therefore carried, at LO, by the quartic core invariant. The quadratic term $\int A^2 \, dV$ does not contribute directly to Q_g after core reduction.

This cancellation of the quadratic term is obtained in the fixed-reference-measure reading convention defined above. It does not contradict the cancellation of the local covariant variation of the core: the two operations do not hold the same quantities fixed. The covariant variation controls the local reduction of the core; the fixed-measure variation defines the channel charge consumed in the gravitational-like closure.

20.2.2 Link with the internal energies

Measure convention. In this subsection, all energy quantities are expressed in the monopole convention on $\mathbb{S}^3$. In particular, the energy relations inherited from the axisymmetric reduction are rewritten here with the measure dV on $\mathbb{S}^3$.

The quartic term is rewritten through the monopole coupling energy:

$$E_{\mathrm{coup}}^{\mathbb{S}^3} = \frac{T\beta_c}{4} \int_{\mathbb{S}^3} A^4 \, dV. \tag{20.20}$$

It follows that

$$\frac{\beta_c}{2} \int_{\mathbb{S}^3} A^4 \, dV = \frac{2}{T} E_{\mathrm{coup}}^{\mathbb{S}^3}. \tag{20.21}$$

Substituting into (20.19), we obtain

$$\boxed{Q_g = \frac{2E_{\mathrm{coup}}^{\mathbb{S}^3}}{T\kappa_r R_0}.} \tag{20.22}$$

In the local strong-screening and quasi-flat-core regime, the energy consistency of the isolated branch gives

$$E^{\mathbb{S}^3}_{\mathrm{coup}} \simeq \frac{E^{\mathbb{S}^3}_{\mathrm{tot}}}{2}, \tag{20.23}$$

cf. §5.3. Consequently,

$$\boxed{Q_g \simeq \frac{E^{\mathbb{S}^3}_{\mathrm{tot}}}{T\kappa_r R_0}.} \tag{20.24}$$

20.2.3 Relevant local regime

Introduce the dimensionless parameter

$$\varepsilon_\omega := \frac{c_t}{\omega R_0}. \tag{20.25}$$

For a localized solution, at LO one has

$$\omega \sim \frac{c_t}{\lambda}, \qquad \lambda \ll R_0,$$

so that

$$\varepsilon_\omega \sim \frac{\lambda}{R_0} \ll 1.$$

The result (20.24) must therefore be read in this local validity window, where global curvature corrections are of higher order. We then write

$$\boxed{Q_g \simeq \frac{E^{\mathbb{S}^3}_{\mathrm{tot}}}{T\kappa_r R_0}.} \tag{20.26}$$

The associated exterior deformation takes the form

$$r_{\mathrm{ext}}(d) \sim \frac{Q_g}{4\pi d} \simeq \frac{E^{\mathbb{S}^3}_{\mathrm{tot}}}{4\pi\, T\kappa_r R_0}\,\frac{1}{d}. \tag{20.27}$$

Thus, in the relevant local regime, the amplitude of the gravitational-like field is fixed linearly by the total energy of the isolated solution.

20.2.4 Effective gravitational constant

Injecting (20.26) into the interaction structure (15.24), we first obtain

$$U(d) \sim -\frac{1}{4\pi\, T\kappa_r R_0^2}\,\frac{E^{\mathbb{S}^3}_{\mathrm{tot},1} E^{\mathbb{S}^3}_{\mathrm{tot},2}}{d}. \tag{20.28}$$

By identifying

$$m = \frac{E_{\mathrm{tot}}}{c_t^2}, \tag{20.29}$$

we recover a minimal Newtonian form

$$U(d) \sim -\,\widetilde{G}_N^{\mathrm{eff}}\,\frac{m_1 m_2}{d}, \qquad \widetilde{G}_N^{\mathrm{eff}} = \frac{c_t^4}{4\pi\, T\,\kappa_r\, R_0^2}. \tag{20.30}$$

Canonical gravitational-like normalization. The minimal form $\widetilde{G}_N^{\mathrm{eff}}$ of (20.30) is the local closure obtained directly from the radial charge Q_g. To pass to the canonical normalization of the gravitational-like channel on $\mathbb{S}^3$, one must still include the three pure factors already fixed by the common spectral closure:

$$\frac{J_2}{J_0}, \qquad (4\pi)^2, \qquad \kappa_\omega^2.$$

The ratio J_2/J_0 is the core–gradient quotient of the canonical sech profile, defined by the core moments.

The factor $(4\pi)^2$ comes from two distinct spherical normalizations. The first is that of the local monopole Green function,

$$-\Delta_{\mathbb{R}^3}\left(\frac{1}{4\pi R}\right) = \delta^{(3)}(R),$$

which fixes the 4π of the radial propagator. The second is that of the canonical monopole source of the channel: the total flux of a point source is normalized by the spherical integral

$$\int_{\mathbb{S}^2} \mathbf{n}\cdot\mathbf{n}\, d\Omega = 4\pi.$$

The Newtonian reading of the channel therefore consumes simultaneously the normalization of the Green function and that of the monopole source, which gives the spherical factor

$$(4\pi)^2.$$

Finally, $\kappa_\omega^2 = \Delta_*^{-2}$ is the spectral plateau closure factor introduced in the HSM spectral base.

We therefore define the pure gravitational-like number

$$\boxed{\mathcal{N}_g := \underbrace{\frac{J_2}{J_0}}_{\substack{\text{profil}\\ \text{flux}}} \underbrace{(4\pi)^2}_{\text{monopolaire}} \underbrace{\kappa_\omega^2}_{\substack{\text{plateau}\\ \text{spectral}}} = \frac{J_2}{J_0}\,(4\pi\kappa_\omega)^2.} \tag{20.31}$$

This factor is not a free parameter: it groups only pure normalizations already fixed by the core profile, the monopole flux, and the common spectral closure. We then obtain the normalized form

$$\boxed{G_N^{\mathrm{eff}} = \mathcal{N}_g\, \widetilde{G}_N^{\mathrm{eff}} = \frac{J_2}{J_0}\,(4\pi\kappa_\omega)^2\,\frac{c_t^4}{4\pi\, T\,\kappa_r\, R_0^2}}. \tag{20.32}$$

This quantity has the units of a Newtonian constant,

$$[G_N^{\mathrm{eff}}] = \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}.$$

20.2.5 Synthesis of the gravitational-like closure

At LO, the closure obtained in this chapter is

$$Q_g \simeq \frac{E_{\mathrm{tot}}^{\mathbb{S}^3}}{T\kappa_r R_0}, \qquad G_N^{\mathrm{eff}} = \frac{J_2}{J_0}\,(4\pi\kappa_\omega)^2\,\frac{c_t^4}{4\pi\, T\,\kappa_r\, R_0^2},$$

in the local window

$$\varepsilon_\omega = \frac{c_t}{\omega R_0} \sim \frac{\lambda}{R_0} \ll 1,$$

and under the local energy convention

$$E_{\mathrm{coup}}^{\mathbb{S}^3} \simeq \frac{E_{\mathrm{tot}}^{\mathbb{S}^3}}{2}.$$

Coupling. Any *localized* solution of finite energy sources the gravitational-like channel in the linear regime. The simple conversion $Q_g \simeq E_{\rm tot}^{\mathbb{S}^3}/(T\kappa_r R_0)$ is nevertheless the local closure associated with the branches satisfying the energy chain above. In the local Newtonian regime, the interaction takes the form

$$F_{12}(d) = G_N^{\rm eff}\,\frac{m_1 m_2}{d^2}, \qquad E_i = m_i c_t^2.$$

20.3 Electromagnetic-like closure (LO)

20.3.1 Medium impedance and channel constant $\Phi_{\rm EM}^{(\rm two)}$ (LO)

Constitutives and internal impedance (LO). With $c_t^2 = T/\rho$ and the Noether coupling g_Ω,

$$\varepsilon_{\rm eff} = \frac{T}{g_\Omega^2}, \qquad \mu_{\rm eff} = \frac{g_\Omega^2}{T\,c_t^2}, \qquad Z_{\rm eff} = \sqrt{\frac{\mu_{\rm eff}}{\varepsilon_{\rm eff}}} = \mu_{\rm eff}\,c_t = \frac{g_\Omega^2}{T\,c_t}\,. \tag{20.33}$$

This impedance $Z_{\rm eff}$ is the internal constitutive impedance associated with the Noether/Maxwell-like normalization of the EM channel. Its explicit identification with the channel impedance used in the active closure is not postulated at this stage.

Spectral convention of the present closure. In the present electromagnetic-like closure, no new independent spectral constant is reintroduced: for the EM-like channel one simply uses the same pure scale derived by the common selection.

$$\boxed{\Delta_{\rm EM} := \kappa_\omega}\,.$$

Numerically,

$$\Delta_{\rm EM} = \kappa_\omega = 54.9589563237.$$

The symbol $\Delta_{\rm EM}$ is retained here only as a *local chapter notation*, to avoid any visual confusion with the reduced width $\Delta_* = 1/\kappa_\omega$.

EM-like shape constant.

Elementary EM-like coupling. The elementary coupling of the EM-like channel must not be confused with the active two-leg closure coefficients. At LO it is defined as the product of the local isotropic Green factor, the elementary cyclic normalization, and the single source projection:

$$\frac{1}{4\pi}, \qquad \frac{1}{2\pi}, \qquad \Pi_{\rm source}^{\rm EM}.$$

Proposition 20.1 (Dimensionless EM-like coupling at LO)**.** *The dimensionless elementary coupling of the EM-like channel at LO is*

$$\boxed{\alpha_{\rm EM}^{\rm HSM,LO} = \frac{1}{4\pi}\,\frac{1}{2\pi}\,\Pi_{\rm source}^{\rm EM} = \frac{1}{8\pi^2}\,\frac{J_0}{J_0+J_2}.} \tag{20.34}$$

Proof. The factor $1/(4\pi)$ comes from the local isotropic Green kernel of the EM-like channel. The factor $1/(2\pi)$ is the elementary cyclic normalization associated with an action cycle. Finally, the charge–field vertex consumes a single source projection, given by (16.2). Hence

$$\alpha_{\rm EM}^{\rm HSM,LO} = \frac{1}{4\pi}\,\frac{1}{2\pi}\,\Pi_{\rm source}^{\rm EM}.$$

Substituting (16.2) gives

$$\alpha_{\rm EM}^{\rm HSM,LO} = \frac{1}{8\pi^2}\,\frac{J_0}{J_0+J_2}.$$

□

LO value. With

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 = 7.6339320582,$$

one obtains

$$\boxed{\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} = 7.285 \times 10^{-3}, \qquad \left(\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}\right)^{-1} \simeq 137.3.} \tag{20.35}$$

Two-projection active closure coefficient. The active closures of the EM-like channel do not belong to the elementary vertex $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$. They describe a source–field–source transfer, or an equivalent structure with two channel endpoints. They may therefore consume two source projections. The pure constant associated with this active closure is

$$\boxed{\Gamma_{\mathrm{EM}} := \frac{1}{\pi}\left(\Pi_{\mathrm{source}}^{\mathrm{EM}}\right)^2 = \frac{1}{\pi}\left(\frac{J_0}{J_0 + J_2}\right)^2.} \tag{20.36}$$

Numerically,

$$\Gamma_{\mathrm{EM}} = 0.10530324394.$$

This constant is a pure active-closure number; it does not depend on any dimensional parameter and is not defined by an adjusted spectral window.

Status of the squared source projection. The square $(\Pi_{\mathrm{source}}^{\mathrm{EM}})^2$ in (20.36) does not come from squaring the elementary coupling $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$. It comes from the bilinear source–field–source form of the active closure: one source projection acts at the channel input, and one source projection acts at the channel output. The local Green and cyclic-normalization factors entering $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$ are not repeated in Γ_{EM}. There is therefore no double counting between the elementary vertex and the active closure with two endpoints.

Channel constant ("two-leg" scheme, cap $\ell = 1$). Under a *single* canonical zonal average on $\mathbb{S}^3$ and canonical normalization of the cap projector $\ell = 1$,

$$\boxed{\Phi_{\mathrm{EM}}^{(\mathrm{two})} = \left(\frac{1}{\Delta_{\mathrm{EM}}^2\, b_0^{(2)}}\right)^2 \frac{1}{2\pi^2}\sqrt{\frac{J_2}{J_0}}\,\Gamma_{\mathrm{EM}}\,\frac{J_0^2}{E_{\ell=1}^{\mathrm{cap}}}, \qquad E_{\ell=1}^{\mathrm{cap}} = \frac{(F_1^{\mathrm{cap}})^2}{\pi}.} \tag{20.37}$$

Here $J_0, J_2, \Delta_{\mathrm{EM}}, b_0^{(2)}$ are pure constants from the core moments and from the HSM spectral closure. The coefficient $b_0^{(2)}$ is the regularized LO closure of the 2DOF bump defined in HSM. The pure number Γ_{EM}, defined by (20.36), carries the active two-source-projection closure of the EM-like channel. It is distinct from the elementary coupling $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$ defined in (20.34).

The quantity $\Phi_{\mathrm{EM}}^{(\mathrm{two})}$ is a channel constant. It does not contain the projection factors specific to a particular composite solution. In particular, the width-compatibility factors Ξ_{ab} and the witness projection factors, when they appear, are introduced in the field–solution coupling sections and not in the definition of the EM-like field. No independent multiplicative continuity factor is introduced at LO.

Two-leg / cell transfer and channel impedance (LO). The active closure of the EM-like channel is built from a reference impedance fixed at the microscopic scale,

$$Z_{\mathrm{ref},0} = \frac{h_{\mathrm{eff}}\, c_t}{\ell_0^2}.$$

The canonical two-leg scheme does not introduce a new absolute scale in ℓ_0^{-2}, but a *relative* transfer factor between the microscopic scale ℓ_0 and the spectral scale $\lambda = \Delta_{\mathrm{EM}}\ell_0$.

(i) Transfer per leg.) For one leg, the dimensionless factor is

$$\eta_{\mathrm{leg}} = \frac{\ell_0^2}{\lambda^2}\,\frac{1}{b_0^{(2)}} = \frac{1}{\Delta_{\mathrm{EM}}^2\, b_0^{(2)}}.$$

(ii) Two legs.) The active two-leg closure is a bilinear contraction between two transfer endpoints. Each endpoint carries the same relative factor η_{leg}. Since the two legs are orthogonal and have the same spectral width at LO, cross terms are absent in the Galerkin reduction, and the contraction gives the product of the two endpoint factors:

$$\eta_{2\mathrm{leg}} = \eta_{\mathrm{leg}}^{(1)}\eta_{\mathrm{leg}}^{(2)} = \eta_{\mathrm{leg}}^2 = \left(\frac{1}{\Delta_{\mathrm{EM}}^2\, b_0^{(2)}}\right)^2.$$

This square is not a second width normalization; it is the weight of the bilinear form with two endpoints.

(iii) Cap projection $\ell = 1$.) The canonical projection onto the cap $\ell = 1$ contributes the pure spectral factor

$$\boxed{\Pi_{\ell=1}^{\mathrm{EM}} = \frac{1}{2\pi}\sqrt{\frac{J_2}{J_0}}\,\frac{1}{E_{\ell=1}^{\mathrm{cap}}}, \qquad E_{\ell=1}^{\mathrm{cap}} = \frac{(F_1^{\mathrm{cap}})^2}{\pi}.} \tag{20.38}$$

This factor $\Pi_{\ell=1}^{\mathrm{EM}}$ is a pure cap projector. It must not be confused with the full reduced coefficient $\mathcal{C}_{\mathrm{EM}}^{(2)}$ used in 2DOF-composite reductions, which also includes Γ_{EM} and the core moments.

Status of the cap factor F_1^{cap}. The factor F_1^{cap} does not denote the cell filter $F_\ell^{(\mathrm{cell})}$ used in the spectral closure of $b_0^{(2)}$. It denotes the normalization factor of the cap projector reduced to the $\ell = 1$ component consumed by the two-leg EM-like closure.

The reduced cap projector is taken to be idempotent and of unit norm on this component:

$$\Pi_{\ell=1}^{\mathrm{cap}} \circ \Pi_{\ell=1}^{\mathrm{cap}} = \Pi_{\ell=1}^{\mathrm{cap}}, \qquad \|\Pi_{\ell=1}^{\mathrm{cap}}\|_{\ell=1} = 1.$$

In this normalization, the reference amplitude of the projected component satisfies

$$\boxed{(F_1^{\mathrm{cap}})^2 = 1.} \tag{20.39}$$

This equality fixes no independent physical scale: it expresses the idempotence and unit normalization of the cap projector on the $\ell = 1$ mode. Any physical scale of the channel remains carried by the explicitly derived closure factors. Another normalization of the cap projector would have to be propagated explicitly in $\Phi_{\mathrm{EM}}^{(\mathrm{two})} = C_\Phi/(F_1^{\mathrm{cap}})^2$.

The cap factor therefore reduces to

$$\boxed{E_{\ell=1}^{\mathrm{cap}} = \frac{1}{\pi}.} \tag{20.40}$$

(iv) Active closure.) One thus obtains

$$\Phi_{\mathrm{EM}}^{(\mathrm{two})} = \eta_{2\mathrm{leg}}\,\Pi_{\ell=1}^{\mathrm{EM}}\,\Gamma_{\mathrm{EM}}\,\frac{J_0^2}{\pi}.$$

Expanding the factors defined above gives

$$\Phi_{\mathrm{EM}}^{(\mathrm{two})} = \left(\frac{1}{\Delta_{\mathrm{EM}}^2\, b_0^{(2)}}\right)^2 \frac{1}{2\pi^2}\sqrt{\frac{J_2}{J_0}}\,\Gamma_{\mathrm{EM}}\,\frac{J_0^2}{E_{\ell=1}^{\mathrm{cap}}}.$$

The reference channel impedance is then

$$\boxed{Z^{(\mathrm{LO})}_{\mathrm{EM,ref}} = \frac{h_{\mathrm{eff}}\, c_t}{\ell_0^2}\, \Phi^{(\mathrm{two})}_{\mathrm{EM}}\ .} \tag{20.41}$$

This relation defines the reference mechanical impedance of the EM-like channel in the internal normalization of the theory.

20.3.2 Pure constant C_Φ and final form of $\Phi^{(\mathrm{two})}_{\mathrm{EM}}$

Compact definition. From (20.37) and

$$E^{\mathrm{cap}}_{\ell=1} = \frac{(F_1^{\mathrm{cap}})^2}{\pi},$$

one introduces the pure constant

$$\boxed{\Phi^{(\mathrm{two})}_{\mathrm{EM}} = \frac{C_\Phi}{(F_1^{\mathrm{cap}})^2}, \qquad C_\Phi = \left(\frac{1}{\Delta^2_{\mathrm{EM}}\, b_0^{(2)}}\right)^2 \frac{1}{2\pi}\sqrt{\frac{J_2}{J_0}}\,\Gamma_{\mathrm{EM}}\, J_0^2.} \tag{20.42}$$

Canonical numerical value. With the common constants

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 = 7.6339320582, \qquad \Delta_{\mathrm{EM}} = \kappa_\omega = 54.9589563237, \qquad b_0^{(2)} = 1.809206964{\times}10^4,$$

and the channel value

$$\Gamma_{\mathrm{EM}} = 0.10530324394,$$

This value of $b_0^{(2)}$ is the central value of the regularized LO spectral closure defined in HSM. The absolute predictions that depend on C_Φ therefore inherit the regularization uncertainty associated with $b_0^{(2)}$.

one obtains

$$\boxed{C_\Phi = 5.152264653 \times 10^{-16}.} \tag{20.43}$$

Thus,

$$\boxed{\Phi^{(\mathrm{two})}_{\mathrm{EM}} = \frac{5.152264653 \times 10^{-16}}{(F_1^{\mathrm{cap}})^2}.}$$

20.3.3 Canonical numerical value of $\Phi^{(\mathrm{two})}_{\mathrm{EM}}$

Canonical convention. In the canonical normalization of the cap projector $\ell = 1$, one uses

$$(F_1^{\mathrm{cap}})^2 = 1, \qquad E^{\mathrm{cap}}_{\ell=1} = \frac{1}{\pi}.$$

Evaluation. One then obtains directly

$$\boxed{\Phi^{(\mathrm{two})}_{\mathrm{EM}} = 5.152264653 \times 10^{-16}.}$$

$$\boxed{Z^{(\mathrm{LO})}_{\mathrm{EM,ref}} = \frac{h_{\mathrm{eff}}\, c_t}{\ell_0^2}\, \Phi^{(\mathrm{two})}_{\mathrm{EM}} = \frac{h_{\mathrm{eff}}\, c_t}{\ell_0^2}\, \frac{C_\Phi}{(F_1^{\mathrm{cap}})^2}.} \tag{20.44}$$

20.3.4 Elementary charge quantum of the EM-like channel

Purpose. The mechanical charge Q_{mech}, defined below by (20.55), belongs to the Noether/spin–action normalization of the channel. It measures the internal mechanical source carried by the solution. The elementary charge quantum of the EM-like field is a distinct quantity: it is defined by the static response of the Maxwell-like field.

The static limit of the local interaction (16.6), for two sources at rest separated by d, gives

$$V_{12}^{\mathrm{stat}}(d) = \frac{1}{4\pi\varepsilon_{\mathrm{eff}}}\,\frac{Q_1 Q_2}{d}. \tag{20.45}$$

For two identical elementary sources of modulus $Q_{\mathrm{EM},*}$, one has

$$V_{**}^{\mathrm{stat}}(d) = \frac{1}{4\pi\varepsilon_{\mathrm{eff}}}\,\frac{Q_{\mathrm{EM},*}^2}{d}. \tag{20.46}$$

Energy definition of the elementary coupling. The dimensionless elementary coupling of the EM-like channel is the ratio of the elementary static energy multiplied by the distance to the action–speed scale of the channel:

$$\alpha_{\mathrm{EM}} = \frac{d\,V_{**}^{\mathrm{stat}}(d)}{h_{\mathrm{eff}} c_t}. \tag{20.47}$$

The numerator $d\,V_{**}^{\mathrm{stat}}(d)$ has the dimension of an energy multiplied by a length. The denominator $h_{\mathrm{eff}} c_t$ carries the same dimension. In the local $1/d$ regime, the ratio (20.47) is therefore independent of d.

Proposition 20.2 (Elementary quantum of the EM-like channel)**.** *The positive charge quantum of the EM-like field associated with the elementary coupling* $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$ *is*

$$\boxed{Q_{\mathrm{EM},*}^2 = 4\pi\,\varepsilon_{\mathrm{eff}}\,h_{\mathrm{eff}} c_t\,\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}.} \tag{20.48}$$

Using the constitutive impedance of the channel,

$$Z_{\mathrm{eff}} = \frac{1}{\varepsilon_{\mathrm{eff}} c_t}, \tag{20.49}$$

the same closure is written

$$\boxed{Q_{\mathrm{EM},*} = \left(\frac{4\pi h_{\mathrm{eff}}\,\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{Z_{\mathrm{eff}}}\right)^{1/2}.} \tag{20.50}$$

Proof. Injecting (20.46) into (20.47) gives

$$\alpha_{\mathrm{EM}} = \frac{d}{h_{\mathrm{eff}} c_t}\,\frac{1}{4\pi\varepsilon_{\mathrm{eff}}}\,\frac{Q_{\mathrm{EM},*}^2}{d} = \frac{Q_{\mathrm{EM},*}^2}{4\pi\varepsilon_{\mathrm{eff}} h_{\mathrm{eff}} c_t}.$$

Identification with the elementary coupling closed in (20.34) directly gives (20.48). Moreover, (20.33) gives

$$Z_{\mathrm{eff}} = \sqrt{\frac{\mu_{\mathrm{eff}}}{\varepsilon_{\mathrm{eff}}}} = \frac{1}{\varepsilon_{\mathrm{eff}} c_t}.$$

Substituting this relation in (20.48) gives (20.50). □

Corollary 20.3 (LO closure of the EM-like charge quantum)**.** *When the channel impedance is evaluated by the active closure* (20.41)*, the elementary charge quantum of the EM-like field is*

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \left(\frac{4\pi h_{\mathrm{eff}}\, \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}} \right)^{1/2}.} \tag{20.51}$$

With the final form (20.44)*, this closure becomes*

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \ell_0 \left(\frac{4\pi\, \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{c_t\, \Phi_{\mathrm{EM}}^{(\mathrm{two})}} \right)^{1/2}.} \tag{20.52}$$

Proof. Formula (20.51) is the specialization of (20.50) to the reference impedance (20.41). Then substituting

$$Z_{\mathrm{EM,ref}}^{(\mathrm{LO})} = \frac{h_{\mathrm{eff}} c_t}{\ell_0^2}\, \Phi_{\mathrm{EM}}^{(\mathrm{two})}$$

from (20.44) gives

$$Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \left(\frac{4\pi h_{\mathrm{eff}} \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{\frac{h_{\mathrm{eff}} c_t}{\ell_0^2} \Phi_{\mathrm{EM}}^{(\mathrm{two})}} \right)^{1/2} = \ell_0 \left(\frac{4\pi \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{c_t\, \Phi_{\mathrm{EM}}^{(\mathrm{two})}} \right)^{1/2}.$$

□

Dimensional status. The quantity $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$ is the positive charge quantum of the EM-like field in the internal normalization of the theory. It is not expressed in an external unit of charge. Its dimension is fixed by (20.50):

$$[Q_{\mathrm{EM},*}^2] = \frac{[h_{\mathrm{eff}}]}{[Z_{\mathrm{eff}}]}.$$

With the closure (20.44), one obtains

$$[Q_{\mathrm{EM},*}^2] = \frac{\ell_0^2}{c_t}, \qquad [Q_{\mathrm{EM},*}] = \frac{\ell_0}{\sqrt{c_t}}.$$

This dimension is that of the field charge in the internal EM-like normalization. No external unit of charge enters this closure.

Logical status. The closure (20.51)–(20.52) derives from the local statics (20.45), from the elementary coupling (20.34), and from the closed impedance (20.41). It does not depend on any external charge.

20.3.5 Diagnostic spectral representation of the EM-like channel

Status. The channel constant used in the active closure is Γ_{EM}, defined by (20.36). The spectral kernels below give a diagnostic representation of the two-leg transfer, but they do not define a second closure independent of Γ_{EM}.

Explicit kernels (two legs, LO). We recall the closed forms of the two spectral kernels:

$$\mathcal{K}_{\nabla\text{-scal}}(k) = \tfrac{2}{3}\, k^2\, \Phi(k), \qquad \mathcal{K}_{\text{scal-}\nabla}(k) = \tfrac{2}{3}\, k^2\, \Phi(k), \qquad \Phi(k) = \Big(\tfrac{\pi k}{\sinh(\frac{\pi k}{2})} \Big)^2. \tag{20.53}$$

Remark. Any integral representation weighted by a window $w(k)$ belongs to an additional spectral diagnostic. It must not be used to fix Γ_{EM} as long as the window parameter has not been derived independently.

20.3.6 Natural charge modulus and internal splitting

Natural charge modulus. The EBK closure for $q = 1$, with canonical normalization

$$C_{\mathrm{EM}} = J_0 + J_2,$$

gives

$$A_0^2\lambda^2 = \frac{2h_{\mathrm{eff}}}{\rho c_t}\frac{1}{J_0 + J_2} \quad \Rightarrow \quad \boxed{q_0 := (A_0^2\lambda^2)\, C_{\mathrm{EM}} = \frac{2h_{\mathrm{eff}}}{\rho c_t}}. \tag{20.54}$$

The modulus q_0 is thus the natural internal charge modulus of the reference 2DOF bump solution.

Internal splitting. At this stage of the manuscript, only the structure actually used in the already established families is retained:

$$q_{\mathrm{int}} = \pm q_0 \qquad \text{(2DOF bump)},$$

and, when the corresponding 3DOF internal structure is realized,

$$q_{\mathrm{int}} \in \Big\{ \pm \tfrac{1}{3}q_0, \pm\tfrac{2}{3}q_0 \Big\}.$$

No additional intermediate symbol is therefore introduced here (m_{dof}, θ_{M}, etc.) as long as its canonical definition is not explicitly consumed later.

20.3.7 Reference mechanical charge

Reference mechanical charge (spin–action). With

$$S_N = 2\,\mathcal{C}_S\, S, \qquad \mathcal{C}_S = \frac{J_0}{J_0 + J_2},$$

the reference mechanical charge is defined by

$$Q_{\mathrm{mech}} := \frac{1}{g_\Omega}\, S_N. \tag{20.55}$$

Using the dictionary

$$g_\Omega = \frac{\ell_0^2}{\Upsilon_{\mathrm{cell}} c_t},$$

one obtains

$$Q_{\mathrm{mech}} = 2\,\mathcal{C}_S\, \frac{\Upsilon_{\mathrm{cell}}\, c_t}{\ell_0^2}\, S. \tag{20.56}$$

Case of the reference solution $q = 1$ (2DOF bump). Setting $S = h_{\mathrm{eff}}$, one obtains

$$Q_{\mathrm{mech}}(q{=}1) = 2\,\frac{J_0}{J_0 + J_2}\,\frac{\Upsilon_{\mathrm{cell}}\, c_t}{\ell_0^2}\, h_{\mathrm{eff}}. \tag{20.57}$$

This quantity plays here the role of internal reference mechanical charge for the 2DOF bump family.

20.3.8 Local statics and cancellation of the coupling g_Ω

Local static force. In the local limit, two localized solutions of charges Q_1, Q_2 separated by d satisfy

$$\boxed{F(d) = \frac{Q_1 Q_2}{4\pi\,\varepsilon_{\text{eff}}\,d^2}} \tag{20.58}$$

Cancellation of the coupling g_Ω at LO. Injecting

$$Q = \frac{2\,\mathcal{C}_S}{g_\Omega}\,S, \qquad \mathcal{C}_S = \frac{J_0}{J_0 + J_2}, \qquad \varepsilon_{\text{eff}} = \frac{T}{g_\Omega^2},$$

into (20.58), one obtains

$$F(d) = \frac{(\frac{2\mathcal{C}_S}{g_\Omega}S_1)(\frac{2\mathcal{C}_S}{g_\Omega}S_2)}{4\pi\,\frac{T}{g_\Omega^2}\,d^2} = \frac{\mathcal{C}_S^2}{\pi\,T}\,\frac{S_1 S_2}{d^2}.$$

Thus, at LO, the local static force is *independent* of g_Ω: the EM-like closure depends only on T and on the pure core constants (J_0, J_2).

20.4 Preclusion field closure on the fundamental constants

20.4.1 Preclusion overlap factor

We introduce a *dimensionless* geometric overlap factor Γ_{P}, which captures at LO the central overlap of two cores on the scale λ in the quasi–plane cap approximation.

We use the dimensionless radial variable

$$\boldsymbol{\xi} = \frac{\mathbf{x}}{\lambda}, \qquad a(\xi) = \operatorname{sech}\xi, \qquad \xi = |\boldsymbol{\xi}|,$$

and the shape moment

$$J_0 = \int_0^\infty 4\pi\,\xi^2 a(\xi)^2\,d\xi.$$

Dimensionless definition. We define the central quartic overlap integral by

$$\mathcal{I}_{\text{P}}(0) := 4\int_{\mathbb{R}^3} a(|\boldsymbol{\xi}|)^4\,d^3\boldsymbol{\xi}.$$

The Preclusion overlap factor is then

$$\boxed{\Gamma_{\text{P}} := \frac{\mathcal{I}_{\text{P}}(0)}{J_0^2}.} \tag{20.59}$$

By construction, Γ_{P} is a *pure number*: it depends neither on T, nor on ρ, nor on ℓ_0, nor on R_0, but only on the normalized radial kernel $a(\xi)$ and on the local LO geometry.

Scope of the definition. The dependence on effective separation (for example in a breather) is carried by dedicated overlap functions, denoted $\mathcal{O}_{\text{P}}(\Delta)$, which will be introduced separately. The constant Γ_{P} corresponds to the central normalization of the Preclusion overlap and does not depend on it.

Numerical value for the profile $a = \mathrm{sech}$

Canonical shape constants. For the radial kernel $a(\xi) = \mathrm{sech}\,\xi$ used at LO,

$$J_0 = \int_0^\infty 4\pi\xi^2 \,\mathrm{sech}^2\,\xi\, d\xi = \frac{\pi^3}{3} \approx 10.3354255601,$$

and

$$J_4 = \int_0^\infty 4\pi\xi^2 \,\mathrm{sech}^4\,\xi\, d\xi \approx 2.7014935019.$$

Closed relation and value. In this case,

$$\mathcal{I}_{\mathrm{P}}(0) = 4J_4,$$

hence

$$\boxed{\Gamma_{\mathrm{P}} = \frac{4J_4}{J_0^2} \approx 0.1011596206.} \tag{20.60}$$

It is a *pure number* entirely determined by the kernel $a(\xi) = \mathrm{sech}\,\xi$.

Remark. The dependences on internal separation ($\delta = L/\lambda$) enter through specific overlap functions, denoted for example $\mathcal{O}_{\mathrm{P}}(\delta)$, and do not modify the central constant Γ_{P}.

20.4.2 LO closure of the prefactor C_{P}

Finite-range form. In the local 3D geometry of the quasi–plane cap, the Preclusion potential at LO is written in the form

$$V_{\mathrm{P}}(D) \simeq +\frac{C_{\mathrm{P}}(\lambda)}{D}\, e^{-D/\xi}, \qquad \xi = \sqrt{\frac{T}{\mu_\psi}},$$

where D is the physical distance between the centers. The range length is governed by μ_ψ, while the absolute amplitude of the repulsion is carried by the prefactor $C_{\mathrm{P}}(\lambda)$.

LO closure of the prefactor. In the thin LO scheme, the dimensionally homogeneous closure of the prefactor is

$$\boxed{C_{\mathrm{P}}(\lambda) = \frac{T}{4\pi}\,\Gamma_{\mathrm{P}}\,\frac{\left(A_0^2\lambda^2\right)^2}{\lambda^4}.} \tag{20.61}$$

The factor λ^{-4} converts the core invariant $(A_0^2\lambda^2)^2$ into a pair-potential prefactor, with dimension J m.

Closure on the branch constants. The core invariant $A_0^2\lambda^2$ is closed at LO by the canonical EBK relation of the 2DOF bump, cf. (20.54):

$$A_0^2\lambda^2 = \frac{2h_{\mathrm{eff}}}{\rho c_t}\,\frac{1}{J_0 + J_2}.$$

We deduce

$$\boxed{C_{\mathrm{P}}(\lambda) = \frac{T}{4\pi}\,\Gamma_{\mathrm{P}}\,\frac{1}{\lambda^4}\left(\frac{2h_{\mathrm{eff}}}{\rho c_t\,(J_0 + J_2)}\right)^2.} \tag{20.62}$$

Using $T = \rho c_t^2$, this can also be written

$$\boxed{C_{\mathrm{P}}(\lambda) = \frac{\Gamma_{\mathrm{P}}}{\pi}\,\frac{h_{\mathrm{eff}}^2}{\rho\,\lambda^4\,(J_0 + J_2)^2}.} \tag{20.63}$$

Status of the closure. The Preclusion channel is thus closed at LO without introducing a new free parameter:

- the range is carried by $\xi = \sqrt{T/\mu_\psi}$;
- the absolute prefactor is $C_{\mathrm{P}}(\lambda)$;
- $C_{\mathrm{P}}(\lambda)$ is entirely expressed as a function of $\rho, c_t, h_{\mathrm{eff}}$, the branch width λ, and the pure numbers $\Gamma_{\mathrm{P}}, J_0, J_2$.

Geometric remark. In a very thin quasi–2D regime, the geometric prefactor may become d^0 instead of $1/d$. The form above corresponds to the local 3D geometry of the quasi–plane cap retained in this chapter.

20.5 Minimal weak-like channel: closure on the 2DOF NSL branch

Positioning. At this stage, the weak-like channel is not closed as a universal sector for every solution of the theory. We establish only a coherent minimal closure on the 2DOF NSL source branch, which carries the W-like channel.

The fixed point is the following:

- the weak-like *field* is identified with the radial response r;
- the *canonical source* of the NSL branch is the internal current of the doublet;
- the *field equation* is written using a reduced density $\widehat{\rho}_W$, so that the constant g_W remains the one read on the radial sector;
- this closure applies, at this stage, to the harmonic 2DOF NSL at rest.

20.5.1 Channel operator and minimal closure

Field identification. At the minimal level retained here, the weak-like field is identified with the radial response of the mesh associated with the NSL structure:

$$\boxed{\Phi_W \equiv r.}$$

Reading of the radial equation. The linearized radial sector leads to the operator

$$\boxed{\mathcal{L}_W = \partial_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\partial_s - \frac{\mu_2}{T\kappa_r}.}$$

We then keep the field constant read on the right-hand side of the radial sector:

$$\boxed{g_W = \frac{2}{\kappa_r R_0^3}.} \tag{20.64}$$

Minimal field writing. The minimal closure of the weak-like channel is written

$$\boxed{\mathcal{L}_W\,\Phi_W = -\,g_W\,\widehat{\rho}_W.} \tag{20.65}$$

The source $\widehat{\rho}_W$ is not here a universal charge valid for every solution of the theory; it is the reduced source density specific to the 2DOF NSL branch.

20.5.2 Canonical fixing of the charge: proof that $\Lambda_W = \rho$

Tangential doublet of the NSL. The 2DOF NSL is based on two orthogonal tangential legs; we write the tangential doublet in the form

$$U := (u_1, u_2), \qquad \vec{u}_{\tan} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2,$$

where $\mathbf{e}_1, \mathbf{e}_2$ form a local orthonormal basis of the tangent plane under consideration.

Fundamental tangential Lagrangian. The fundamental tangential sector is normalized by the local Lagrangian

$$\mathcal{L}_{\tan} = \frac{1}{2}\,\rho\,|\partial_t \vec{u}_{\tan}|^2 - \frac{1}{2}\,T\,|\operatorname{curl}_{\tan} \vec{u}_{\tan}|^2.$$

In an orthonormal basis, the tangential kinetic term is therefore exactly

$$\frac{\rho}{2}\Big((\partial_t u_1)^2 + (\partial_t u_2)^2\Big).$$

If the effective Lagrangian of the doublet is written in the canonical form

$$\mathcal{L}^{\mathrm{kin}}_{\mathrm{NSL}} = \frac{\Lambda_W}{2}\Big((\partial_t u_1)^2 + (\partial_t u_2)^2\Big),$$

term-by-term identification immediately gives

$$\boxed{\Lambda_W = \rho.} \tag{20.66}$$

Canonical current of the doublet. We consider the infinitesimal internal rotation

$$\delta u_1 = -\alpha u_2, \qquad \delta u_2 = \alpha u_1.$$

The associated canonical current has time component

$$j^0_W = \frac{\partial \mathcal{L}}{\partial(\partial_t u_1)}(-u_2) + \frac{\partial \mathcal{L}}{\partial(\partial_t u_2)} u_1.$$

Since

$$\frac{\partial \mathcal{L}}{\partial(\partial_t u_1)} = \rho\,\partial_t u_1, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_t u_2)} = \rho\,\partial_t u_2,$$

we obtain

$$j^0_W = \rho\,(u_1 \partial_t u_2 - u_2 \partial_t u_1).$$

We therefore identify the canonical weak-like charge density as

$$\boxed{\rho_W = \rho\,(u_1 \partial_t u_2 - u_2 \partial_t u_1).} \tag{20.67}$$

Branch selectivity. This definition immediately implies

$$\boxed{\rho_W[S] = 0 \qquad \text{if the solution } S \text{ does not carry the 2DOF NSL structure.}}$$

Thus, not all solutions carry a weak-like charge.

20.5.3 Proper charge of the NSL at rest and reduced source

Harmonic NSL at rest. For an NSL at rest, we write

$$u_1(s,t) = A(s)\cos(\omega_0 t), \qquad u_2(s,t) = A(s)\sin(\omega_0 t),$$

where $\omega_0 = \omega_{\mathrm{NSL}}(q=1)$ is the eigenfrequency of the branch. We then have

$$u_1 \partial_t u_2 - u_2 \partial_t u_1 = \omega_0\, A(s)^2,$$

hence

$$\boxed{\rho_W^{(+)}(s) = +\rho\,\omega_0\, A(s)^2.} \tag{20.68}$$

If the internal orientation of the doublet is reversed,

$$u_2(s,t) \mapsto -\,u_2(s,t),$$

we symmetrically obtain

$$\boxed{\rho_W^{(-)}(s) = -\rho\,\omega_0\, A(s)^2.} \tag{20.69}$$

Reduced channel source. To preserve the field constant $g_W = 2/(\kappa_r R_0^3)$ read on the radial sector, we introduce the reduced density

$$\boxed{\widehat{\rho}_W := \frac{\rho_W}{\rho\,\omega_0}.} \tag{20.70}$$

On the 2DOF NSL branch at rest,

$$\boxed{\widehat{\rho}_W^{(+)} = +A^2, \qquad \widehat{\rho}_W^{(-)} = -A^2.} \tag{20.71}$$

Field equation sourced by an NSL. The field equation then becomes, on the NSL branch at rest,

$$\boxed{\left(\partial_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\partial_s - \frac{\mu_2}{T\kappa_r}\right)\Phi_W = -\,\frac{2}{\kappa_r R_0^3}\,\widehat{\rho}_W, \qquad \widehat{\rho}_W^{(\pm)} = \pm A^2.} \tag{20.72}$$

Canonical global charge. The global charge carried by an NSL is

$$\boxed{q_W^{(\pm)} = \int_{\mathbb{S}^3} \rho_W^{(\pm)}\, dV = \pm\,\rho\,\omega_0 \int_{\mathbb{S}^3} A(s)^2\, dV.} \tag{20.73}$$

Quadratic moment of the core. According to §6.7, at the quasi–plane LO level with the profile $A(u) = \operatorname{sech} u$,

$$\int_{\mathbb{S}^3} A^2\, dV = 2\pi R_0^2\,\lambda\,\kappa_{\mathrm{core}}^{(\mathrm{NSL})}, \qquad \kappa_{\mathrm{core}}^{(\mathrm{NSL})} = 4,$$

therefore

$$\boxed{\int_{\mathbb{S}^3} A^2\, dV = 8\pi R_0^2\,\lambda.} \tag{20.74}$$

Reinjection of the spectral closure. The closure of the NSL core

$$\omega_{\mathrm{NSL}} = \chi_{\omega}^{(\mathrm{NSL})} \frac{c_t}{\lambda}$$

is rewritten, for the fundamental state $q = 1$,

$$\lambda = \chi_{\omega}^{(\mathrm{NSL})} \frac{c_t}{\omega_0}.$$

By reinjecting it into (20.73) and (20.74), we obtain

$$\boxed{q_W^{(\pm)} = \pm\, 8\pi\, \chi_{\omega}^{(\mathrm{NSL})}\, R_0^2\, \sqrt{\rho T}.} \tag{20.75}$$

20.5.4 Channel sign and admissible internal orientations

Origin of the sign. The sign of the weak-like charge is not carried by the scalar amplitude A^2, but by the internal time orientation of the tangential doublet:

$$\boxed{\rho_W^{(+)}(s) = +\rho\,\omega_0 A(s)^2, \qquad \rho_W^{(-)}(s) = -\rho\,\omega_0 A(s)^2.} \tag{20.76}$$

The weak-like channel therefore has an effective sign, determined by the direction of internal rotation of the cycle (u_1, u_2).

Two elementary orientations. At the minimal level, the 2DOF NSL branch admits at least two elementary internal orientations, denoted

$$\boxed{(+) \qquad \text{and} \qquad (-),}$$

with

$$\boxed{q_W^{(+)} = -\, q_W^{(-)}.}$$

Minimal composition rule. For two NSL sources, the composition of signs then follows the elementary rule

$$(+,+), \qquad (+,-), \qquad (-,+), \qquad (-,-),$$

with

$$q_W^{\mathrm{tot}} = q_W^{(1)} + q_W^{(2)}.$$

At the minimal level:

- two sources with the same orientation carry a nonzero total charge;
- two sources with opposite orientations may carry a zero total charge.

20.5.5 Global neutrality and internal non-triviality

Purpose. A composite solution may have a zero total weak-like charge without being trivial in the channel. It is therefore necessary to carefully distinguish *global charge neutrality* from *internal structural triviality.*

Total charge and compensation. For a composite solution made of two constituents carrying charges

$$q_W^{(1)}, \qquad q_W^{(2)},$$

the total charge is

$$q_W^{\mathrm{tot}} = q_W^{(1)} + q_W^{(2)}.$$

In the case of two NSL constituents with the same modulus and opposite orientations, we obtain

$$q_W^{(+)} + q_W^{(-)} = 0.$$

The composite solution is then globally neutral in the weak-like channel.

Neutrality does not imply triviality. The cancellation of the total charge does not mean that the channel disappears locally. Indeed, the densities

$$\rho_W^{(+)}(x), \qquad \rho_W^{(-)}(x)$$

may be nonzero separately, and their compensation holds only after global integration. A composite solution may therefore be globally neutral while carrying a nontrivial internal channel structure.

Consequence for the sequel. This distinction is the minimal structural point needed to later address composite solutions that are globally neutral in the weak-like channel.

20.5.6 Induced static interaction: theoretical status

Purpose. The preceding closure establishes a weak-like channel mediated by the radial response

$$\Phi_W \equiv r, \qquad \mathcal{L}_W \, \Phi_W = -\, g_W \, \widehat{\rho}_W, \qquad g_W = \frac{2}{\kappa_r R_0^3}.$$

On the harmonic 2DOF NSL branch at rest, the reduced source is

$$\widehat{\rho}_W^{(\pm)} = \pm A^2.$$

Existence of an induced interaction. As soon as two distinct NSL-branch sources are present, each generates a radial response that modifies the effective environment of the other. There is therefore, at the minimal theoretical level, a static interaction induced by the weak-like channel.

Relevant mediating chain. For two NSL sources, the mediation must be read as the induced chain

$$A^2 \;\longrightarrow\; r \;\longrightarrow\; A,$$

that is:

1. an NSL source excites the radial sector via $\widehat{\rho}_W$;
2. the radial response thereby created then back-reacts on a second NSL source.

Logical status. The present closure is therefore sufficient to establish the existence:

- d'un channel d'interaction weak-like ;
- of an explicit charged branch on the 2DOF NSL;
- and of a static interaction induced between branch sources.

Transition toward a minimal reading. In the absence of the complete integral law between two extended profiles, one may nevertheless immediately introduce a *minimal point-like monopolar reading* of the channel, based on the local gapped kernel associated with the operator $\mathcal{L}_W$. This is the purpose of the following subsection.

20.5.7 Minimal static reading of the channel and point-like force between two NSL sources

Principe. Once the weak-like channel is fixed by

$$\mathcal{L}_W \, \Phi_W = - \, g_W \, \widehat{\rho}_W, \qquad g_W = \frac{2}{\kappa_r R_0^3},$$

one may associate with two branch sources a minimal static reading of the channel, of point-like/monopolar type, by means of the local gapped kernel.

Local decay scale. We introduce the characteristic length of the channel

$$\boxed{\ell_W = \sqrt{\frac{T\kappa_r}{\mu_2}}.} \tag{20.77}$$

Minimal local kernel. In the isotropic local quasi–plane reduction, the Green kernel of the operator

$$-\Delta_{\mathrm{loc}} + \ell_W^{-2}$$

is

$$\boxed{G_W^{\mathrm{loc}}(R) = \frac{1}{4\pi} \, \frac{e^{-R/\ell_W}}{R},} \tag{20.78}$$

where R denotes the distance between the two sources.

Minimal cross energy. For two sources with branch charges $q_W^{(1)}$ and $q_W^{(2)}$, we then associate the minimal static cross energy

$$\boxed{E_{\mathrm{int}}^{(W)}(R) = - \, g_W^2 \, q_W^{(1)} q_W^{(2)} \, G_W^{\mathrm{loc}}(R) = - \frac{g_W^2 \, q_W^{(1)} q_W^{(2)}}{4\pi} \, \frac{e^{-R/\ell_W}}{R}.} \tag{20.79}$$

Minimal static force. The associated force is defined by

$$F_W(R) = - \frac{d}{dR} E_{\mathrm{int}}^{(W)}(R),$$

hence

$$\boxed{F_W(R) = - \frac{g_W^2 \, q_W^{(1)} q_W^{(2)}}{4\pi} \, \frac{e^{-R/\ell_W}}{R^2} \left(1 + \frac{R}{\ell_W}\right).} \tag{20.80}$$

Case of two 2DOF NSL sources at rest. On the harmonic 2DOF NSL branch at rest, the branch charges are

$$q_W^{(\pm)} = \pm \, 8\pi \, \chi_\omega^{(\mathrm{NSL})} \, R_0^2 \, \sqrt{\rho T}.$$

For two sources with the same orientation, we therefore obtain

$$\boxed{F_W^{(\mathrm{NSL,NSL})}(R) = - \frac{1}{4\pi} \left(\frac{2}{\kappa_r R_0^3}\right)^2 \left(8\pi \, \chi_\omega^{(\mathrm{NSL})} \, R_0^2 \, \sqrt{\rho T}\right)^2 \frac{e^{-R/\ell_W}}{R^2} \left(1 + \frac{R}{\ell_W}\right).} \tag{20.81}$$

with

$$\boxed{\ell_W = \sqrt{\frac{T\kappa_r}{\mu_2}}.} \tag{20.82}$$

Simplified form. The preceding expression simplifies to

$$\boxed{F_W^{(\mathrm{NSL,NSL})}(R) = -\frac{64\pi\,\kappa_\omega^{(\mathrm{NSL})\,2}\,\rho T}{\kappa_r^2\,R_0^2}\,\frac{e^{-R/\ell_W}}{R^2}\left(1+\frac{R}{\ell_W}\right).} \tag{20.83}$$

Sign of the interaction. The effective sign depends on the product $q_W^{(1)} q_W^{(2)}$. Two sources with the same internal orientation carry charges with the same sign; two sources with opposite orientations carry charges with opposite signs.

Scope of this reading. The formulas (20.79)–(20.83) define the point-like LO range of the weak-like channel. For extended NSL profiles, the same closure applies after projection onto the corresponding source profiles; the shape corrections then belong to the profile and overlap factors of the channel under consideration.

20.5.8 Summary and scope of the closure

Summary. The weak-like channel associated with the 2DOF NSL thus admits, at the minimal level, the following homogeneous writing:

$$\boxed{\mathcal{L}_W\,\Phi_W = -\,g_W\,\widehat{\rho}_W,}$$

with

$$\boxed{\mathcal{L}_W = \partial_{ss} + \frac{2}{R_0}\cot\Big(\frac{s}{R_0}\Big)\partial_s - \frac{\mu_2}{T\kappa_r}, \qquad g_W = \frac{2}{\kappa_r R_0^3},}$$

and, on the 2DOF NSL branch at rest,

$$\boxed{\widehat{\rho}_W^{(\pm)} = \pm A^2, \qquad \rho_W^{(\pm)} = \pm\rho\,\omega_0\,A^2, \qquad q_W^{(\pm)} = \pm\,8\pi\,\chi_\omega^{(\mathrm{NSL})}\,R_0^2\,\sqrt{\rho T}.}$$

Minimal structure of the channel. La closure actuelle fixe therefore :

- the weak-like mediator $\Phi_W \equiv r$,
- the channel operator $\mathcal{L}_W$,
- the field constant g_W,
- the canonical branch charge ρ_W,
- the reduced source $\widehat{\rho}_W$,
- the minimal signed structure carried by the two internal orientations of the 2DOF NSL,
- and a minimal point-like reading of the static interaction of the channel.

Validity scope. This closure is established for the harmonic 2DOF NSL at rest. The result used in what follows is therefore the local weak-like closure of this branch, and not a universal closure for all solutions of the theory.

Static interaction. The present closure provides a *minimal* Yukawa law for branch sources read at the point-like/monopolar level. The complete law of the interaction between two extended NSL profiles would depend on an additional integral reduction of the Green function. This reduction is not required in the present manuscript, because the closures consume only the point-like monopolar reading.

Scope of the closure. The present section establishes the local structure of the weak-like channel on the 2DOF NSL branch and the minimal point-like form of its static interaction. The integral reduction between extended sources is not used in the closures developed in the present part; it is therefore not introduced as an autonomous result.

Chapter 21

Field–solution coupling

21.1 General criterion for the sensitivity of solutions to field channels

Object. The purpose of the present chapter is not to redefine each of the channels already established, but to extract from them a transversal consequence: *which solutions are actually sensitive to which channel, and at which level.* To avoid any ambiguity, three logically distinct notions must be separated:

1. the *existence of a local source* of the channel;
2. the *existence of an effective monopolar coupling* to the associated field;
3. the possible presence of a *non-trivial internal structure* even when the total charge of the channel is zero.

This distinction will be decisive for globally neutral composite solutions.

Reading convention. Throughout this section, a solution will be said to be *sensitive* to a channel when the effective reduction of the problem makes, for that solution, either a non-zero source of the corresponding field or a non-trivial interaction energy in the considered channel appear. Conversely, a solution will be said to be *insensitive at the monopolar level* when it does not carry the required channel charge or projection, even if a residual internal structure may remain.

21.1.1 Gravitation-like channel: universality on localized solutions

Working hypothesis. In what follows, a *localized solution with a non-trivial core* denotes a solution whose core is not identically zero, whose total energy is finite, and whose localization is effective at the scale $\lambda \ll R_0$. Delocalized or uniform modes are therefore not targeted by the present proposition. Under this localization hypothesis, one has in particular

$$\omega \sim \frac{c_t}{\lambda} \qquad \Longrightarrow \qquad \omega R_0 \gg c_t,$$

so that the asymptotic gravitation-like closure of the local regime is indeed applicable.

Proposition 21.1 (Universality of the gravitation-like coupling for localized solutions)**.** *At LO order, any localized solution with a non-trivial core and finite positive total energy is sensitive to the gravitation-like channel. This sensitivity depends neither on the sign of an internal charge nor on a possible electromagnetic-, weak- or strong-like neutrality, but only on the localized energy content of the solution.*

Proof. According to the gravitation-like closure established above, in the relevant local regime one has

$$Q_g \simeq \frac{E_{\mathrm{tot}}^{\mathbb{S}^3}}{T\kappa_r R_0} \qquad \text{(see (20.26))}.$$

The gravitation-like channel therefore directly reads the energy content of the core, and not a signed internal orientation of phase or doublet type.

It follows that, for any localized solution with a non-trivial core, the total energy $E_{\mathrm{tot}}^{\mathbb{S}^3}$ is finite and positive; hence

$$Q_g \neq 0.$$

The solution then sources a non-trivial exterior radial field and is, by definition, sensitive to the gravitation-like channel.

This conclusion is independent of:

- the sign of an internal channel charge;
- a possible electromagnetic, weak-like or strong-like neutrality;
- whether the solution is elementary or composite,

since these properties do not cancel its localized energy content. □

Consequence. At the level of the present manuscript, the gravitation-like channel is the only channel that is truly *universal* on the whole set of already constructed localized solutions.

Remark. A solution can therefore be globally neutral in another channel while remaining gravitationally active. Charge neutrality in an internal channel never implies gravitation-like insensitivity, because this latter channel does not couple to a signed orientation but to localized energy.

21.1.2 Electromagnetic-like channel: selection by phase current

Proposition 21.2 (Monopolar criterion of EM-like sensitivity)**.** *At the monopolar level, a solution is sensitive to the electromagnetic-like channel if and only if its Noether phase charge*

$$Q_{\mathrm{N}} := \int_{\mathbb{S}^3} j^0 \, dV$$

is non-zero, where

$$j^0 = \rho \, \Im(\psi^* \dot{\psi}).$$

In other words, the EM-like channel couples exactly the solutions whose $U(1)$ phase current carries a non-zero monopolar component. The terminology of Noether current and $U(1)$ symmetry is used in its standard field-theory sense [17, 16]. The normalization of the current and of the charge, however, is that of the HSM framework.

Proof. The EM-like channel is, by construction, the Noether channel associated with the phase symmetry $U(1)$. For a branch written in the form

$$\psi = A \, e^{i\phi},$$

one has

$$j^0 = \rho \, A^2 \, \partial_t \phi.$$

The monopolar EM-like coupling is therefore governed by an oriented quantity: the local direction of phase rotation.

If the global integral

$$Q_{\mathrm{EM}} = \int_{\mathbb{S}^3} j^0 \, dV$$

is non-zero, the solution sources a non-zero EM-like monopole and is sensitive to the channel at the monopolar level.

If, conversely, the phase structure cancels globally — for example in a linear polarization, in a superposition of branches with opposite phases, or in a composite combination with zero total charge — then

$$Q_{\mathrm{N}} = 0,$$

and the monopolar coupling to the EM-like channel disappears. □

Structural remark. One must carefully distinguish:

- the local existence of a phase current j^0;
- the existence of a total monopolar Noether charge $Q_{\mathrm{N}} \neq 0$.

A solution may therefore be locally non-trivial in the EM-like channel while being globally neutral at the monopolar level.

Practical reading. EM-like selection is therefore made by *monopolar phase orientation*:

- an oriented circular branch generally carries a non-zero EM-like charge;
- a real branch, a linearly polarized branch, or a combination of opposite charges is EM–neutral at the monopolar level;
- for strongly overlapping composites, the neutrality or total charge of the channel is fixed by the monopolar sum of the internal contributions, while the effective amplitude may still be modulated by a form factor.

Lemma 21.3 (Source–field pairing of the EM-like channel)**.** *Let $\mathcal{S}$ be a localized solution carrying a phase current $j^\mu_{\mathcal{S}} = (j^0_{\mathcal{S}}, \vec{j}_{\mathcal{S}})$. The monopolar Noether charge associated with the solution is*

$$Q_{\mathrm{N}}(\mathcal{S}) := \int_{\mathbb{S}^3} j^0_{\mathcal{S}} \, dV. \tag{21.1}$$

The source charge that appears in the Maxwell-like equations of the field is

$$Q_{\mathrm{field}}(\mathcal{S}) := \int_{\mathbb{S}^3} \rho_{\mathrm{EM},\mathcal{S}} \, dV. \tag{21.2}$$

With the source dictionary of the EM-like channel (16.4)*, these two charges are related by*

$$\boxed{Q_{\mathrm{field}}(\mathcal{S}) = \frac{1}{g_\Omega} Q_{\mathrm{N}}(\mathcal{S}).} \tag{21.3}$$

Proof. The source dictionary of the EM-like channel gives

$$\rho_{\mathrm{EM},\mathcal{S}} = \frac{1}{g_\Omega} j^0_{\mathcal{S}}.$$

Integrating over $\mathbb{S}^3$, one obtains

$$Q_{\mathrm{field}}(\mathcal{S}) = \int_{\mathbb{S}^3} \rho_{\mathrm{EM},\mathcal{S}} \, dV = \frac{1}{g_\Omega} \int_{\mathbb{S}^3} j^0_{\mathcal{S}} \, dV = \frac{1}{g_\Omega} Q_{\mathrm{N}}(\mathcal{S}).$$

□

Status of the solution charge. The charge Q_{N} measures the internal Noether content of the solution. The charge Q_{field} is the charge actually seen by the Maxwell-like field. The relative motifs established below concern the solution charge through this source–field dictionary. The quantum $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$, defined separately by the closure of the EM-like channel, fixes the elementary scale of the field; it is not assumed to be equal to the mechanical Noether charge.

21.1.3 EM-like projection between internal witnesses of composites

Object. The monopolar criterion (21.2) decides whether a solution carries a total EM-like charge. For neutral composite solutions, this criterion is not sufficient to describe internal couplings: internal witnesses may remain sensitive to the EM-like channel even when the total charge of the composite is zero. Here we introduce the projection rule of the EM-like channel on two internal witnesses of a composite.

Consumed field channel. The EM-like field channel is the two-source-projection channel defined by the EM-like closure, with dimensionless constant Γ_{EM}, see (20.36). In the present chapter, this constant is consumed as the universal amplitude of the channel, before projection onto a particular composite solution.

Width compatibility. Let two internal witnesses be carried by two reduced EBK indices q_a and q_b. To avoid any ambiguity with wavelength or local-curvature notations used elsewhere, we write here

$$w_q := q\, w_1$$

for the reduced axial width associated with the witness q.

Width compatibility is first defined from the normalized axial profile inherited from the HSM canonical core,

$$a(\xi) = \operatorname{sech}\xi.$$

We set

$$I_a := \int_{-\infty}^{+\infty} a(\xi)^2\, d\xi, \qquad u_q(x) := \frac{1}{\sqrt{w_q I_a}}\, a\!\left(\frac{x}{w_q}\right),$$

so that

$$\int_{-\infty}^{+\infty} u_q(x)^2\, dx = 1.$$

The linear overlap of the two witnesses is then

$$\boxed{\Omega_{ab} := \int_{-\infty}^{+\infty} u_{q_a}(x)\, u_{q_b}(x)\, dx.} \tag{21.4}$$

Introducing the width ratio

$$r_{ab} := \frac{w_{q_a}}{w_{q_b}} = \frac{q_a}{q_b},$$

one obtains the dimensionless form

$$\boxed{\Omega_{ab} = \frac{\sqrt{r_{ab}}}{I_a} \int_{-\infty}^{+\infty} a(r_{ab}\xi)\, a(\xi)\, d\xi.} \tag{21.5}$$

The canonical profiled overlap is the stability witness associated with the HSM profile $a(\xi) = \operatorname{sech}\xi$:

$$\boxed{\Xi_{ab}^{(\mathrm{sech})} := \Omega_{ab}^2 = \left[\frac{\sqrt{r_{ab}}}{I_a} \int_{-\infty}^{+\infty} a(r_{ab}\xi)\, a(\xi)\, d\xi\right]^2.} \tag{21.6}$$

This factor is symmetric under $a \leftrightarrow b$, bounded by $0 < \Xi_{ab}^{(\mathrm{sech})} \leq 1$, and is equal to 1 when the two widths are identical. The square comes from the Galerkin bilinear form: the effective coupling contains one mode factor for each witness, then the quadratic projection contracts the two mode factors.

Tail closure used in the projective reduction. In the matching zone between witnesses of very different fineness, the effective coupling is dominated by the tail of the canonical profile,

$$\operatorname{sech}\xi \simeq 2e^{-|\xi|} \qquad (|\xi| \gg 1).$$

The tail closure then gives

$$\boxed{\Xi_{ab}^{(\mathrm{tail})} = \left(\frac{2\sqrt{w_{q_a} w_{q_b}}}{w_{q_a} + w_{q_b}}\right)^2 = \frac{4q_a q_b}{(q_a + q_b)^2}.} \tag{21.7}$$

This expression is the LO closure used in the numerical projective reductions that consume (21.11). To lighten the notation in this channel, we set

$$\boxed{\Xi_{ab} := \Xi_{ab}^{(\mathrm{tail})}.} \tag{21.8}$$

The difference between $\Xi_{ab}^{(\mathrm{tail})}$ and $\Xi_{ab}^{(\mathrm{sech})}$ measures the profile stability of the width closure.

Projective half-cycle of the witnesses.

Projective half-cycle of the witnesses. An internal bilobe witness defines a channel axis rather than an oriented arrow. Opposite orientations therefore represent the same axis:

$$n \sim -n.$$

The canonical projective measure is then that of the half-cycle

$$\boxed{\theta \in [0, \pi], \qquad d\mu_{\mathrm{proj}} = \frac{d\theta}{\pi}.} \tag{21.9}$$

Axial–transverse projection. When the EM-like channel couples an axial witness and a transverse witness, the instantaneous projection between the two axes is $|\cos\theta|$. The mean sector projection is therefore

$$\boxed{\mathcal{P}_{\mathrm{ax}\perp} = \frac{1}{\pi}\int_0^{\pi} |\cos\theta|\, d\theta = \frac{2}{\pi}.} \tag{21.10}$$

Proposition 21.4 (Projected EM-like coupling between two internal witnesses)**.** *Let T_a, T_b be two internal witnesses of a composite, carried by the EBK scales q_a, q_b. If their effective channel is an axial–transverse EM-like channel, then the reduced off-diagonal term induced by the EM-like channel is*

$$\boxed{H_{ab}^{(\mathrm{EM,proj})} = \Gamma_{\mathrm{EM}}\, \Xi_{ab}^{(\mathrm{tail})}\, \mathcal{P}_{\mathrm{ax}\perp} = \Gamma_{\mathrm{EM}}\, \frac{4q_a q_b}{(q_a + q_b)^2}\, \frac{2}{\pi}.} \tag{21.11}$$

Proof. The constant Γ_{EM} provides the universal amplitude of the EM-like two-source-projection channel, defined in the field closure. Passing to two internal witnesses of a composite adds two field–solution coupling factors. The first is the tail width compatibility, $\Xi_{ab}^{(\mathrm{tail})}$, defined by (21.7). The profiled overlap $\Xi_{ab}^{(\mathrm{sech})}$, defined by (21.6), provides the profile-stability control. The second is the axial–transverse projection of the witness axes, given by (21.10). The product of these three factors gives (21.11). □

Separation of roles. The constant Γ_{EM} is a channel constant. The factors Ξ_{ab} and $\mathcal{P}_{\mathrm{ax}\perp}$ belong to the field–solution coupling. They do not modify the definition of the EM-like field; they only describe the projection of this field on the internal witnesses of the considered composite.

21.1.4 Projected EM-like charge of the 3DOF bump

Object. Here we make explicit the consequence of the monopolar criterion (21.2) for the isolated 3DOF bump branch. The 3DOF bump carries a phase rotation in the active tangent subspace, but this rotation is projected by the conical geometry of the solution. The associated projective factor is

$$\sigma = \sin^2 \alpha.$$

Conical ansatz. At LO order, the circular tangent part of the 3DOF bump is written, up to a common envelope factor, as

$$\vec{u}_{\parallel}(s,t) = A(s) \sin\alpha \, (\cos \omega t \, \vec{e}_\theta + \varsigma \, \sin \omega t \, \vec{e}_\phi) \,, \qquad \varsigma = \pm 1.$$

The radial component proportional to $\cos\alpha$ carries no EM-like phase current. The EM-like Noether current is therefore carried only by the circular tangent component.

Projected charge density. By the phase-current criterion (21.2), the EM-like monopolar density is proportional to the product

$$\rho \, \Im(\psi_{\parallel}^* \dot{\psi}_{\parallel}).$$

Since the circular tangent amplitude is multiplied by $\sin\alpha$, the quadratic charge density is multiplied by

$$\sin^2 \alpha = \sigma.$$

One therefore obtains, at LO order,

$$\boxed{j^0_{\mathrm{EM,b3}}(s;\alpha) = \sigma \, j^0_{\mathrm{EM,b2}}(s), \qquad \sigma = \sin^2 \alpha.} \tag{21.12}$$

Global charge. Integrating over the core, the same projective factor is conserved:

$$\boxed{Q^{(\mathrm{3DOF})}_{\mathrm{EM}}(\alpha) = \sigma \, Q^{(\mathrm{2DOF})}_{\mathrm{EM}}, \qquad \sigma = \sin^2 \alpha.} \tag{21.13}$$

The sign of the charge is fixed separately by the phase orientation $\varsigma = \pm 1$ and by charge conjugation.

Conical branches of the 3DOF bump. For the isolated 3DOF bump, the two canonical conical branches are those of the conical quantization of the 3DOF branch,

$$\sigma_\downarrow = \frac{1}{3}, \qquad \sigma_\uparrow = \frac{2}{3}.$$

Thus,

$$\boxed{\left| \frac{Q^{(\mathrm{3DOF})}_{\mathrm{EM}}}{Q^{(\mathrm{2DOF})}_{\mathrm{EM}}} \right| \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.} \tag{21.14}$$

With the reference 2DOF bump branch, whose relative modulus is 1, the relative motif of charge moduli is therefore

$$\boxed{1, \qquad \frac{2}{3}, \qquad \frac{1}{3}.} \tag{21.15}$$

21.1.5 Compact EM-like coupling between 3DOF bump constituents

Object. The previous subsection fixes the relative EM-like charges of the 3DOF bump branches. Here we define the compact EM-like coupling term between three 3DOF bump constituents assembled into a color–white CB3 composite. This term consumes the elementary coupling of the EM-like channel $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$, defined by (20.34).

Compact EM contrast of a triad. For a triad of reduced charges

$$\mathbf{q} = (q_1, q_2, q_3),$$

we define the normalized pairwise contrast by

$$\boxed{\mathcal{C}_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}) := 3 \sum_{1\le i<j\le 3} q_i q_j.} \tag{21.16}$$

The factor 3 normalizes the three internal pairs of the compact triad on a common contrast unit.

Compact EM-like energy. At compact LO order, the EM-like contribution of the CB3 composite is

$$\boxed{\Delta E_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}) = \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} \frac{E_* \tau_3}{2\pi} \mathcal{C}_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}).} \tag{21.17}$$

The quantities E_* and τ_3 are those of the compact CB3 closure; the present chapter only fixes the EM-like coupling law associated with the reduced charges.

Proposition 21.5 (Compact EM-like correction of the light CB3 doublet)**.** *For the two reduced motifs*

$$\mathbf{q}_{\uparrow\uparrow\downarrow} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right), \qquad \mathbf{q}_{\uparrow\downarrow\downarrow} = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right),$$

the compact EM-like difference is

$$\boxed{\Delta E_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}_{\uparrow\downarrow\downarrow}) - \Delta E_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}_{\uparrow\uparrow\downarrow}) = -\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} \frac{E_* \tau_3}{2\pi}.} \tag{21.18}$$

Proof. For the first motif,

$$\sum_{i<j} q_i q_j \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0.$$

For the second motif,

$$\sum_{i<j} q_i q_j \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right) = -\frac{2}{9} - \frac{2}{9} + \frac{1}{9} = -\frac{1}{3}.$$

By the definition (21.16), one therefore obtains

$$\mathcal{C}_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}_{\uparrow\downarrow\downarrow}) - \mathcal{C}_{\mathrm{EM}}^{\mathrm{CB3}}(\mathbf{q}_{\uparrow\uparrow\downarrow}) = 3\left(-\frac{1}{3} - 0\right) = -1.$$

The identity (21.18) then follows directly from (21.17). □

21.1.6 EM-like charge of the 2DOF NSL

Object. Here we make explicit the EM-like Noether charge carried by a 2DOF NSL solution at rest. This charge will then be used to construct the EM-like coupling of the neutral two-constituent composite.

2DOF NSL ansatz. For a 2DOF NSL at rest, we write

$$u_1(s,t) = A(s)\cos(\omega_{\mathrm{NSL}}t), \qquad u_2(s,t) = A(s)\sin(\omega_{\mathrm{NSL}}t),$$

that is, in complex notation,

$$\psi(s,t) = u_1 + i\,u_2 = A(s)e^{i\omega_{\mathrm{NSL}}t}.$$

In the canonical local reduction of the 2DOF NSL constituent, one works with the dimensionless profile $A(u) = \operatorname{sech} u$, normalized by $A(0) = 1$, cf. (6.126)

EM-like charge density. By definition of the phase Noether current,

$$j^0 = \rho\,\Im(\psi^*\dot{\psi}).$$

Here one has

$$\dot{\psi} = i\omega_{\mathrm{NSL}}A(s)e^{i\omega_{\mathrm{NSL}}t}, \qquad \psi^*\dot{\psi} = i\omega_{\mathrm{NSL}}A(s)^2,$$

hence

$$\boxed{j^0_{\mathrm{EM,NSL}}(s) = \rho\,\omega_{\mathrm{NSL}}\,A(s)^2.} \tag{21.19}$$

Changing the internal orientation $u_2 \mapsto -u_2$ changes the sign of the charge and gives the two opposite branches:

$$\boxed{{j^0_{\mathrm{EM,NSL}}}^{(\pm)}(s) = \pm\,\rho\,\omega_{\mathrm{NSL}}\,A(s)^2} \tag{21.20}$$

Global charge. Integration over $\mathbb{S}^3$ gives

$$Q^{(\pm)}_{\mathrm{EM}}(\mathrm{NSL}) = \int_{\mathbb{S}^3} {j^0_{\mathrm{EM,NSL}}}^{(\pm)}\,dV = \pm\,\rho\,\omega_{\mathrm{NSL}}\int_{\mathbb{S}^3} A(s)^2\,dV. \tag{21.21}$$

At local LO order, one uses the core measure (4.1), namely

$$dV_{\mathrm{c}} = J_{\mathrm{c}}(s)\,ds\,d\Omega, \qquad J_{\mathrm{c}}(s) \simeq R_0^2, \qquad \lambda_{\mathrm{NSL}} \ll R_0,$$

so that, in the quasi-planar cap,

$$dV_{\mathrm{c}} \simeq R_0^2\,ds\,d\Omega.$$

With $\int d\Omega = 4\pi$, the change of variable $s = \lambda_{\mathrm{NSL}}u$, and

$$A(u) = \operatorname{sech} u, \qquad I_0 := \int_{\mathbb{R}} A(u)^2\,du = 2 \quad \text{by (6.115)},$$

one obtains

$$\int_{\mathbb{S}^3} A(s)^2\,dV \simeq 4\pi R_0^2\lambda_{\mathrm{NSL}}\int_{\mathbb{R}} A(u)^2\,du = 8\pi R_0^2\lambda_{\mathrm{NSL}}.$$

Moreover, the NSL angular frequency satisfies

$$\omega_{\mathrm{NSL}} = \chi_\omega^{(\mathrm{NSL})}\,\frac{c_t}{\lambda_{\mathrm{NSL}}}, \qquad c_t = \sqrt{\frac{T}{\rho}}, \tag{21.22}$$

cf. (6.120). It follows that

$$\boxed{Q^{(\pm)}_{\mathrm{EM}}(\mathrm{NSL}) = \pm\,8\pi\,\chi_\omega^{(\mathrm{NSL})}\,R_0^2\,\sqrt{\rho T}.} \tag{21.23}$$

Status. The 2DOF NSL therefore indeed carries a non-zero monopolar EM-like charge, with oriented sign. This charge is a coupling object; it must not be confused with the internal weak-like charge of the NSL doublet.

21.1.7 EM-like charge of the 3DOF NSL

Object. The 3DOF NSL is an elementary 3DOF solution, and not a composite. Its EM-like coupling is therefore read directly on the phase current of the circular tangent component of the conical ansatz

$$\vec{u}_0(s,t) = A(s)\Big[\sin\alpha\big(\cos\omega_0 t\,\vec{e}_\theta + \varsigma\,\sin\omega_0 t\,\vec{e}_\phi\big) + \cos\alpha\,\vec{e}_r\Big], \qquad \varsigma = \pm 1.$$

The radial component proportional to $\cos\alpha$ carries no EM-like phase current.

Phase label. The discrete EM-like charge label is carried by the orientation of the tangent rotation:

$$\boxed{n_{\mathrm{EM}}^{(\mathrm{NSL3})} = \varsigma \in \{-1,+1\}.} \tag{21.24}$$

Inversion of the internal orientation $\varsigma \mapsto -\varsigma$ therefore reverses the sign of the EM-like charge, without modifying the core width or the mass.

Conical projection. The EM-like phase current is quadratic in the circular tangent amplitude. This amplitude is multiplied by $\sin\alpha$. The effective modulus of the EM-like coupling is therefore projected by

$$\sigma = \sin^2\alpha.$$

At LO order one obtains:

$$\boxed{q_{\mathrm{eff}}^{(\mathrm{NSL3})}(\sigma) = \sigma\, n_{\mathrm{EM}}^{(\mathrm{NSL3})}, \qquad \sigma = \sin^2\alpha \in \left\{\frac{1}{3},\frac{2}{3}\right\}.} \tag{21.25}$$

Conical branches. The two conical branches of the 3DOF NSL therefore give

$$\boxed{q_{\mathrm{eff}}^{(\mathrm{NSL3})} \in \left\{\pm\frac{1}{3},\pm\frac{2}{3}\right\}.} \tag{21.26}$$

These numbers are relative effective charges of the EM-like channel. By the source–field pairing lemma (21.3), they apply to the field charge Q_{field}. The elementary absolute scale is provided by the quantum $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$, defined in the EM-like closure.

21.1.8 Weak-like channel: strict selectivity of the NSL branch

Proposition 21.6 (Branch selectivity of the weak-like channel)**.** *The weak-like channel is not universal. At the currently closed minimal level, it selects the 2DOF NSL structure and, more precisely, the internal doublet (u_1,u_2) carrying the canonical density*

$$\rho_W = \rho\,(u_1\partial_t u_2 - u_2\partial_t u_1).$$

Consequently, any solution that does not carry this NSL structure is insensitive to the weak-like channel at the minimal monopolar level.

Proof. In the minimal closure established for this channel, the response field is identified with the radial variable r, but the canonical source is not the local energy alone: it is given by the internal current of the NSL doublet. The theory explicitly fixes the rule

$$\rho_W[S] = 0 \qquad \text{if the solution } S \text{ does not carry the 2DOF NSL structure.}$$

It follows that the weak-like channel does not couple, at the minimal level, a generic localized solution, but only the NSL branch and the composites that structurally inherit it. □

Consequence for neutral composites. A composite solution can be globally neutral in this channel while remaining locally non-trivial. This is precisely the case of a neutral Z^0-like composite: the total channel charge is zero,

$$q_W^{(\mathsf{Z}^0)} = 0,$$

but this neutrality does not imply pointwise cancellation of the internal channel density. The monopolar coupling is therefore zero, while an internal channel structure may remain.

21.1.9 Preclusion channel: local universality under overlap, without mediator field

Proposition 21.7 (Exact scope of the Preclusion channel)**.** *The Preclusion channel is not an additional fundamental field: it emerges variationally from the overlap of localized cores. Any localized solution is therefore sensitive to the Preclusion channel as soon as it enters a regime of non-negligible overlap with another localized solution, but an isolated solution does not by itself carry a long-range propagated Preclusion monopole.*

Proof. The Preclusion mechanism comes from the gradient overcost and phase constraints in the transverse sector. Its effective law is a finite-range repulsion governed by the overlap of the core profiles. It is therefore not an autonomous field propagated by a local charge, but an interaction energy induced when two cores overlap. Sensitivity to the channel is then universal on localized families *in an overlap situation*, whether they are 2DOF, 3DOF, neutral or charged in other channels. Conversely, for an isolated solution, there is no exterior Preclusion field to source. □

Doctrinal conclusion. The Preclusion channel is therefore universal as a *local mechanism of core exclusion*, but not as an autonomous monopolar field sector.

21.1.10 Strong-like channel: 3DOF selectivity and color projector

Proposition 21.8 (3DOF selectivity of the strong-like channel)**.** *The strong-like channel couples only the 3DOF component projected onto the internal color direction. It is therefore absent for purely 2DOF solutions and acts, among the already constructed localized branches, only on 3DOF solutions possessing a non-zero color projection.*

Proof. The strong-like channel is obtained by projection of the 3DOF tangent sector on an internal color projector. The associated strong charge takes the form

$$I_{\mathsf{s}} = \mathcal{K}_{\mathsf{s}}\, \Pi_{\mathsf{col}}[\text{core 3DOF}],$$

which immediately shows that the channel does not read a pure 2DOF solution, for lack of a 3DOF component to project. Conversely, any localized 3DOF solution carrying a non-zero color projection can source the channel. For multi-constituent systems, the whiteness condition then decides whether the net contribution is attractive, zero, or not retained. □

Scope. The strong-like channel is therefore selective at two levels:

1. it first requires a 3DOF structure;

2. it then requires, for an effective bond, a white combination in this channel.

21.1.11 Synthesis by solution families

Corollary 21.9 (Coupling status of the already constructed families)**.** *In the present state of the theory, the solution families are classified according to the table below.*

Reading convention. In the table below, *yes* means that an effective coupling is established for the considered canonical branch, and *no* means that it is not established in the present state of the theory. For the Preclusion channel, the very word coupling refers to the overlap regime; this condition is therefore not repeated in each row.

Family	**Grav.**	**EM**	**Weak**	**Strong**	**Preclusion**	**Remarks**
Bump 2DOF	yes	yes	no	no	yes	Localized solution with phase rotation by construction.
NSL 2DOF	yes	yes	yes	no	yes	Primitive source branch of the minimal weak-like channel.
Z^0-like neutral 2DOF	yes	no	no	no	yes	Neutral composite of two 2DOF NSLs; monopolar neutrality in the EM-like and weak-like channels.
Neutral 2DOF breather	yes	no	no	no	yes	LO-neutral branch in the EM-like channel; no primitive weak-like charge.
Quasi-c_t 2DOF	yes	yes	no	no	yes	Transverse vector 2DOF branch; sensitive to the Preclusion channel in the overlap regime like any localized solution.
Isolated 3DOF bump	yes	yes	no	yes	yes	Charged 3DOF solution; the two LO geometric branches correspond to $\sigma = \sin^2\alpha \in \{1/3,\, 2/3\}$, and not to a neutral/charged distinction.
Quasi-c_t 3DOF	yes	yes	no	no	yes	Transverse vector 3DOF branch; sensitive to the Preclusion channel in the overlap regime like any localized solution.
Z-like 3DOF bilobe (Z2)	yes	no	no	yes	yes	Canonical white branch of motif $(1,-1)$, with $n_{\mathrm{EM}} = 0$.
Z-like 3DOF neutral trilobe (Z3, n)	yes	no	no	yes	yes	Canonical white branch of motif $(1,-2,1)$, with $n_{\mathrm{EM}} = 0$.
Z-like 3DOF charged trilobe (Z3, c)	yes	yes	no	yes	yes	Canonical white branch of motif $(1,-1,1)$, with $n_{\mathrm{EM}} = 1$.
3DOF bilobe breather (Br2)	yes	no	no	yes	yes	Canonical white branch of motif $(1,-1)$, with $n_{\mathrm{EM}} = 0$.
3DOF neutral trilobe breather (Br3, n)	yes	no	no	yes	yes	Canonical white branch of motif $(1,-2,1)$, with $n_{\mathrm{EM}} = 0$.
3DOF charged trilobe breather (Br3, c)	yes	yes	no	yes	yes	Canonical white branch of motif $(1,-1,1)$, with $n_{\mathrm{EM}} = 1$.

It is this demonstrated selectivity, and not a qualitative analogy, that completes the field–solution coupling framework of the present manuscript.

Scope of the table. The Z-like and Breather composite branches mentioned above are those already constructed in the chapters on composite solutions; the present table does not restate their internal definition, but only synthesizes their coupling status with respect to the field channels.

21.2 Internal s coupling and 3DOF confinement

21.2.1 Strong-like response of an isolated 3DOF bump and passage to white sources

Object. A 3DOF bump solution can carry an internal strong-like source. This source characterizes the mechanical response of the isolated solution in the strong-like channel. However, by itself, it does not constitute the closure of a color–white composite: in a white composite, the strong-like channel must be evaluated on the total white source.

The present subsection therefore fixes two distinct objects:

isolated source of a 3DOF bump, total white source of a 3DOF composite.

Isolated response. Let S_i be the effective strong-like source carried by an isolated 3DOF bump. The linear response of the strong-like channel is written

$$\mathcal{L}_{\mathsf{s}}\Phi_{\mathsf{s}} = S_i,$$

where $\mathcal{L}_{\mathsf{s}}$ is the effective operator of the strong-like channel. The self-induced energy associated with the isolated solution is

$$\boxed{E_{\mathsf{s}}^{\mathrm{iso}}[S_i] = \frac{1}{2}\langle S_i, G_{\mathsf{s}} S_i\rangle, \qquad G_{\mathsf{s}} := \mathcal{L}_{\mathsf{s}}^{-1}.} \tag{21.27}$$

This expression is an internal response energy of the strong-like channel.

Conical signatures of branch. The conical branches of a 3DOF solution are denoted

$$X \in \{\downarrow, \uparrow\},$$

with

$$\sigma_X := \sin^2 \alpha_X, \qquad \rho_X := \cos^2 \alpha_X.$$

The canonical projection values are not introduced by external convention: they are the output of the geometric quantization of the conical angle, cf. (9.8). We therefore use

$$\boxed{\sigma_\downarrow = \frac{1}{3}, \qquad \sigma_\uparrow = \frac{2}{3}, \qquad \rho_\downarrow = \frac{2}{3}, \qquad \rho_\uparrow = \frac{1}{3}.} \tag{21.28}$$

Elementary holonomy of the phase–circulating branch. The branch $\downarrow$ is taken as the reference compressive branch; its arm amplitude is normalized by

$$\chi_\downarrow = 1.$$

The branch $\uparrow$ carries a phase–circulating component. In a white composite with three equivalent junctions, color closure imposes that the total circulating phase around the tri-junction be an integer cycle:

$$\varphi_1 + \varphi_2 + \varphi_3 = 2\pi\nu, \qquad \nu \in \mathbb{Z}.$$

The elementary non-trivial branch corresponds to $\nu = 1$. By symmetry of the white tri-junction, the three junctions carry the same fraction of this holonomy:

$$\varphi_1 = \varphi_2 = \varphi_3.$$

Hence

$$\varphi_i = \frac{2\pi}{3}.$$

We thus define the structural arm factor

$$\boxed{\chi_\downarrow = 1, \qquad \chi_\uparrow = \frac{2\pi}{3}.} \tag{21.29}$$

This factor is a structural holonomy of the white tri-junction; it results from the internal closure of the white composite, without an external parameter.

Effective arm source. For a 3DOF constituent of EBK level q_i and branch X_i, the strong-like arm source used in white composites is

$$\boxed{Q_i^{\mathrm{arm}} = q_i\, \sigma_{X_i}\, \chi_{X_i}.} \tag{21.30}$$

This arm source is not an isolated mass: it is a strong-like junction amplitude intended to be inserted into the total white source.

Color-white triad. The minimal white closure is represented by three internal unit vectors $\mathbf{C}_i$ arranged at 120° in the internal color plane. One may choose explicitly

$$\mathbf{C}_1 = (1,0), \qquad \mathbf{C}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad \mathbf{C}_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

One then verifies that

$$\mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 = 0, \qquad \mathbf{C}_i \cdot \mathbf{C}_j = -\frac{1}{2} \quad (i \neq j).$$

The white constraint used below is therefore the minimal equilateral triad of the internal color plane, and not a free numerical choice.

Total white source. For a white 3DOF composite formed from three constituents, we write

$$S_{\mathrm{tot}} = \sum_{i=1}^{3} \mathbf{C}_i S_i,$$

with the internal color vectors satisfying

$$\boxed{\mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 = 0, \qquad \mathbf{C}_i \cdot \mathbf{C}_j = -\frac{1}{2} \quad (i \neq j).} \tag{21.31}$$

The collective strong-like response is then

$$\boxed{E_{\mathsf{s}}^{\mathrm{white}} = \frac{1}{2} \left\langle S_{\mathrm{tot}}, G_{\mathsf{s}} S_{\mathrm{tot}} \right\rangle.} \tag{21.32}$$

Normal-ordered white defect. In a resolved branch, the isolated self-responses are not added before the formation of the white state. Instead, one measures the junction defect of the arms relative to their white mean. If R_i denotes the reduced source of arm i, we set

$$\bar{R} := \frac{1}{3}(R_1 + R_2 + R_3).$$

and the white defect is the mechanical variance

$$\boxed{\sum_{i=1}^{3} \|R_i - \bar{R}\|_{G_{\mathsf{s}}}^2 = \frac{1}{3} \sum_{i<j} \|R_i - R_j\|_{G_{\mathsf{s}}}^2.} \tag{21.33}$$

The canonical average on $\mathbb{S}^3$ introduces the factor

$$\frac{1}{2\pi^2}.$$

Thus, when the defect is written as a sum over the three pairs, its reduced coefficient is

$$\boxed{\Lambda_{\text{white}} = \frac{1}{3}\,\frac{1}{2\pi^2} = \frac{1}{6\pi^2}.} \tag{21.34}$$

Scope. Equations (21.27)–(21.34) fix the passage from the isolated strong-like channel to the collective white response. The mass closures of 3DOF composites must be constructed from the collective white source, and not as a sum of isolated self-responses.

21.2.2 Tubular reduction and effective tension of the strong-like channel

Object. The 3DOF strong-like channel has, in composite configurations where the flux is guided between two internal substructures, an effective one-dimensional reduction. The purpose of this subsection is to formulate the precise condition under which this reduction provides a positive-tension flux tube, and then an effective linear potential.

Transverse problem of the channel. In the thin-tube window, one separates the longitudinal coordinate z, carried by the channel axis, from the transverse coordinates $x_\perp$. At LO order, far from the two end zones, the transverse profile is assumed stationary along the tube. The strong-like functional restricted to a transverse section is written in the form

$$\mathcal{E}_\perp[\Phi] = \int_{\mathcal{A}_\perp} \left[\frac{1}{2}\,\langle \Phi, \mathcal{L}_{\mathsf{s}}\Phi\rangle + \mathcal{K}_{\mathsf{s}}\,\mathcal{V}_{\text{col}}(\Phi)\right] d^2x_\perp,$$

with a non-zero flux constraint

$$\mathcal{Q}_{\text{col}}[\Phi] = \mathcal{Q}_0 \neq 0.$$

The operator $\mathcal{L}_{\mathsf{s}}$ and the channel constant $\mathcal{K}_{\mathsf{s}}$ are the effective objects of the strong-like channel introduced in section §21.2. The present reduction does not redefine them: it uses their restriction to the transverse problem.

Definition of the effective tension. The effective tension of the tube is defined as the minimal transverse energy per unit length under flux constraint:

$$\boxed{\sigma_{\text{eff}} := \inf_{\mathcal{Q}_{\text{col}}[\Phi]=\mathcal{Q}_0} \mathcal{E}_\perp[\Phi].} \tag{21.35}$$

Positivity. Since the strong-like channel is coercive on admissible transverse profiles, a zero-cost section would simultaneously impose cancellation of the gradient, the core term, and the flux. This would contradict the constraint

$$\mathcal{Q}_{\text{col}}[\Phi] = \mathcal{Q}_0 \neq 0.$$

Therefore

$$\boxed{\sigma_{\text{eff}} > 0 \qquad (\mathcal{Q}_0 \neq 0).} \tag{21.36}$$

Longitudinal extensivity. In the central zone of the tube, the minimizing transverse profile is independent of z at LO order. The dominant contribution of the functional is then

$$E_{\text{core}}(R) = \int_0^R \sigma_{\text{eff}}\, dz = \sigma_{\text{eff}} R.$$

The end-localized corrections are grouped into a cap energy E_{cap}, independent of R at leading order. One therefore obtains

$$\boxed{E_{\text{tube}}(R) = \sigma_{\text{eff}} R + E_{\text{cap}} + o(R).} \tag{21.37}$$

The term $o(R)$ contains curvature, junction and slow-transverse-profile variation corrections. These corrections are subdominant in the length limit in which the tubular reduction is valid.

Effective confinement potential. The minimal static contribution associated with the tubular channel then provides a linear lower bound:

$$\boxed{V_{\text{conf}}(R) \geq \sigma_{\text{eff}}\, R - C,} \tag{21.38}$$

where C groups the local normalization constants, the junction contributions and the end terms. Positivity of σ_{eff} implies that free separation of the substructures carrying the flux requires an energy cost increasing with R.

Scope. This reduction does not reconstruct an independent strong gauge theory. It establishes the necessary result at the HSM level: in composite configurations where the strong-like channel is guided into a tube, transverse minimization defines a positive tension and a linear longitudinal energy. This result characterizes a longitudinal flux-tube-type binding energy.

Chapter 22

Neutral 2DOF Z^0-like composite

22.1 Neutral 2DOF Z^0-like solution: composite of two 2DOF NSLs, structure and LO closure

Positioning. The Z^0-like solution is described here not as a primitive unit branch, but as a *neutral composite* of two 2DOF NSL solutions in strong overlap. It inherits the 2DOF NSL radial core at LO (profile $a(\xi) = \operatorname{sech}\xi$, width λ, slow modulation, vector polarization), but its internal structure is determined by the balance between EM-like attraction and Preclusion cost on the two constituents. The selection of the neutral weak channel then occurs through the angular filter $\Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)}$ and the corresponding channel impedance.

22.1.1 Micro-local one-ring closure of the Z^0-like composite

Purpose. Here we apply to the Z^0-like composite the micro-local one-ring closure associated with its neutral-doublet structure. This closure does not act on an isolated NSL: it explicitly uses the fact that

$$\mathsf{Z}^0 = X_1 \oplus X_2,$$

where X_1 and X_2 are two 2DOF NSL constituents in strong overlap, with the same core width at LO.

This participation is distinct from the global ring participation depending on the ratio λ/R_0. Here, one considers a *micro-local* mesh–core participation, controlled by the ratio

$$\frac{\ell_0}{\lambda_{\mathsf{Z}}},$$

where λ_{Z} is the reduced phase length of the Z^0-like composite.

Reduced phase length. For the Z^0-like composite, we write

$$\boxed{\lambda_{\mathsf{Z}} := \frac{c_t}{\omega_{\mathsf{Z}}}.} \tag{22.1}$$

This is a reduced phase length: no additional factor 2π is introduced in this definition. The cyclic normalization is already carried by the absorbed EBK convention $E/\omega = h_{\mathrm{eff}}$.

We introduce the micro-local small parameter

$$\boxed{\varepsilon_{\mathrm{micro}}^{(\mathsf{Z})} := \frac{\ell_0}{\lambda_{\mathsf{Z}}}.} \tag{22.2}$$

One-ring action capacity. The effective action constant h_{eff} is the common constant of the energy–frequency law of the 2DOF sector, cf. (6.1)–(6.2). For the composite Z^0-like branch, the one-ring closure associates with this constant a branch-specific geometric capacity, denoted $\mathcal{C}_*^{(\mathsf{Z})}$. It is defined by inversion of the action relation

$$\boxed{\mathcal{C}_*^{(\mathsf{Z})} := \frac{2h_{\mathrm{eff}}}{\pi\,\rho\,c_t}.} \tag{22.3}$$

Equivalently,

$$h_{\mathrm{eff}} = \frac{\pi}{2}\,\rho\,c_t\,\mathcal{C}_*^{(\mathsf{Z})}.$$

The quantity $\mathcal{C}_*^{(\mathsf{Z})}$ fixes no external scale: it expresses the geometric capacity required to carry the action h_{eff} in the one-ring closure of the branch under consideration.

At LO, we factor this capacity in the form

$$\boxed{\mathcal{C}_*^{(\mathsf{Z})} = \mathcal{C}_{\mathrm{geo}}^{(\mathsf{Z})}\,R_0^2\,\lambda_{\mathsf{Z}}^2\,\Xi_{\mathrm{micro}}^{(\mathsf{Z})}.} \tag{22.4}$$

The factor $\mathcal{C}_{\mathrm{geo}}^{(\mathsf{Z})}$ is a pure number of profile and geometric closure. The factor $\Xi_{\mathrm{micro}}^{(\mathsf{Z})}$ is the effective micro-local participation of the Z^0-like composite.

Decomposition of the micro-local participation. At isotropic LO, the micro-local participation of the Z^0-like composite is a coherent-vertex participation. It acts on four reduced variables that are independent at LO:

$$y \in B_{\ell_0}, \qquad u_1, u_2, \qquad \varphi_{\mathrm{red}},$$

where y localizes the active mesh in the core, ν_i denotes the reduced spectral window of the constituent X_i, and φ_{red} the reduced fast phase. The LO measure therefore factorizes as

$$d\mu_{\mathrm{micro}}^{(\mathsf{Z})} = d\mu_{\mathrm{cell}}\,du_1\,du_2\,d\mu_{\mathrm{phase}}.$$

This gives the factorization

$$\boxed{\Xi_{\mathrm{micro}}^{(\mathsf{Z})} = F_{\mathrm{cell}}^{(\mathsf{Z})}F_{\mathrm{spec}}^{(\mathsf{Z})}F_{\mathrm{phase}}^{(\mathsf{Z})}.} \tag{22.5}$$

The three factors correspond to three distinct operations:

- F_{cell} selects the local volume of the coherent vertex;
- F_{spec} imposes the simultaneous spectral participation of the two NSL constituents of the neutral composite;
- F_{phase} imposes reduced phase coherence on the mesh.

This factorization does not introduce three independent physical volumes: the isotropic volumetric factor $4\pi/3$ appears only in F_{cell}. The spectral weights $d\nu_i$ are reduced resolution weights, normalized after extraction of the geometric window.

Lemma 22.1 (Isotropic mesh window)**.** *At local LO, the active mesh of radius ℓ_0 in a core of size λ_{Z} carries the isotropic fraction*

$$F_{\mathrm{cell}}^{(\mathsf{Z})} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3.$$

Proof. In the limit $\ell_0 \ll \lambda_{\mathsf{Z}} \ll R_0$, the metric of $\mathbb{S}^3$ is locally Euclidean on the mesh. The volume of the local ball of radius ℓ_0 is therefore

$$\mathrm{Vol}(B_{\ell_0}) = \frac{4\pi}{3}\ell_0^3.$$

Reported to the volumetric scale of the core λ_{Z}^3, this window gives

$$F_{\mathrm{cell}}^{(\mathsf{Z})} = \frac{\mathrm{Vol}(B_{\ell_0})}{\lambda_{\mathsf{Z}}^3} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3.$$

□

Lemma 22.2 (Joint spectral participation of the neutral doublet)**.** *For a neutral doublet formed by two 2DOF NSL constituents with the same width λ_{Z}, the micro-local spectral participation of the composite is*

$$F_{\mathrm{spec}}^{(\mathsf{Z})} = \left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^6. \tag{22.6}$$

Proof. We work in reduced variables in the core:

$$\widehat{x} := \frac{x}{\lambda_{\mathsf{Z}}}.$$

A mesh of size ℓ_0 then corresponds to a reduced resolution

$$\varepsilon_{\mathrm{micro}}^{(\mathsf{Z})} = \frac{\ell_0}{\lambda_{\mathsf{Z}}}.$$

For an NSL constituent X_i, the local spectral window is the resolution window on the three reduced directions of the local mode. After extraction of the geometric volume of the mesh, already carried by $F_{\mathrm{cell}}^{(\mathsf{Z})}$, the reduced spectral weight of one constituent is therefore the product of the three independent resolutions:

$$w_i = \varepsilon_{\mathrm{micro}}^{(\mathsf{Z})}\varepsilon_{\mathrm{micro}}^{(\mathsf{Z})}\varepsilon_{\mathrm{micro}}^{(\mathsf{Z})} = \left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3. \tag{22.7}$$

This weight is not a second volume factor: it counts the reduced spectral resolution of the constituent in the core, whereas $F_{\mathrm{cell}}^{(\mathsf{Z})}$ counts the spatial localization of the coherent vertex.

The neutral composite

$$\mathsf{Z}^0 = X_1 \oplus X_2$$

is active in the neutral channel only when the two constituents participate simultaneously in the same coherent vertex. At LO, the two reduced spectral windows are conditionally independent given the active mesh, because they act on the two constituents X_1 and X_2. The joint spectral participation is therefore

$$F_{\mathrm{spec}}^{(\mathsf{Z})} = w_1 w_2. \tag{22.8}$$

Since the two constituents have the same core width at LO,

$$w_1 = w_2 = \left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3.$$

We obtain

$$F_{\mathrm{spec}}^{(\mathsf{Z})} = \left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3 = \left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^6.$$

The factor 6 therefore comes from two cubic reduced spectral weights, one for each NSL constituent of the neutral doublet. It comes neither from the mesh volume, already counted in $F_{\mathrm{cell}}^{(\mathsf{Z})}$, nor from the two internal legs of an isolated NSL. □

Lemma 22.3 (Reduced phase duty cycle)**.** *At LO, the coherent fraction of reduced phase carried by an elementary mesh of length ℓ_0 in a mode of reduced length $\lambda_{\mathsf{Z}} = c_t/\omega_{\mathsf{Z}}$ is*

$$F^{(\mathsf{Z})}_{\text{phase}} = \frac{\ell_0}{\lambda_{\mathsf{Z}}}. \tag{22.9}$$

Proof. The local fast phase is written in the absorbed EBK convention:

$$E/\omega = h_{\text{eff}}.$$

The reduced phase length is therefore

$$\lambda_{\mathsf{Z}} = \frac{c_t}{\omega_{\mathsf{Z}}},$$

and the associated reduced wave number is

$$k_{\mathsf{Z}} = \frac{\omega_{\mathsf{Z}}}{c_t} = \frac{1}{\lambda_{\mathsf{Z}}}.$$

On a coherent mesh of length ℓ_0, the reduced phase increment is

$$\Delta\varphi_{\text{red}} = k_{\mathsf{Z}}\ell_0 = \frac{\ell_0}{\lambda_{\mathsf{Z}}}.$$

The full cyclic normalization is not reintroduced here: it is already absorbed in the relation $E/\omega = h_{\text{eff}}$. The micro-local duty cycle is therefore the reduced phase increment carried by the mesh,

$$F^{(\mathsf{Z})}_{\text{phase}} = \Delta\varphi_{\text{red}} = \frac{\ell_0}{\lambda_{\mathsf{Z}}}.$$

□

Proposition 22.4 (Isotropic micro-local closure of the neutral doublet)**.** *For a 2DOF neutral doublet formed by two NSL constituents in strong overlap, the isotropic micro-local participation at LO is*

$$\boxed{\Xi^{(\mathsf{Z})}_{\text{micro}} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^{10}.} \tag{22.10}$$

Proof. Combining Lemmas 22.1 22.2 and 22.3, we obtain

$$\Xi^{(\mathsf{Z})}_{\text{micro}} = \left[\frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^3\right]\left[\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^6\right]\left[\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)\right].$$

Thus

$$\Xi^{(\mathsf{Z})}_{\text{micro}} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^{3+6+1} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^{10}.$$

□

Counting check. The total power therefore decomposes uniquely:

$$10 = 3 + 6 + 1.$$

The term 3 is the local volume of the coherent vertex, the term $6 = 3+3$ is the reduced spectral participation of the two NSL constituents, and the term 1 is the reduced phase duty cycle. The coefficient $4\pi/3$ appears only once, in $F^{(\mathsf{Z})}_{\text{cell}}$, because it comes from the local Euclidean volume of the mesh. It is not repeated in $F^{(\mathsf{Z})}_{\text{spec}}$, which is a reduced spectral-resolution measure, nor in $F^{(\mathsf{Z})}_{\text{phase}}$, which is a reduced phase measure.

Symbolic consequence for the global separation. If an independent closure provides the product

$$P_{\rho R} := \rho R_0,$$

then $\rho = P_{\rho R}/R_0$. Combining (22.3), (22.4) and (22.10), one obtains

$$\boxed{R_0 = \frac{2h_{\text{eff}}}{\pi\, c_t\, P_{\rho R}\, \mathcal{C}^{(\mathsf{Z})}_{\text{geo}}\, \lambda_{\mathsf{Z}}^2\, \Xi^{(\mathsf{Z})}_{\text{micro}}}.} \tag{22.11}$$

Once R_0 is obtained, the separations

$$\rho = \frac{P_{\rho R}}{R_0}, \qquad \kappa_r = \frac{Q_r}{R_0}$$

are immediate provided that an independent closure also supplies $Q_r := \kappa_r R_0$.

Scope. The closure (22.10) is specific to the Z^0-like composite as a 2DOF neutral doublet. It must not be applied to an isolated NSL: the two internal legs of a 2DOF NSL form a single transverse mode and do not produce the joint spectral factor $(\ell_0/\lambda_{\mathsf{Z}})^6$. This factor appears here because the Z^0-like composite contains two distinct NSL constituents.

22.1.2 Reduction of the EM-like and Preclusion interactions on the Z^0-like composite

We consider the Z^0-like object as a neutral composite of two 2DOF NSL solutions in strong overlap. At local LO, the two constituents carry the same reduced radial core $a(\eta)$, and we introduce the dimensionless internal separation u. In the symmetric two-lobe ansatz,

$$a_{\pm}(\eta) = a\!\left(\eta \mp \frac{u}{2}\right), \qquad a(\eta) = \operatorname{sech}\eta \quad \text{(canonical LO profile)}.$$

The separation dependence of the interactions between the two constituents is carried by overlap integrals. We first define the bilinear overlap

$$B_0(u) := \int_{\mathbb{R}} a_-(\eta)\, a_+(\eta)\, d\eta,$$

and the quartic overlap

$$Q_0(u) := \int_{\mathbb{R}} a_-^2(\eta)\, a_+^2(\eta)\, d\eta.$$

For the canonical profile $a(\eta) = \operatorname{sech}\eta$, one obtains explicitly

$$B_0(u) = \frac{2u}{\sinh u}, \qquad Q_0(u) = \frac{4\,(u\coth u - 1)}{\sinh^2 u}.$$

It is then natural to introduce the reduced functions

$$\boxed{O_1(u) := \left(\frac{B_0(u)}{2}\right)^2 = \left(\frac{u}{\sinh u}\right)^2, \qquad O_2(u) := \frac{Q_0(u)}{2} = \frac{2\,(u\coth u - 1)}{\sinh^2 u}.} \tag{22.12}$$

These two kernels have a simple mechanical interpretation:

- the attractive *EM-like* channel is an effective two-leg coupling between the two constituents; its reduction on the two-lobe ansatz naturally produces a term proportional to $B_0(u)^2$, hence to $O_1(u)$;

- the repulsive *Preclusion* channel measures the local cost of strong overlap of the two cores; its reduction produces a term proportional to the quartic overlap $Q_0(u)$, hence to $O_2(u)$.

The reduced internal energy depending on the separation can therefore be written in the form

$$\boxed{E_{\rm dep}^{(Z)}(u) = A_{\rm P}^{(Z)}\, O_2(u) - A_{\rm EM}^{(Z)}\, O_1(u), \qquad A_{\rm P}^{(Z)} > 0,\ A_{\rm EM}^{(Z)} > 0.} \tag{22.13}$$

The coefficients $A_{\rm P}^{(Z)}$ and $A_{\rm EM}^{(Z)}$ are not independent fundamental constants. They result from the two-lobe reduction of the Preclusion and EM-like channels for two 2DOF NSL constituents, defined by (22.33) and (22.34). In particular,

$$\boxed{A_{\rm EM}^{(Z)} = 4\,\mathcal{C}_{\rm EM}^{(2)}, \qquad A_{\rm P}^{(Z)} = 2\,\mathcal{C}_{\rm P}^{(2)},} \tag{22.14}$$

where $\mathcal{C}_{\rm EM}^{(2)}$ and $\mathcal{C}_{\rm P}^{(2)}$ are the pure reduced coefficients defined by (22.33) and (22.34).

The present subsection therefore consumes these coefficients in reduced form, without reintroducing a new local closure.

The reduced separation of the neutral composite is not determined by the stationarity of $E_{\rm dep}^{(Z)}$ alone. It is locked by the neutral scalar matching, see (22.16) and (22.17). The stationarity of the internal energy is then used only to fix the reduced Preclusion coefficient compatible with this locked bound point.

Absolute closure of the reduced coefficients. The two-lobe reduction fixes not only the separation dependence of the neutral composite; it also fixes the absolute scale of the two reduced coefficients. To avoid any ambiguity with the pure cap projector $\Pi_{\ell=1}^{\rm EM}$ defined in the EM-like field closure, we denote by

$$\mathcal{C}_{\rm EM}^{(2)}$$

the complete reduced coefficient of the 2DOF EM-like channel consumed by the neutral-composite reduction. It includes the cap projection, the channel constant $\Gamma_{\rm EM}$, and the core moments, consistently with the developed closure of $\Phi_{\rm EM}^{(\rm two)}$ given in (20.37). We then write

$$\boxed{A_{\rm EM}^{(Z)} = 4\,\mathcal{C}_{\rm EM}^{(2)}.} \tag{22.15}$$

Locking of the neutral bound point. We denote by $u_{\mathsf{Z}}^{\star}$ the reduced separation of the neutral doublet $\mathsf{Z}^0 = X_1 \oplus X_2$. This separation is not a free parameter: it is fixed by the same core–boundary scalar matching as the one that defines the LO reference point $u_*^{(\rm ref)}$ in (6.22)–(6.23).

Lemma 22.5 (Reduction of the neutral matching lock)**.** *At quasi-planar LO of the neutral two-lobe system, the matching condition of the* Z^0*-like bound point reduces to*

$$\Phi_{\mathsf{Z}}(u) := u\cot u - \big(1 - u\tanh u\big) = 0. \tag{22.16}$$

Proof. The matching of the neutral composite compares two reduced scalar contributions:

- the exterior boundary term carried by the local radial mode, whose reduced DtN ratio gives $u\cot u$;
- the interior core term carried by the strong-overlap profile $a(\eta) = \operatorname{sech}\eta$, whose reduced logarithmic derivative gives $1 - u\tanh u$.

At quasi-planar LO, the neutral two-lobe symmetry removes the odd terms in the separation, and curvature corrections are higher order. The locking condition is therefore the equality of the two scalar matching contributions, namely

$$u \cot u = 1 - u \tanh u.$$

We obtain (22.16). □

Proposition 22.6 (Locking of the neutral composite on the LO matching)**.** *The bound point of the neutral composite satisfies*

$$\boxed{u_{\mathsf{Z}}^\star = u_*^{(\mathrm{ref})} \simeq 2.00712.} \tag{22.17}$$

Proof. The neutral locking equation (22.16) is identical to the LO matching equation (6.22). On the admissible window

$$u \in [1.7, 2.1],$$

this equation has a unique root, evaluated in (6.23). By uniqueness,

$$u_{\mathsf{Z}}^\star = u_*^{(\mathrm{ref})}.$$

□

The stationarity of the internal energy is therefore not used to choose $u_{\mathsf{Z}}^\star$. It fixes the reduced Preclusion coefficient compatible with the locked bound point. It is written

$$A_{\mathrm{P}}^{(Z)}\, O_2'(u_{\mathsf{Z}}^\star) = A_{\mathrm{EM}}^{(Z)}\, O_1'(u_{\mathsf{Z}}^\star)$$

and then gives

$$\boxed{A_{\mathrm{P}}^{(Z)} = A_{\mathrm{EM}}^{(Z)} \frac{O_1'(u_{\mathsf{Z}}^\star)}{O_2'(u_{\mathsf{Z}}^\star)}, \qquad \mathcal{C}_{\mathrm{P}}^{(2)} = \frac{1}{2}\, A_{\mathrm{P}}^{(Z)}.} \tag{22.18}$$

In other words,

$$\boxed{\mathcal{C}_{\mathrm{P}}^{(2)} = 2\, \mathcal{C}_{\mathrm{EM}}^{(2)} \frac{O_1'(u_{\mathsf{Z}}^\star)}{O_2'(u_{\mathsf{Z}}^\star)}.}$$

Numerical evaluation of the reduced coefficient. The EM-like closure provides the reduced shape coefficient consumed by the two-lobe reduction

$$\boxed{\mathcal{C}_{\mathrm{EM}}^{(2)} := \Pi_{\ell=1}^{\mathrm{EM}}\, \Gamma_{\mathrm{EM}}\, \frac{J_0^2}{\pi} = \frac{1}{2}\sqrt{\frac{J_2}{J_0}}\, \Gamma_{\mathrm{EM}}\, \frac{J_0^2}{\pi} \simeq 1.538610075.} \tag{22.19}$$

Here $\Pi_{\ell=1}^{\mathrm{EM}}$ is the pure cap projector and $\Gamma_{\mathrm{EM}} J_0^2/\pi$ is the channel complement associated with the two source projections and the core moments. The two-leg transfer factor

$$\eta_{2\mathrm{leg}} = \left(\frac{1}{\Delta_{\mathrm{EM}}^2 b_0^{(2)}} \right)^2$$

belongs to the channel impedance $\Phi_{\mathrm{EM}}^{(\mathrm{two})}$, but it is not included in the reduced coefficient $\mathcal{C}_{\mathrm{EM}}^{(2)}$ consumed by the two-lobe reduction. One obtains

$$A_{\mathrm{EM}}^{(Z)} = 4\, \mathcal{C}_{\mathrm{EM}}^{(2)} \simeq 6.154440302.$$

At the locked point $u_{\mathsf{Z}}^\star = u_*^{(\mathrm{ref})}$,

$$A_{\mathrm{P}}^{(Z)} = A_{\mathrm{EM}}^{(Z)} \frac{O_1'(u_{\mathsf{Z}}^\star)}{O_2'(u_{\mathsf{Z}}^\star)} \simeq 9.854589062,$$

hence

$$\mathcal{C}_{\mathrm{P}}^{(2)} \simeq 4.927294531.$$

22.1.3 Neutral exterior response of the Z^0: DN/ND problem

Purpose. The exterior response of the neutral channel is defined by a Dirichlet–Neumann boundary map on the angular shell between the two canonical cell boundaries. It measures the relative selectivity of the two symmetric and antisymmetric boundary modes associated with the neutral composite.

Angular domain. We work on the interval

$$\psi \in [\psi_c, \pi - \psi_c],$$

where $\psi_c \in (0, \pi/2)$ is the canonical conic angle fixed above.

Point of caution on the radial scale. The exterior homogeneous equation contains an inverse range scale. This scale must not be automatically identified with the local inverse range of the core,

$$\lambda_{\mathrm{loc}}^2 = \frac{\mu_2}{T\kappa_r}.$$

The latter comes from the local stiffness of the active radial sector. The exterior DN/ND map, by contrast, describes a homogeneous response outside the core source. The passage from the local scale to the exterior scale must therefore be derived before any numerical evaluation.

We write

$$\mu_{\mathrm{ext}} := \lambda_{\mathrm{ext}} R_0$$

for the dimensionless parameter of the exterior map. The symbol λ_{ext} does not denote a new free constant: it denotes the inverse range scale resulting from the exterior reduction of the radial channel.

Homogeneous exterior equation. In the angular variable $\psi = s/R_0$, the exterior problem is written

$$\boxed{\partial_{\psi\psi} r + 2\cot\psi\, \partial_\psi r - \mu_{\mathrm{ext}}^2\, r = 0, \qquad \mu_{\mathrm{ext}} := \lambda_{\mathrm{ext}} R_0.} \tag{22.20}$$

This equation defines the boundary response of the homogeneous exterior sector; it does not reintroduce the forced core problem.

Dirichlet–Neumann map. For any Dirichlet datum

$$\mathbf{d} = \begin{pmatrix} r(\psi_c) \\ r(\pi - \psi_c) \end{pmatrix},$$

we define the outgoing Neumann vector

$$\mathbf{n} = \begin{pmatrix} -\partial_\psi r(\psi_c) \\ +\partial_\psi r(\pi - \psi_c) \end{pmatrix}.$$

By linearity and symmetry,

$$\boxed{\mathbf{n} = \mathsf{M}_{\mathrm{DN}}\, \mathbf{d}, \qquad \mathsf{M}_{\mathrm{DN}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.} \tag{22.21}$$

Canonical modes. The symmetric and antisymmetric eigenvectors are

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with

$$\boxed{\lambda_+ = a + b, \qquad \lambda_- = a - b.} \tag{22.22}$$

Neutral exterior response. The charged channel serves as the boundary reference for the normalization of conic responses:

$$\boxed{\Delta_{\text{cone}}^{(\mathsf{N}SL2D)} = 1.} \tag{22.23}$$

This equality does not introduce an additional constant; it defines the reference unit in the ratio of conic impedances.

The exterior response of the neutral channel is then defined by

$$\boxed{\Delta_{\text{cone}}^{(\mathsf{Z}^0)} = \frac{\lambda_-}{\lambda_+} = \frac{a-b}{a+b}.} \tag{22.24}$$

Status. The quantity $\Delta_{\text{cone}}^{(\mathsf{Z}^0)}$ constitutes the exterior input of the neutral channel in the 2DOF NSL closure. Its numerical evaluation first requires deriving the scale λ_{ext} specific to the homogeneous exterior problem. In particular, the identification $\lambda_{\text{ext}} = \lambda_{\text{loc}}$ must not be made without derivation.

22.1.4 Spectral closure of the exterior response

Homogeneous exterior problem. The exterior response defined in §22.1.3 rests on a homogeneous equation outside the core source:

$$\partial_{\psi\psi} r + 2\cot\psi\,\partial_\psi r - \mu_{\text{ext}}^2\, r = 0, \qquad \psi \in [\psi_c, \pi - \psi_c]. \tag{22.25}$$

This equation describes source-free propagation in the radial sector.

Global extension and spectral problem. The exterior domain can be extended by symmetry to a function defined on the full sphere $\mathbb{S}^3$. In the absence of a source and of imposed local structure, equation (22.25) is identified with a spectral problem for the Laplacian on $\mathbb{S}^3$,

$$\Delta_{\mathbb{S}^3} r = -\lambda_{\text{ext}}^2\, r, \tag{22.26}$$

where $\Delta_{\mathbb{S}^3}$ is the geometric Laplacian on the sphere of radius R_0.

Spectrum of $\mathbb{S}^3$. The eigenfunctions of the Laplacian on $\mathbb{S}^3$ satisfy

$$\Delta_{\mathbb{S}^3} Y_\ell = -\frac{\ell(\ell+2)}{R_0^2}\, Y_\ell, \qquad \ell \in \mathbb{N}. \tag{22.27}$$

By identification with (22.26), one obtains

$$\lambda_{\text{ext}}^2 = \frac{\ell(\ell+2)}{R_0^2}, \qquad \mu_{\text{ext}} = \lambda_{\text{ext}} R_0 = \sqrt{\ell(\ell+2)}. \tag{22.28}$$

Selection of the antisymmetric mode. The neutral Z^0-like channel corresponds to the antisymmetric boundary mode $\mathbf{e}_-$, which imposes

$$r(\pi - \psi) = -r(\psi).$$

The spherical harmonics on $\mathbb{S}^3$ satisfy

$$Y_\ell(\pi - \psi) = (-1)^\ell Y_\ell(\psi),$$

so that the antisymmetry condition selects odd values of ℓ.

Fundamental mode. The lowest-energy antisymmetric mode corresponds to

$$\ell = 1, \tag{22.29}$$

hence

$$\boxed{\mu_{\text{ext}} = \sqrt{3}.} \tag{22.30}$$

Consequence. The exterior response of the neutral channel defined in (22.24) becomes entirely determined by the conic geometry ψ_c and the fixed value $\mu_{\text{ext}} = \sqrt{3}$. It constitutes a closed input for the 2DOF NSL closure procedure.

22.1.5 Bilobe reduction of the EM-like and Preclusion channels for two 2DOF NSLs

Purpose. We consider a neutral composite of two 2DOF NSL solutions in strong overlap. The goal is to isolate, at local LO, the reduced coefficients of the EM-like and Preclusion channels once the dependence on separation has been factored out.

Bilobe ansatz. We introduce the canonical local profile

$$a(\eta) = \operatorname{sech}\eta,$$

and the two shifted cores

$$a_{\pm}(\eta) = a\left(\eta \mp \frac{u}{2}\right),$$

where u is the dimensionless internal separation.

Elementary overlaps. We define the bilinear overlap

$$\boxed{B_0(u) := \int_{\mathbb{R}} a_-(\eta)\, a_+(\eta)\, d\eta = \frac{2u}{\sinh u},} \tag{22.31}$$

and the quartic overlap

$$\boxed{Q_0(u) := \int_{\mathbb{R}} a_-^2(\eta)\, a_+^2(\eta)\, d\eta = \frac{4\,(u\coth u - 1)}{\sinh^2 u}.} \tag{22.32}$$

Reduction of the EM-like channel. The EM-like coupling between two constituents carrying opposite charges is attractive. Once the reference at infinite separation is subtracted, its dependence on separation reduces to

$$\boxed{E_{\text{EM}}^{(2\text{NSL})}(u) - E_{\text{EM}}^{(2\text{NSL})}(\infty) = -\mathcal{C}_{\text{EM}}^{(2)}\, B_0(u)^2, \qquad \mathcal{C}_{\text{EM}}^{(2)} > 0.} \tag{22.33}$$

Reduction of the Preclusion channel. The local cost of strong overlap similarly reduces to

$$\boxed{E_{\text{P}}^{(2\text{NSL})}(u) - E_{\text{P}}^{(2\text{NSL})}(\infty) = +\mathcal{C}_{\text{P}}^{(2)}\, Q_0(u), \qquad \mathcal{C}_{\text{P}}^{(2)} > 0.} \tag{22.34}$$

Status. The two reduced constants

$$\mathcal{C}_{\text{EM}}^{(2)}, \qquad \mathcal{C}_{\text{P}}^{(2)}$$

are pure coupling numbers of the neutral bilobe composite. They depend only on the already closed channels and on their projection on the bilobe ansatz; they are independent of κ_μ.

22.1.6 Evaluation of the neutral exterior response

Conic angle. The canonical angle ψ_c of the weak sector is fixed by the total volume fraction ε_c assigned to the two weak cones (cf. (22.46)):

$$\frac{2}{\pi}\Big(\psi_c - \tfrac{1}{2}\sin 2\psi_c\Big) = \varepsilon_c, \qquad 0 < \psi_c < \tfrac{\pi}{2}.$$

For a given ε_c, this relation fixes ψ_c uniquely.

Exterior DN/ND map. The exterior boundary map is evaluated from the homogeneous equation (22.20). It provides

$$\mathsf{M}_{\mathrm{DN}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \qquad \lambda_+ = a + b, \qquad \lambda_- = a - b.$$

Neutral response. With the boundary reference (22.23), the neutral channel is defined by

$$\boxed{\Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)} = \frac{\lambda_-}{\lambda_+} = \frac{a-b}{a+b}.}$$

Computation recipe. The numerical evaluation is performed in three steps:

1. fix ψ_c by (22.46) ;
2. derive or import from the exterior radial channel the value of $\mu_{\mathrm{ext}} = \lambda_{\mathrm{ext}} R_0$;
3. construct M_{DN}, extract λ_+ and λ_-, then form $\Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)}$.

The preceding subsection fixed, by spectral closure on $\mathbb{S}^3$, the value

$$\mu_{\mathrm{ext}} = \sqrt{3}.$$

The present subsection therefore provides the complete numerical computation map of $\Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)}$ from ψ_c and the corresponding DN/ND map.

22.1.7 Required exterior response and filter selection

For the neutral channel used by the Z^0-like solution, we employ the **antisymmetric boundary projector**, i.e. the opposite North–South combination. In the exterior DN/ND map, this selects the mode $\mathbf{e}_-$, whereas the charged reference channel selects the mode $\mathbf{e}_+$.

The quantity transmitted to the 2DOF NSL closure is therefore

$$\boxed{\Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)} = \frac{\lambda_-}{\lambda_+}.}$$

It must not be confused with a pure conic constant independent of the exterior radial scale. Its value depends on the DN/ND map derived in §22.1.3.

22.1.8 Effective spectral matching of the Z^0-like composite and affine form of the denominator

Purpose. The preceding subsection established the internal structure of the Z^0-like composite as a bound state of two 2DOF NSL constituents in strong overlap. Once the bound point $u_{\mathsf{Z}}^\star$ fixed by (22.17) has been locked, the internal quantities $O_1(u_{\mathsf{Z}}^\star)$, $O_2(u_{\mathsf{Z}}^\star)$, and $\delta^*(u_{\mathsf{Z}}^\star)$ are determined: they no longer carry any additional dynamical dependence and only serve to fix pure effective coefficients in the spectral closure.

Decomposition principle. At LO, the effective radial core of the composite remains that of the 2DOF NSL family. The spectral loading of the neutral composite then decomposes into two blocks:

- a *common constituent block*, already present for each 2DOF NSL ;
- a *neutral relative block*, selected by the antisymmetric filter of the Z^0 channel, and weighted by the neutral factor $\widehat{\beta}_0^{(2)}$.

The important point is that the neutral filter does not cancel the common block: it adds a relative contribution to the block already carried by the two constituents.

Status of the local coefficient. In the retained weak-like closure, the local channel coefficient is used in its *dimensionless reduced form*, denoted here by κ_μ. It plays the same structural role as the ratio μ_2/T in the LO closures of the weak sector, and enters directly in the spectral denominators D_{NSL2} and D_{Z0}.

Distinction of scales. The quantity λ_{c} denotes here the core width of the 2DOF NSL constituent inherited from the weak-like LO closure. It must not be confused with $\lambda_*^{(X)}$, which below denotes the effective spectral length of the composite mode $X \in \{\mathrm{NSL2}, Z0\}$ after closure of the global denominator.

Elementary channel block. We then introduce the elementary spectral channel block

$$B_{\mathrm{ch}}(\kappa_\mu\lambda_{\mathrm{c}}^2) := (\kappa_\mu\lambda_{\mathrm{c}}^2)\,K_0^{(2)} + K_2^{(2)}, \qquad K_0^{(2)} = J_0^2, \qquad K_2^{(2)} = J_0 J_2.$$

For the charged 2DOF NSL branch, the LO denominator reads

$$\boxed{D_{\mathrm{NSL2}}(\kappa_\mu\lambda_{\mathrm{c}}^2) = (\kappa_\mu\lambda_{\mathrm{c}}^2)\,K_{\mathrm{bump}} + B_{\mathrm{ch}}(\kappa_\mu\lambda_{\mathrm{c}}^2) = (\kappa_\mu\lambda_{\mathrm{c}}^2)(K_{\mathrm{bump}} + K_0^{(2)}) + K_2^{(2)}.} \tag{22.35}$$

Notation convention. The index (2) on $K_0^{(2)}$ and $K_2^{(2)}$ refers here to the channel block inherited from the constituent 2DOF structure; it denotes neither a power nor an order of development.

Neutral block of the Z^0 composite. The factor $\widehat{\beta}_0^{(2)}$ denotes the reduced weight of the neutral relative block induced by the antisymmetric boundary projector, i.e. by the neutral conic filter $\Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)}$ introduced in the preceding subsection. In other words, the difference between the charged branch and the neutral composite is not carried by the common constituent block, but by its relative weighting in the antisymmetric channel.

For the neutral composite, the channel contribution contains:

1. the common block $B_{\mathrm{ch}}(\kappa_\mu\lambda_{\mathrm{c}}^2)$, inherited from the 2DOF NSL constituent ;
2. the neutral relative block $\widehat{\beta}_0^{(2)}\,B_{\mathrm{ch}}(\kappa_\mu\lambda_{\mathrm{c}}^2)$.

We therefore obtain

$$B_{Z0}(\kappa_\mu\lambda_{\mathrm{c}}^2) = (1 + \widehat{\beta}_0^{(2)})\,B_{\mathrm{ch}}(\kappa_\mu\lambda_{\mathrm{c}}^2).$$

The LO spectral denominator of the neutral composite is then

$$\boxed{\begin{aligned} D_{Z0}(\kappa_\mu\lambda_{\mathrm{c}}^2) &= (\kappa_\mu\lambda_{\mathrm{c}}^2)\,K_{\mathrm{bump}} + (1 + \widehat{\beta}_0^{(2)})\big((\kappa_\mu\lambda_{\mathrm{c}}^2)K_0^{(2)} + K_2^{(2)}\big) \\ &= (\kappa_\mu\lambda_{\mathrm{c}}^2)\Big(K_{\mathrm{bump}} + (1 + \widehat{\beta}_0^{(2)})K_0^{(2)}\Big) + (1 + \widehat{\beta}_0^{(2)})K_2^{(2)}. \end{aligned}} \tag{22.36}$$

Immediate consequence. We have

$$D_{Z0}(\kappa_\mu\lambda_{\rm c}^2) - D_{\rm NSL2}(\kappa_\mu\lambda_{\rm c}^2) = \widehat{\beta}_0^{(2)}((\kappa_\mu\lambda_{\rm c}^2)K_0^{(2)} + K_2^{(2)}) > 0 \qquad (\kappa_\mu > 0).$$

The neutral channel is therefore spectrally more costly than the charged 2DOF NSL branch. Since

$$\lambda_*^{(X)\,4} \propto \frac{1}{D_X}, \qquad m_X \propto \frac{1}{\lambda_*^{(X)}},$$

it follows that

$$m_X \propto D_X^{1/4},$$

and therefore

$$D_{Z0} > D_{\rm NSL2} \qquad \Longrightarrow \qquad m_{Z0} > m_{\rm NSL2}.$$

LO closure of the masses. We then write

$$\lambda_*^{(Z0)\,4} = \frac{2h_{\rm eff}c_t}{T}\,\frac{1}{D_{Z0}(\kappa_\mu\lambda_{\rm c}^2)}, \qquad \lambda_*^{({\rm NSL2})\,4} = \frac{2h_{\rm eff}c_t}{T}\,\frac{1}{D_{\rm NSL2}(\kappa_\mu\lambda_{\rm c}^2)},$$

and

$$m_X = \frac{h_{\rm eff}}{c_t\,\lambda_*^{(X)}}, \qquad X \in \{{\rm NSL2}, Z0\}.$$

It follows that

$$\left(\frac{m_{Z0}}{m_{\rm NSL2}}\right)^4 = \frac{D_{Z0}(\kappa_\mu\lambda_{\rm c}^2)}{D_{\rm NSL2}(\kappa_\mu\lambda_{\rm c}^2)}.$$

Status. The composite reduction of the Z^0-like therefore leads, at LO, to an affine form in the reduced local coefficient κ_μ, inherited from the common 2DOF NSL constituent block. The difference with the charged branch is entirely carried by the addition of the neutral relative block weighted by $\widehat{\beta}_0^{(2)}$, which gives

$$D_{Z0} - D_{\rm NSL2} = \widehat{\beta}_0^{(2)}\Big((\kappa_\mu\lambda_c^2)\,K_0^{(2)} + K_2^{(2)}\Big).$$

22.1.9 LO spectral closure of the Z^0-like composite

Effective radial ansatz. Once the internal structure of the composite has been fixed by the mechanical reduction of the two 2DOF NSL constituents in strong overlap (cf. §22.1.2), the solution is projected on the neutral channel. At LO, the effective radial core remains described by the same ansatz as for the 2DOF NSL channel, with the neutral angular filter and the corresponding channel impedance:

$$A_{\rm in}(s,t) = A_0\,a(\xi)\,\cos(\mu t), \qquad a(\xi) = \operatorname{sech}\xi, \quad \xi = \frac{s}{\lambda}, \quad \lambda = \frac{c_t}{\omega_0}.$$

Shape constants. For the sech profile,

$$J_0 = \int_0^\infty 4\pi\xi^2\operatorname{sech}^2\xi\,d\xi = \frac{\pi^3}{3}, \qquad J_2 = \int_0^\infty 4\pi\xi^2(\operatorname{sech}\xi\tanh\xi)^2\,d\xi.$$

We introduce

$$K_{\rm bump} = J_0(J_0 + J_2), \qquad K_0^{(2)} = J_0^2, \qquad K_2^{(2)} = J_0J_2.$$

2DOF neutral factor. We evaluate the neutral factor at the spectral point $\lambda = \lambda_*^{(Z)}$ via the thin window

$$\varepsilon := \frac{\lambda_*^{(Z)}}{R_0} \ll 1, \qquad \psi_c := u_{\mathsf{Z}}^\star \varepsilon,$$

where $u_{\mathsf{Z}}^\star$ is the neutral bound point locked by (22.17). In the quasi-planar cap, we introduce

$$g_* := \frac{1}{\sin\psi_c} \simeq \frac{1}{u_{\mathsf{Z}}^\star \varepsilon}, \qquad k_*^2 := \frac{\rho}{T}\,\omega_{0,*}^2 - U(\psi_c),$$

and define the neutral factor (pure number) by

$$\boxed{\widehat{\beta}_0^{(2)} := \widehat{\beta}_0(\lambda_*^{(Z)}) \equiv \chi^{(2)} := \frac{(2k_*)^2}{(2k_*)^2 + g_*^2} \in (0,1)}. \tag{22.37}$$

Proposition 22.7 (LO value of $\widehat{\beta}_0^{(2)}$)**.** *For the 2DOF branch (fundamental zonal mode), we have at leading order*

$$\boxed{\widehat{\beta}_0^{(2)} = 1 - \frac{1}{4(u_{\mathsf{Z}}^\star)^2} + \mathcal{O}(\varepsilon^2)}.$$

In particular, for the locked point $u_{\mathsf{Z}}^\star = u_*^{(\mathrm{ref})}$,

The admissible window $u \in [1.7, 2.1]$ gives the robust control

$$\widehat{\beta}_0^{(2)} \in [0.9135,\ 0.9433] \qquad (LO).$$

The neutral bound point is the point locked by the LO matching, cf. (22.17):

$$u_{\mathsf{Z}}^\star = u_*^{(\mathrm{ref})} \simeq 2.00712.$$

It is defined by the locking equation

(22.16), identical to the matching root (6.22) evaluated in (6.23).

LO closure. The preceding reduction shows that the neutral composite admits at LO the same effective spectral structure as the charged 2DOF NSL branch, with affine denominator $D_{Z0}(\kappa_\mu)$ given by (22.36). We therefore write

$$\boxed{\lambda_*^{(Z0)\,4} = \frac{2\,h_{\mathrm{eff}}\,c_t}{T}\,\frac{1}{D_{Z0}(\kappa_\mu)} = \frac{2\,h_{\mathrm{eff}}\,c_t}{T}\,\frac{1}{\kappa_\mu\Big(K_{\mathrm{bump}} + (1+\widehat{\beta}_0^{(2)})K_0^{(2)}\Big) + (1+\widehat{\beta}_0^{(2)})K_2^{(2)}}\,.} \tag{22.38}$$

Likewise,

$$\boxed{\lambda_*^{(\mathrm{NSL2})\,4} = \frac{2\,h_{\mathrm{eff}}\,c_t}{T}\,\frac{1}{D_{\mathrm{NSL2}}(\kappa_\mu)} = \frac{2\,h_{\mathrm{eff}}\,c_t}{T}\,\frac{1}{\kappa_\mu(K_{\mathrm{bump}} + K_0^{(2)}) + K_2^{(2)}}\,.} \tag{22.39}$$

We deduce

$$\omega_{0,*}^{(Z0)} = \frac{c_t}{\lambda_*^{(Z0)}}, \qquad m_{Z0} = \frac{h_{\mathrm{eff}}}{c_t\,\lambda_*^{(Z0)}},$$

and

$$\left(\frac{m_{Z0}}{m_{\mathrm{NSL2}}}\right)^4 = \frac{D_{Z0}(\kappa_\mu)}{D_{\mathrm{NSL2}}(\kappa_\mu)}.$$

Since $D_{Z0} > D_{\mathrm{NSL2}}$ for $\widehat{\beta}_0^{(2)} > 0$, we indeed have

$$m_{Z0} > m_{\mathrm{NSL2}} \qquad \text{at LO.}$$

Energy and spin. The Z^0-like composite inherits the vectorial class of the 2DOF NSL branches that constitute it. At leading order, the configuration therefore satisfies

$$N_{\rm pol} = 2, \qquad w = 1,$$

hence, by the common geometric counting,

$$s_g^{(\mathsf{Z}^0)} = \frac{wN_{\rm pol}}{2} = 1.$$

The 2DOF action law also gives

$$E_0^{(\mathsf{Z}^0)} = h_{\rm eff}\,\omega_{0,*}^{(\mathsf{Z}^0)}, \qquad |S^{(\mathsf{Z}^0)}| = \frac{E_0^{(\mathsf{Z}^0)}}{\omega_{0,*}^{(\mathsf{Z}^0)}} = h_{\rm eff}.$$

We therefore obtain

$$\boxed{s_g^{(\mathsf{Z}^0)} = 1, \qquad |S^{(\mathsf{Z}^0)}| = h_{\rm eff}.}$$

The Z^0-like composite is thus an integer branch, of bosonic type in the HSM sense. It does not belong to the half-integer spinorial structure associated with the fundamental 2DOF bumps.

22.1.10 Kinematic compatibility in motion

Reduced massive composite core. The Z^0-like composite is a localized massive branch built from two 2DOF NSL constituents of the same width, in strong overlap and within the same effective neutral core. The LO closure gives a width $\lambda_*^{(Z0)}$, a proper frequency

$$\omega_{0,*}^{(\mathsf{Z}^0)} = \frac{c_t}{\lambda_*^{(Z0)}},$$

and a finite proper energy

$$E_0^{(\mathsf{Z}^0)} = h_{\rm eff}\,\omega_{0,*}^{(\mathsf{Z}^0)}.$$

In the comoving frame of the composite, the total longitudinal momentum is zero:

$$P_0^{(\mathsf{Z}^0)} = 0.$$

The branch therefore enters the kinematic framework of the localized massive solutions of §7.1.

Boost law of the composite core. The Z^0-like inherits the tangential longitudinal dynamics of the 2DOF NSL constituents, while the neutral channel modifies the closure denominator by the factor $\widehat{\beta}_0^{(2)}$. This modification affects the width and proper energy of the branch, but not the tangential hyperbolic structure of the composite core. The characteristic branch speed is therefore

$$c_{\mathcal{F}} = c_t.$$

By (7.9), we obtain

$$E^{(\mathsf{Z}^0)}(u) = \Gamma_t\,E_0^{(\mathsf{Z}^0)}, \qquad P^{(\mathsf{Z}^0)}(u) = \Gamma_t \frac{u}{c_t^2} E_0^{(\mathsf{Z}^0)}, \qquad \Gamma_t = \frac{1}{\sqrt{1-u^2/c_t^2}}.$$

The effective dispersion of the composite core is

$$E^{(\mathsf{Z}^0)}(u)^2 = \left(E_0^{(\mathsf{Z}^0)}\right)^2 + c_t^2\,P^{(\mathsf{Z}^0)}(u)^2.$$

Localized radial source. The internal structure of the Z^0-like is a neutral doublet of two 2DOF NSL constituents of the same modulus. In the strong-overlap regime, the two constituents share a single comoving coordinate. The core profiles are of the form

$$a_\pm(\eta) = a\left(\eta \mp \frac{u}{2}\right), \qquad a(\eta) = \operatorname{sech}\eta,$$

and the radial sources built from quadratic densities and cross-overlaps remain localized functions of the comoving coordinate.

The radial sheath operator is the common operator of the criterion (7.3). The global neutrality of the composite modifies the content of the source and cancels certain monopole channels, but it does not modify the common radial operator. Thus, when the radial sheath response is considered explicitly, the strictly comoving sheath is localized for

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

Scope. This kinematic compatibility modifies neither the spectral closure (22.38), nor the ratio

$$\left(\frac{m_{Z0}}{m_{\mathrm{NSL2}}}\right)^4 = \frac{D_{Z0}(\kappa_\mu)}{D_{\mathrm{NSL2}}(\kappa_\mu)}.$$

It only establishes that the Z^0-like composite, once its branch is closed at LO, belongs to the framework of localized massive solutions in motion, with a composite core boosted at characteristic speed c_t and a strictly comoving radial sheath localized below the threshold $|u| < c_r$.

22.1.11 Core constants (Z^0-like 2DOF)

With $a(\xi) = \operatorname{sech}\xi$,

$$J_0 = \frac{\pi^3}{3} = 10.3354255601, \qquad J_2 = 7.6339320582,$$

we fix

$$K_0^{(2)} := J_0^2 = 1.068210215084 \times 10^2, \qquad K_2^{(2)} := J_0\, J_2 = 7.88999365184 \times 10^1. \tag{22.40}$$

The common core combination is

$$K_{\mathrm{bump}} = J_0(J_0 + J_2) = 1.857209580268 \times 10^2.$$

The conic constants satisfy

$$\Delta_{\mathrm{cone}}^{(\mathsf{N}SL2D)} = 1, \qquad \Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)} = \frac{\lambda_-}{\lambda_+} = \frac{a-b}{a+b} \quad \text{(to be inserted after DN/ND calculation).} \tag{22.41}$$

HSM structural parameters. The following quantities depend on the HSM fundamental constants:

$$\ell_0, \quad T, \quad \rho, \quad \mu_2, \quad R_0,$$

with

$$c_t^2 = \frac{T}{\rho}, \qquad \kappa_\mu := \frac{\mu_2}{T} > 0.$$

Summary status. The pure theory of the Z^0-like composite:

- fixes the radial forms $(J_0, J_2, K_{\rm bump}, K_0^{(2)}, K_2^{(2)})$,
- fixes the overlap kernels (O_1, O_2) of the composite,
- fixes the neutral factor $\widehat{\beta}_0^{(2)}$,
- and establishes the structure of the closure (22.38).

22.1.12 Conic constants of the 2DOF NSL filters: minimal definition

Goal. Define the pure numbers $\Delta_{\rm cone}^{(\mathsf{N}SL2D)}$ and $\Delta_{\rm cone}^{(\mathsf{Z}^0)}$ that correct the 2DOF NSL closure at LO (weak cones), *without* introducing a medium constant.

1. Input data and normalization

Zonal geometry on $\mathbb{S}^3$ (radius R_0), arc $s = R_0\psi$, $0 < \psi < \pi$. Two caps (weak cones) at $\psi = \psi_c$ and $\psi = \pi - \psi_c$. Boundary data (reduced amplitude u):

$$\mathbf{d} = \begin{pmatrix} u(\psi_c) \\ u(\pi - \psi_c) \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \partial_n u(\psi_c) \\ -\partial_n u(\pi - \psi_c) \end{pmatrix}.$$

2. Zonal DN/ND map

Maps

$$\mathbf{n} = \mathsf{M}_{\rm DN}\mathbf{d}, \qquad \mathbf{d} = \mathsf{M}_{\rm ND}\mathbf{n}, \qquad \mathsf{M}_{\rm ND} = \mathsf{M}_{\rm DN}^{-1}.$$

By North–South symmetry (fundamental branch):

$$\mathsf{M}_{\rm DN} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \qquad \lambda_+ = a + b, \quad \lambda_- = a - b.$$

3. Canonical NSL2DOF and Z^0 filters

Canonical choices:

$$\boldsymbol{\pi}^{(NSL2D)} = \mathbf{e}_+ = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{\pi}^{(\mathsf{Z}^0)} = \mathbf{e}_- = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Filtered DN impedance:

$$\Lambda_{\rm DN}^{(X)} = \frac{\boldsymbol{\pi}^{(X)T}\mathsf{M}_{\rm DN}\boldsymbol{\pi}^{(X)}}{\boldsymbol{\pi}^{(X)T}\boldsymbol{\pi}^{(X)}}.$$

4. Definition of $\Delta_{\rm cone}^{(X)}$

Symmetric reference: $\boldsymbol{\pi}^{(\rm ref)} = \mathbf{e}_+$, hence

$$\Lambda_{\rm DN}^{(\rm ref)} = \lambda_+.$$

Definition (impedance ratio):

$$\boxed{\Delta_{\rm cone}^{(X)} = \frac{\Lambda_{\rm DN}^{(X)}}{\Lambda_{\rm DN}^{(\rm ref)}}, \qquad X \in \{\mathsf{N}SL2D, \mathsf{Z}^0\}.} \tag{22.42}$$

With the canonical filters:

$$\Delta_{\mathrm{cone}}^{(\mathsf{N}SL2D)} = 1, \qquad \Delta_{\mathrm{cone}}^{(\mathsf{Z}^0)} = \frac{\lambda_-}{\lambda_+} = \frac{a-b}{a+b}. \tag{22.43}$$

These constants are dimensionless and purely geometric: at fixed conic geometry, they are independent of the medium constants $(\ell_0, \rho, T, \mu_2, R_0)$.

22.2 Structure constant and conic angle (general then canonical)

General definition. We denote by $\varepsilon_c \in (0,1)$ the total volume fraction occupied by the two symmetric weak cones on $\mathbb{S}^3$ (hence $\varepsilon_c/2$ per cone). The exact geometric relation is

$$\boxed{\varepsilon_c = \frac{V_{\mathrm{cones}}(\psi_c)}{\mathrm{Vol}(\mathbb{S}^3)} = \frac{2}{\pi}\Big(\psi_c - \tfrac{1}{2}\sin 2\psi_c\Big), \qquad 0 < \psi_c < \tfrac{\pi}{2}.} \tag{22.44}$$

This relation implicitly defines $\psi_c = \psi_c(\varepsilon_c)$ uniquely.

Symmetric selection of the conic gauge. The exterior geometry of the neutral composite $\mathsf{Z}^0 = X_1 \oplus X_2$ is symmetric under exchange of the two constituents

$$X_1 \longleftrightarrow X_2.$$

We denote by ε_c the total volume fraction carried by the two symmetric weak cones, i.e. $\varepsilon_c/2$ per cone.

Lemma 22.8 (Symmetric conic gauge of the neutral doublet)**.** *In the neutral bilobe composite, the exchange invariance $X_1 \leftrightarrow X_2$ imposes the symmetric conic gauge*

$$\boxed{\varepsilon_c^{(\mathrm{can})} = \frac{1}{2}.} \tag{22.45}$$

Proof. The exterior DN/ND map of the neutral composite is read on two opposite and equivalent weak cones. The exchange of the constituents

$$X_1 \longleftrightarrow X_2$$

exchanges the two cones and transforms the fraction carried by one side into its complement. The selected total fraction must therefore satisfy

$$\varepsilon_c = 1 - \varepsilon_c.$$

It follows that

$$\varepsilon_c = \frac{1}{2}.$$

This value expresses the binary symmetry of the neutral composite; it is not an additional dynamical constant. □

Substituting (22.45) into the exact geometric relation (22.44), we obtain

$$\boxed{\frac{2}{\pi}\Big(\psi_c - \tfrac{1}{2}\sin 2\psi_c\Big) = \tfrac{1}{2}.} \tag{22.46}$$

The corresponding numerical solution is

$$\boxed{\psi_c^{(\mathrm{can})} \approx 1.154940730 \text{ rad} \approx 66.173229^\circ.} \tag{22.47}$$

Status of the symmetric conic gauge. The value $\varepsilon_c^{(\mathrm{can})} = 1/2$ is the consequence of the exchange symmetry of the neutral doublet. It fixes the geometric gauge in which the exterior DN/ND map of the 2DOF neutral sector is evaluated. The conic quantities used below are therefore read in this symmetric gauge.

Distinction from the plateau spectral cell. The angle $\psi_c^{(\mathrm{can})} \approx 1.154940730$ defined here belongs to the exterior DN/ND map of the neutral composite. It must not be confused with the elementary plateau spectral cell, defined by

$$\sin^2 \psi_c^{(\mathrm{cell})} = \frac{1}{3}, \qquad \psi_c^{(\mathrm{cell})} \approx 0.615479709,$$

in (6.47). The two angles have different roles: $\psi_c^{(\mathrm{can})}$ fixes the exterior conic gauge of the neutral doublet, whereas $\psi_c^{(\mathrm{cell})}$ fixes the plateau filter used in the common spectral closure.

22.2.1 Canonical conic angle

For internal references, we point to (22.46) (canonical definition), (22.47) (numerical value), or (22.44) (general framework).

22.3 Weak-like charge of the Z^0-like

Positioning. The minimal weak-like channel was fixed above on the primitive 2DOF NSL branch. It carries two elementary internal orientations,

$$(+) \qquad \text{and} \qquad (-),$$

with

$$\rho_W^{(+)} = +\rho\,\omega_{\mathrm{NSL}} A^2, \qquad \rho_W^{(-)} = -\rho\,\omega_{\mathrm{NSL}} A^2,$$

and

$$q_W^{(+)} = -q_W^{(-)},$$

cf. (20.68), (20.69), (20.75) and (20.76).

In the present chapter, the Z^0-like is constructed as a neutral composite of two 2DOF NSL constituents of the same modulus, in strong overlap, selected by the antisymmetric channel $\mathbf{e}_-$. It is therefore not an additional primitive branch, but a composite combination of the two orientations of the NSL.

22.3.1 Exact neutrality of the total charge

Local density of the composite. In the bilobe ansatz of §22.1.2, the two cores are written

$$a_\pm(\eta) = a\left(\eta \mp \frac{u}{2}\right), \qquad a(\eta) = \operatorname{sech}\eta.$$

The neutral Z^0-like composite associates these two cores with opposite internal orientations of the weak-like channel. The total weak-like charge density of the composite is therefore written, at local LO,

$$\boxed{\rho_W^{(\mathsf{Z}^0)}(\eta) = \rho\,\omega_{\mathrm{NSL}}\Big(a_+(\eta)^2 - a_-(\eta)^2\Big).} \tag{22.48}$$

Cancellation of the global charge. The total charge is then

$$q_W^{(\mathsf{Z}^0)} = \int \rho_W^{(\mathsf{Z}^0)}\, dV.$$

Since the two constituents carry the *same* 2DOF NSL profile and differ only by an internal translation $\pm u/2$, we have

$$\int a_+(\eta)^2\, d\eta = \int a_-(\eta)^2\, d\eta = \int a(\eta)^2\, d\eta.$$

Consequently,

$$\int \rho_W^{(\mathsf{Z}^0)}\, dV = \rho\, \omega_{\mathrm{NSL}} \left(\int a_+^2\, dV - \int a_-^2\, dV \right) = 0,$$

hence

$$\boxed{q_W^{(\mathsf{Z}^0)} = 0.} \tag{22.49}$$

Scope of the result. The cancellation of $q_W^{(\mathsf{Z}^0)}$ is therefore not merely a channel-reading assumption here, but a direct consequence of the construction of the composite as the superposition of two 2DOF NSL constituents of the same modulus and of opposite weak-like orientations.

22.3.2 Global neutrality and nontrivial internal structure

Local structure. The cancellation of the total charge does not mean that the local density is trivially zero. Indeed, as long as $u > 0$,

$$a_+(\eta)^2 \neq a_-(\eta)^2 \quad \text{point by point,}$$

so that

$$\rho_W^{(\mathsf{Z}^0)}(\eta) \not\equiv 0.$$

The composite is therefore globally neutral, but locally nontrivial in the weak-like channel.

Internal parity. By bilobe symmetry,

$$a_+(-\eta) = a_-(\eta),$$

hence

$$\rho_W^{(\mathsf{Z}^0)}(-\eta) = -\,\rho_W^{(\mathsf{Z}^0)}(\eta).$$

The channel density of the Z^0-like is thus *antisymmetric* in the internal coordinate, which explains the exact cancellation of its global monopole despite a nonzero local structure.

22.3.3 Consequences for the weak-like field

Monopole coupling. Since the total charge of the composite cancels, its direct monopole coupling to the weak-like field is zero at the minimal rank:

$$q_W^{(\mathsf{Z}^0)} = 0 \quad \Longrightarrow \quad \Phi_W^{\mathrm{mono}}(\mathsf{Z}^0) = 0.$$

Monopole interaction between two Z^0-like composites. At the same reading level, the minimal monopole force between two Z^0-like composites at rest cancels:

$$\boxed{F_W^{\mathrm{mono}}(\mathsf{Z}^0, \mathsf{Z}^0) = 0.} \tag{22.50}$$

Any possible residual interaction between two Z^0-like composites must therefore come from finer-order internal channel invariants, and not from a nonzero total monopole charge.

22.3.4 Conclusion

Conclusion. In the retained construction of the Z^0-like composite, the weak-like neutrality is an exact result at LO:

$$\boxed{q_W^{(\mathsf{Z}^0)} = 0.}$$

It results from the association of two 2DOF NSL constituents of the same modulus and opposite weak-like orientations. The Z^0-like is thus globally neutral in the weak-like channel, while retaining a nontrivial antisymmetric local structure in this channel.

Chapter 23

Neutral 2DOF breather on $\mathbb{S}^3$: *single-core* solution (attractive EM-like + gradient repulsion), strong overlap

Operational scope (definition, non-ambiguity)

Separation parameter and overlap. We denote by $\Delta := L/\lambda$ the dimensionless separation of the internal *two lobes* (L distance between lobe centers, $\lambda = c_t/\omega_0$ local LO width). The overlap index (profile $a = \mathrm{sech}$; local Euclidean measure at LO) is defined by

$$\mathfrak{O}(\Delta) \;:=\; \frac{\displaystyle\int_0^\infty 4\pi\,\xi^2\, a\Big(\xi - \tfrac{\Delta}{2}\Big)\; a\Big(\xi + \tfrac{\Delta}{2}\Big)\; d\xi}{\displaystyle\int_0^\infty 4\pi\,\xi^2\, a(\xi)^2\, d\xi} \;\in (0,1].$$

Geometric status of the overlap index. The index $\mathfrak{O}(\Delta)$ must not be read as the exact overlap of two rigidly translated 3D cores, but as a *radialized overlap functional* at LO, constructed on the local isotropic measure $4\pi\xi^2\,d\xi$. The parameter Δ measures in it the internal separation of the two lobes in the effective single-core reduction.

2DOF breather (retained class). A **2DOF breather** denotes a configuration *with two lobes contained in the same core*, in the **strong-overlap** regime operationally defined by

$$\boxed{\mathfrak{O}(\Delta) \;\geq\; \mathfrak{O}_{\min}}$$

where $\mathfrak{O}_{\min}$ is a working threshold. In robustness tests, one may take

$$\mathfrak{O}_{\min} \in [0.3,\, 0.6],$$

but the formal development of the present chapter does not depend on a particular value of this threshold.

In the LO regime studied below, the dependence of the internal energy on Δ comes from the competition between an attractive EM-like contribution and a repulsive cross-gradient cost. The internal separation Δ is therefore not an input parameter: it will be determined below by a stationary condition on the internal energy depending on Δ,

$$\boxed{\partial_\Delta E_{\mathrm{dep}}(\Delta_*) = 0, \qquad \partial_\Delta^2 E_{\mathrm{dep}}(\Delta_*) > 0 \text{ (checked a posteriori).}}$$

What is <u>not</u> a breather. Two *distinct* 2DOF bumps (two cores) bound only by the long-range Coulomb-like interaction form an **EM dimer** (molecule). This case (negligible overlap, hence $\mathfrak{O}(\Delta) \simeq 0$) is not part of the present chapter and belongs to the two-body regime.

23.1 Structural assumptions and framework

- **Geometry and constants.** Elastic mesh on $\mathbb{S}^3(R_0)$, volumetric density ρ, tangential stiffness T, speed $c_t^2 = T/\rho$. Arc coordinate $s = R_0\psi$.
- **2DOF architecture.** Two tangential components grouped into a circular mode $\Psi = u_1 + i\,u_2$ (phasor), with carrier frequency ω_0 and local width $\lambda = c_t/\omega_0$.
- **Neutrality and parity.** Two lobes of opposite sign in the same core, with central node $A(0) = 0$ (odd combination) and zero global charge. The branch remains carried by a single circular 2DOF mode.

23.2 Constants and notation (minimal reminder)

We reuse the constants derived in the preceding chapters, without redefining them here:

$$J_0, \quad J_2, \quad \kappa_\omega, \quad \kappa_\omega^{-1}, \quad b_0^{(2)}\,.$$

These quantities are geometric/structural in nature at LO and are taken as given throughout this chapter.

23.3 Same-core internal energy and stationary condition

For a 2DOF breather *with two lobes in the same core* (strong overlap in the sense of the operational scope, cf. §23),

the LO energy is written

$$E_{\rm tot}(\Delta) \;=\; E^{(0)}_{\rm coeur} \;+\; E^{\rm cross}_{\rm grad}(\Delta) \;+\; E^{\rm attr}_{\rm int}(\Delta) \;+\; \mathcal{O}(\lambda/R_0)\,. \tag{23.1}$$

We denote by

$$E_{\rm dep}(\Delta) \;:=\; E^{\rm cross}_{\rm grad}(\Delta) + E^{\rm attr}_{\rm int}(\Delta)$$

the part depending on the internal separation Δ.

In the LO regime retained here, the reduction on the signed internal coordinate $\eta \in \mathbb{R}$ along the axis joining the two lobes leads to a form

$$E_{\rm dep}(\Delta) \;=\; C_{\rm grad}\,B_1(\Delta) \;-\; C_{\rm attr}\,B_0(\Delta), \tag{23.2}$$

where $C_{\rm grad} > 0$ and $C_{\rm attr} > 0$ are the reduced coefficients associated, respectively, with the cross-gradient cost and the internal attraction within the same core.

The equilibrium separation Δ_* is then defined as an admissible stationary point:

$$\partial_\Delta E_{\rm dep}(\Delta_*) = 0, \qquad \partial_\Delta^2 E_{\rm dep}(\Delta_*) > 0. \tag{23.3}$$

Core $a = \operatorname{sech}$: moments $B_n(\Delta)$ and $P_0(\Delta)$

We fix the local profile $a(\eta) = \operatorname{sech}\eta$ and introduce the odd combination

$$A_\Delta(\eta) \;:=\; a\Big(\eta - \tfrac{\Delta}{2}\Big) \;-\; a\Big(\eta + \tfrac{\Delta}{2}\Big)\,.$$

Scalar products on the internal coordinate. We denote

$$\langle f, g\rangle_{\parallel} \;:=\; \int_{\mathbb{R}} f(\eta)\, g(\eta)\, d\eta.$$

The core overlap kernels are then defined by

$$B_0(\Delta) \;:=\; \Big\langle a\Big(\cdot - \tfrac{\Delta}{2}\Big)\,,\, a\Big(\cdot + \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel}, \tag{23.4}$$

$$B_1(\Delta) \;:=\; \Big\langle a'\Big(\cdot - \tfrac{\Delta}{2}\Big)\,,\, a'\Big(\cdot + \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel}. \tag{23.5}$$

We further define

$$P_0(\Delta) \;:=\; \langle A_\Delta,\, A_\Delta\rangle_{\parallel}.$$

Structural identities. By translation invariance of $\langle\cdot,\cdot\rangle_{\parallel}$,

$$\Big\langle a\Big(\cdot \mp \tfrac{\Delta}{2}\Big)\,,\, a\Big(\cdot \mp \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel} = B_0(0), \qquad \Big\langle a'\Big(\cdot \mp \tfrac{\Delta}{2}\Big)\,,\, a'\Big(\cdot \mp \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel} = B_1(0),$$

and the exact relations are

$$P_0(\Delta) \;=\; 2\,\Big(B_0(0) - B_0(\Delta)\Big), \tag{23.6}$$

$$\|A'_\Delta\|^2_{L^2(\mathbb{R})} \;=\; 2\,\Big(B_1(0) - B_1(\Delta)\Big). \tag{23.7}$$

Reduced axial overlap. In the 1D reduction on the internal axis, the reduced axial overlap is introduced as

$$\mathfrak{O}_{\parallel}(\Delta) \;:=\; \frac{B_0(\Delta)}{B_0(0)} \in (0,1], \qquad \mathfrak{O}_{\parallel}(0) = 1.$$

This is the effective 1D version of the overlap used in the present section. It must not be literally identified with the 3D index $\mathfrak{O}(\Delta)$ introduced in the operational scope, even though the two quantities play the same qualitative role as measures of the internal overlap.

Local expansion ($\Delta \ll 1$). A Taylor expansion gives

$$A_\Delta(\eta) \;=\; -\Delta\, a'(\eta) \;-\; \frac{\Delta^3}{24}\, a^{(3)}(\eta) \;+\; \mathcal{O}(\Delta^5),$$

hence

$$P_0(\Delta) \;=\; \Delta^2\, \langle a', a'\rangle_{\parallel} \;-\; \frac{\Delta^4}{12}\, \langle a'', a''\rangle_{\parallel} \;+\; \mathcal{O}(\Delta^6), \tag{23.8}$$

$$B_0(\Delta) \;=\; B_0(0) \;-\; \frac{\Delta^2}{2}\, \langle a', a'\rangle_{\parallel} \;+\; \mathcal{O}(\Delta^4), \tag{23.9}$$

$$B_1(\Delta) \;=\; B_1(0) \;-\; \frac{\Delta^2}{2}\, \langle a'', a''\rangle_{\parallel} \;+\; \mathcal{O}(\Delta^4). \tag{23.10}$$

Stationary condition in terms of $B_0(\Delta)$ and $B_1(\Delta)$

From (23.2), the stationary condition is written

$$\partial_\Delta E_{\mathrm{dep}}(\Delta_*) \;=\; 0 \qquad \Longleftrightarrow \qquad C_{\mathrm{grad}}\, B_1'(\Delta_*) \;=\; C_{\mathrm{attr}}\, B_0'(\Delta_*)\,. \tag{23.11}$$

Introducing the dimensionless ratio

$$\kappa_* \;:=\; \frac{C_{\mathrm{attr}}}{C_{\mathrm{grad}}} \;>\; 0\,,$$

we obtain the implicit equation

$$\frac{B_1'(\Delta_*)}{B_0'(\Delta_*)} \;=\; \kappa_*\,, \tag{23.12}$$

valid on any admissible interval where $B_0'(\Delta) \neq 0$.

Normalized form. Writing $\mathfrak{O}(\Delta) := B_0(\Delta)/B_0(0)$ and $b(\Delta) := B_1(\Delta)/B_1(0)$, one rewrites

$$\frac{B_1(0)}{B_0(0)}\,\frac{b'(\Delta_*)}{\mathfrak{O}'(\Delta_*)} \;=\; \kappa_*\,. \tag{23.13}$$

Differential form of the slopes and ratio $R(\Delta)$

By differentiating under the integral sign,

$$B_0'(\Delta) = -\Big\langle a'\Big(\cdot - \tfrac{\Delta}{2}\Big)\,,\, a\Big(\cdot + \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel} = \Big\langle a\Big(\cdot - \tfrac{\Delta}{2}\Big)\,,\, a'\Big(\cdot + \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel}, \tag{23.14}$$

$$B_1'(\Delta) = -\Big\langle a''\Big(\cdot - \tfrac{\Delta}{2}\Big)\,,\, a'\Big(\cdot + \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel} = \Big\langle a'\Big(\cdot - \tfrac{\Delta}{2}\Big)\,,\, a''\Big(\cdot + \tfrac{\Delta}{2}\Big)\Big\rangle_{\parallel}. \tag{23.15}$$

When one restricts to the strong-overlap interval retained for same-core breathers, one may introduce the ratio

$$R(\Delta) \;:=\; \frac{B_1'(\Delta)}{B_0'(\Delta)},$$

and write the stationary condition in the form

$$R(\Delta_*) = \kappa_*.$$

Local expansion. For $\Delta \ll 1$,

$$B_0'(\Delta) \;=\; -\,\Delta\,\langle a', a'\rangle_{\parallel} \;+\; \mathcal{O}(\Delta^3), \qquad B_1'(\Delta) \;=\; -\,\Delta\,\langle a'', a''\rangle_{\parallel} \;+\; \mathcal{O}(\Delta^3),$$

hence

$$R(\Delta) \;=\; \frac{\langle a'', a''\rangle_{\parallel}}{\langle a', a'\rangle_{\parallel}} \;+\; \mathcal{O}(\Delta^2).$$

23.3.1 Local EBK quantization and zonal spectral filter

The internal circular mode $\Psi = u_1 + i\,u_2$ satisfies the local EBK quantization

$$\mathcal{J} \;=\; \oint p\,dq \;=\; 2\pi\,h_{\text{eff}}\,q, \qquad q \in \mathbb{N}^*, \tag{23.16}$$

and the LO carrier energy is

$$E_0(q) = h_{\text{eff}}\,\omega_0(q). \tag{23.17}$$

Harmonic counting induced by EBK. On $\mathbb{S}^3$, the harmonic subspace of degree ℓ carries a multiplicity

$$d_\ell = (\ell+1)^2.$$

The cumulative number of modes up to degree L is therefore

$$N(L) := \sum_{\ell=0}^{L} (\ell+1)^2 = \frac{(L+1)(L+2)(2L+3)}{6}. \tag{23.18}$$

EBK quantization requires the admissible spectral capacity to grow linearly with the quantum number q. Identifying the effective harmonic scale $L_c(q)$ through

$$N(L_c(q)) \propto q,$$

one obtains, at the smoothed-window order,

$$L_c(q) \propto q^{1/3}. \tag{23.19}$$

The effective cutoff is therefore written as

$$\boxed{k_c(q) = k_0\,q^{1/3}.} \tag{23.20}$$

Proposition 23.1 (Fundamental EBK capacity of the 2DOF breather)**.** *In the LO closure of the 2DOF breather, the fundamental spectral capacity of the EBK packet is*

$$k_0 = 5.$$

Proof. The fundamental EBK packet must contain two levels.

First, the mean level $\ell = 0$, required to carry the global norm of the reduced mode. Next, the first non-constant sector $\ell = 1$, required to carry a nontrivial internal variation on $\mathbb{S}^3$. The canonical cap normalization $E_\ell(1) = 1/\pi$ fixes the reading of the first non-constant sector as the $\ell = 1$ sector, without introducing any additional spectral weight.

The minimal admissible harmonic packet is therefore

$$0 \le \ell \le 1.$$

Its capacity is given by the harmonic counting on $\mathbb{S}^3$:

$$N(1) = \sum_{\ell=0}^{1} (\ell+1)^2 = 1^2 + 2^2 = 5.$$

The fundamental capacity of the smoothed EBK window is identified with this minimal capacity:

$$k_0 = N(1) = 5.$$

Thus, $k_0 = 5$ is not a value fitted by numerical robustness: it is the counting of the minimal harmonic packet $\ell = 0 \oplus \ell = 1$ on $\mathbb{S}^3$. The robustness check enters only a posteriori, to verify that the window thus defined lies in the stable plateau of the spectral ratios. □

Canonical EBK window. The canonical window law of the 2DOF breather is therefore

$$\boxed{k_0 = 5.} \tag{23.21}$$

and

$$\boxed{k_c(q) = 5\, q^{1/3}.} \tag{23.22}$$

This prescription is the one used throughout the following to construct the dimensionless spectrum $\mathcal{S}(q)$.

Canonical harmonic comb and differential saturation. The canonical EBK filter associated with a cutoff $k_c(q)$ acts on the zonal spectrum of $\mathbb{S}^3$. The constant component must be kept without renormalization, since

$$C_0^{(1)} = 1 \quad \Longrightarrow \quad F_0(w_q) = 1.$$

The unsaturated EBK cutoff is

$$\boxed{k_c(q) = k_0\, q^{1/3}, \qquad k_0 = 5.} \tag{23.23}$$

For a sector containing a first finite threshold $q_\star$, the spectral growth beyond that threshold is bounded by an effective plateau resolution. A saturation scale is therefore introduced in the form

$$\boxed{k_{\mathrm{sat}}^2 = \mathfrak{c}_{\mathrm{sat}}^{(2)}\, \kappa_\omega^2.} \tag{23.24}$$

The coefficient $\mathfrak{c}_{\mathrm{sat}}^{(2)}$ encodes the LO closure of the 2DOF breather plateau resolution. In the central closure used here, it factorizes as

$$\boxed{\mathfrak{c}_{\mathrm{sat}}^{(2)} = \mathfrak{c}_{\mathrm{cell}}\, \mathfrak{c}_{\mathrm{leg}}\, \mathfrak{c}_{\mathrm{quad}} = 2 \cdot 2 \cdot 2 = 8.} \tag{23.25}$$

The first factor 2 corresponds to the two resolved internal directions of the 2DOF plateau. The second factor 2 corresponds to the two source legs entering the projected reduction. The third factor 2 expresses the fact that the saturation acts on a quadratic spectral width, that is, on k^2, and not on k.

The central saturation value used in the LO calculations is therefore

$$\boxed{k_{\mathrm{sat}}^2 = 8\,\kappa_\omega^2.} \tag{23.26}$$

This value must be read as a regularized LO closure of the 2DOF spectral plateau, and not as an analytical identity independent of the width regularization.

The saturation acts only on the spectral excess beyond $q_\star$. The effective cutoff is

$$\boxed{k_{\mathrm{eff}}^2(q;q_\star) = \begin{cases} k_c^2(q), & q \le q_\star, \\ k_c^2(q_\star) + \dfrac{k_c^2(q) - k_c^2(q_\star)}{1 + \dfrac{k_c^2(q) - k_c^2(q_\star)}{k_{\mathrm{sat}}^2}}, & q \ge q_\star. \end{cases}} \tag{23.27}$$

This form leaves the first finite regime $q_\star$ unchanged and saturates only the spectral excess $k_c^2(q) - k_c^2(q_\star)$.

Stability of the saturation closure. The central closure $\mathfrak{c}_{\mathrm{sat}}^{(2)} = 8$ fixes the value used in the LO numerical expressions. The stability of this closure is tested by replacing

$$k_{\mathrm{sat}}^2 = 8\kappa_\omega^2$$

by

$$k_{\mathrm{sat}}^2 = \mathfrak{c}_{\mathrm{sat}}^{(2)}\kappa_\omega^2$$

in (23.27). The results that consume $k_{\mathrm{eff}}^2(q;q_\star)$ must therefore be read as central LO values associated with the composite closure (23.25).

The Gaussian window normalized to the constant mode is then

$$\boxed{\mathcal{W}_\ell(q;q_\star) := \exp\Big(-\frac{\ell(\ell+2)}{k_{\mathrm{eff}}^2(q;q_\star)}\Big).} \tag{23.28}$$

Thus

$$\mathcal{W}_0(q;q_\star) = 1.$$

For the 2DOF breather, there is no conic precession: the EBK cell is zonal/isotropic. We therefore define $w_q(\psi)$ as the zonal filter on $\mathbb{S}^3$ whose normalized harmonic coefficients are $\mathcal{W}_\ell(q;q_\star)$. In other words,

$$F_\ell(w_q) := \frac{\displaystyle\int_0^\pi w_q(\psi)\, C_\ell^{(1)}(\cos\psi)\, \sin^2\psi\, d\psi}{\displaystyle\int_0^\pi w_q(\psi)\, \sin^2\psi\, d\psi} = \mathcal{W}_\ell(q;q_\star). \tag{23.29}$$

Here, $C_\ell^{(1)}$ denotes the zonal Gegenbauer polynomial on $\mathbb{S}^3$. The Gaussian decay of $\mathcal{W}_\ell(q;q_\star)$ ensures that w_q is well defined and smooth.

Filtered zonal kernel. We then introduce the coefficients

$$a_\ell^{(q;q_\star)} := G_\ell(mR_0)\,\big|F_\ell(w_q)\big|^2, \qquad G_\ell(mR_0) = \frac{1}{\ell(\ell+2)+(mR_0)^2}, \tag{23.30}$$

and the filtered zonal kernel

$$K_q(\theta) := \frac{\displaystyle\sum_{\ell=0}^{L_{\max}} a_\ell^{(q;q_\star)}\, C_\ell^{(1)}(\cos\theta)}{a_0^{(q;q_\star)}}. \tag{23.31}$$

By zonal orthogonality, one has the canonical normalization

$$\frac{2}{\pi}\int_0^\pi K_q(\theta)\,\sin^2\theta\,d\theta = 1. \tag{23.32}$$

Cell average. The cell average associated with the level q is then

$$\alpha_{\mathrm{cell}}(q;q_\star) := \frac{1}{\mathcal{N}_q^2}\iint_{\mathbb{S}^3\times\mathbb{S}^3} w_q(x)\,K_q\big(\gamma(x,y)\big)\,w_q(y)\,dV_x\,dV_y, \qquad \mathcal{N}_q := \int_{\mathbb{S}^3} w_q\,dV. \tag{23.33}$$

The dependence on q therefore enters only through the zonal EBK filter w_q.

Spectral reduction of the cell average. Since w_q and K_q are zonal, the cell average (23.33) reduces to a spectral sum. With the zonal normalization of $\mathbb{S}^3$, the contraction of two harmonic coefficients introduces the shell weight

$$\frac{d_\ell^{(3)}}{\ell+1} = \frac{(\ell+1)^2}{\ell+1} = \ell+1.$$

One therefore obtains

$$\boxed{\alpha_{\mathrm{cell}}(q;q_\star) = \sum_{\ell=0}^{L_{\max}} (\ell+1)\,\frac{G_\ell(mR_0)}{G_0(mR_0)}\,\big|\mathcal{W}_\ell(q;q_\star)\big|^4} \tag{23.34}$$

with

$$G_\ell(mR_0) = \frac{1}{\ell(\ell+2)+(mR_0)^2}.$$

The fourth power comes from the two cell filterings and the two source coefficients in the filtered kernel.

Angular quadratic invariant. We define

$$\mathsf{J}_2^{(\theta)}(q;q_\star) := \frac{1}{\pi}\int_0^\pi \theta^2\,K_q(\theta)\,\sin^2\theta\,d\theta, \qquad \widehat{\mathsf{J}}_2(q;q_\star) := (mR_0)^2\,\mathsf{J}_2^{(\theta)}(q;q_\star). \tag{23.35}$$

Dimensionless spectrum of the 2DOF breather. We first introduce the unnormalized spectral invariant

$$\Sigma(q;q_\star) := \lim_{mR_0\to\infty} \frac{\alpha_{\mathrm{cell}}(q;q_\star)}{\widehat{\mathsf{J}}_2(q;q_\star)}. \tag{23.36}$$

The normalized dimensionless spectrum of the 2DOF breather is then

$$\boxed{\mathcal{S}(q;q_\star) := \frac{\Sigma(q;q_\star)}{\Sigma(1;q_\star)}.} \tag{23.37}$$

In particular,

$$\mathcal{S}(1;q_\star) = 1. \tag{23.38}$$

Reproducible numerical procedure. For fixed q and a fixed threshold $q_\star$, the unnormalized quantity $\Sigma(q;q_\star)$ is evaluated numerically by truncating at

$$L_{\max} = c\, mR_0, \qquad c \gtrsim 8,$$

then jointly increasing $(mR_0, L_{\max})$ until a relative stabilization better than 10^{-3} is reached.

Status of the spectral hierarchy. This section defines the normalized spectral function $\mathcal{S}(q;q_\star)$ for every admissible integer q, relative to the first finite threshold $q_\star$ of the sector under consideration. It fixes the relative part of the 2DOF breather spectrum, that is, the ratios of squared-mass splittings, but it does not fix the common absolute prefactor M_*^2.

The spectral hierarchy is determined by the evaluation of (23.36), with the spectral reduction (23.34) and the angular invariant (23.35). It is not introduced as an independent assumption: for any admissible triplet $q_1 < q_2 < q_3$, and for the finite threshold $q_\star$ retained in the sector, the values

$$\mathcal{S}(q_1;q_\star), \qquad \mathcal{S}(q_2;q_\star), \qquad \mathcal{S}(q_3;q_\star)$$

are obtained by calculating

$$\mathcal{S}(q;q_\star) = \frac{\Sigma(q;q_\star)}{\Sigma(1;q_\star)}.$$

All the q-dependence of the spectrum $\mathcal{S}(q;q_\star)$ is derived from the chain

$$q \longmapsto k_c(q) \longmapsto k_{\mathrm{eff}}(q;q_\star) \longmapsto \mathcal{W}_\ell(q;q_\star) \longmapsto w_q \longmapsto F_\ell(w_q) \longmapsto K_q,\ \alpha_{\mathrm{cell}}(q;q_\star),\ \widehat{\mathrm{J}}_2(q;q_\star) \longmapsto \Sigma(q;q_\star) \longmapsto \mathcal{S}(q;$$

The construction is therefore independent of any particular numerical choice of the indices q_i. The numerical values of the triplet retained for comparison with observables belong to the observational anchoring procedure, outside the present theoretical framework. Here one fixes only the relative spectral law and the ratios that follow from it.

23.3.2 Geometric spin of the 2DOF breather

The 2DOF breather is carried by a single tangential circular mode

$$\Psi = u_1 + i\, u_2,$$

with carrier frequency ω_0, modulated by an odd two-lobe envelope in a single core. This structure does not introduce an additional independent circular polarization; therefore

$$N_{\mathrm{pol}} = 1.$$

Since the remaining fast cycle is fundamental at leading order, one has

$$w = 1.$$

The common geometric counting then gives

$$s_g = \frac{wN_{\mathrm{pol}}}{2} = \frac{1}{2}.$$

The local EBK law

$$E_0(q) = h_{\mathrm{eff}}\, \omega_0(q)$$

fixes the orbital action of the fundamental 2DOF branch, while the geometric counting above fixes its *geometric spin*. To avoid any ambiguity with the EBK action, the spin angular momentum will therefore be denoted here by J_{spin}. One obtains

$$\boxed{s_g = \frac{1}{2}, \qquad |J_{\mathrm{spin}}| = \frac{1}{2}\, h_{\mathrm{eff}}.}$$

Scope. The 2DOF breather is therefore a half-integer spinorial branch. The two internal lobes correspond to an odd spatial structure of the same circular mode, and not to two independent polarizations whose spins would compensate each other. The EBK orbital action of the branch remains governed separately by the law $E_0 = h_{\text{eff}}\,\omega_0$.

23.3.3 Kinematic compatibility in motion

Reduced massive core. The 2DOF breather is a neutral two-lobe configuration contained in a single core, in strong overlap. At LO, it is not read as two separated cores but as a single effective core. The branch therefore has a comoving frame as soon as one considers a massive level of its internal spectrum, with a finite proper energy and a total longitudinal momentum equal to zero in that frame.

The reduced longitudinal dynamics of the breather core then falls within the framework of §7.1. The boost law (7.9) applies to the reduced massive core, with the tangential characteristic speed

$$c_{\mathcal{F}} = c_t.$$

Thus,

$$E(u) = \Gamma_t E_0, \qquad P(u) = \Gamma_t \frac{u}{c_t^2} E_0, \qquad \Gamma_t = \frac{1}{\sqrt{1 - u^2/c_t^2}}.$$

The associated effective dispersion is that of (7.10):

$$E(u)^2 = E_0^2 + c_t^2 P(u)^2.$$

Composite structure and localized source. The internal structure of the breather is carried by two lobes of opposite sign inside a single core, with a localized profile. The strong overlap guarantees that the composite remains described by a single comoving coordinate at LO. The internal contributions that depend on the separation Δ, in particular the crossed-gradient terms and the internal attraction terms, therefore remain localized functionals of the comoving coordinate of the core.

If one explicitly considers the radial sheath response associated with this branch, the corresponding radial source is a combination of localized core profiles and localized overlap terms. It then satisfies the localized-comoving source hypothesis of the criterion (7.3). The strictly comoving sheath is localized for

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

Scope. This block does not modify the spectral closure of the 2DOF breather, nor the canonical form of the squared-mass gaps. It only establishes that the massive levels of the breather core, when viewed in uniform motion, fall within the kinematic framework of localized massive solutions. A possible massless case of the minimal spectrum does not fall under the law $E(u) = \Gamma E_0$, since it has no massive rest frame.

23.3.4 Passage of the 2DOF breather through the admissible-return filter

The neutral 2DOF breather is a two-lobe configuration in strong overlap, contained in a single core. The point to be established is that this internal structure does not create, at LO, a new admissible-return filter. The comparison must therefore concern the normalized return map read at the boundary, and not the internal detail of the two lobes.

Lemma 23.2 (Boundary reading of a breather in strong overlap)**.** *At LO, the 2DOF breather in strong overlap is read at the boundary as a single core. Its local return map is written*

$$\mathcal{E}_{\mathrm{ret}}^{(\mathrm{br})}(\psi;q) = \mathcal{C}_{\mathrm{ext}}^{(\mathrm{br})}(q)\,\Gamma_{\mathrm{ret}}(u;q), \qquad u := \frac{\psi - \psi_{\mathrm{ret}}(q)}{\delta\psi_c(q)}. \tag{23.39}$$

The reduced function $\Gamma_{\mathrm{ret}}(u;q)$ is the same reduced function as that of the 2DOF bump at LO; only the external amplitude $\mathcal{C}_{\mathrm{ext}}^{(\mathrm{br})}(q)$ may differ.

Proof. In the strong-overlap regime, the internal separation of the two lobes is strictly smaller than the boundary-reading resolution. The boundary therefore does not resolve two distinct centers; it sees a single main return centered at $u = 0$. After local normalization,

$$\Gamma_{\mathrm{ret}}(0;q) = 1, \qquad \partial_u\Gamma_{\mathrm{ret}}(0;q) = 0.$$

The even part of the boundary expansion is then

$$\Gamma_{\mathrm{ret}}(u;q) = 1 - \frac{\kappa_{\mathrm{ret}}(q)}{2}u^2 + \frac{\lambda_{\mathrm{ret}}(q)}{24}u^4 + \mathcal{O}(u^6).$$

The strong overlap modifies the global return amplitude, but it does not modify the even reduced map read at the boundary. This global amplitude is carried by $\mathcal{C}_{\mathrm{ext}}^{(\mathrm{br})}(q)$, while the reduced map is identical to that of the 2DOF bump at LO. □

Lemma 23.3 (Invariance of the first isotropic threshold)**.** *The first admissible isotropic threshold of the 2DOF breather coincides with that of the 2DOF bump:*

$$q_2^{(\mathrm{br}\,2DOF)} = q_2^{(\mathrm{bump}\,2DOF)}. \tag{23.40}$$

Proof. The first isotropic threshold depends on the ratio of the two reduced projections of the main return. For the breather, one may write

$$R_1^{(\mathrm{br})} = r_1\,\mathcal{C}_{\mathrm{ext}}^{(\mathrm{br})}(q)\,M_{\mathrm{core}}^{(\mathrm{br})}(q),$$

$$R_2^{(\mathrm{br})} = r_2\,\mathcal{C}_{\mathrm{ext}}^{(\mathrm{br})}(q)\,M_{\mathrm{core}}^{(\mathrm{br})}(q).$$

The coefficients r_1, r_2 are those of the reduced map $\Gamma_{\mathrm{ret}}(u;q)$. By Lemma 23.2, this reduced map is the same as that of the 2DOF bump at LO. The global factor

$$\mathcal{C}_{\mathrm{ext}}^{(\mathrm{br})}(q)\,M_{\mathrm{core}}^{(\mathrm{br})}(q)$$

eliminates in the ratio:

$$\frac{R_2^{(\mathrm{br})}}{R_1^{(\mathrm{br})}} = \frac{r_2}{r_1}.$$

The isotropic-threshold condition is therefore identical to that of the 2DOF bump, which gives (23.40). □

Lemma 23.4 (Invariance of the second weak anisotropic threshold)**.** *The second weak anisotropic threshold of the 2DOF breather coincides with that of the 2DOF bump:*

$$q_3^{(\mathrm{br}\,2DOF)} = q_3^{(\mathrm{bump}\,2DOF)}. \tag{23.41}$$

Proof. The second threshold corresponds to the first weak anisotropic splitting of the main return. The reduced expansion is written

$$\Gamma_{\mathrm{ret}}(u,\theta;q) = \Gamma_0(u;q) + \varepsilon_2\,\Gamma_2(u;q)P_2(\cos\theta) + \mathcal{O}(\varepsilon_2^2).$$

In a 2DOF breather in strong overlap, the two lobes are not resolved as two independent centers by the boundary reading. They therefore do not create a new quadrupolar channel at LO. The anisotropic correction effectively read at the boundary is the same reduced block $\Gamma_2(u;q)$ as for the 2DOF bump, multiplied by the common external amplitude. This amplitude eliminates in the ratio that defines the splitting condition. The weak anisotropic condition is therefore identical to that of the 2DOF bump, hence (23.41). □

Proposition 23.5 (LO equality of the admissible bump–breather sets)**.** *At LO, the neutral 2DOF breather in strong overlap and the 2DOF bump have the same set of admissible structural thresholds:*

$$\boxed{\mathcal{Q}_{\mathrm{br}}^{(2D,\mathrm{LO})} = \mathcal{Q}_{\mathrm{bump}}^{(2D,\mathrm{LO})}.} \tag{23.42}$$

In particular, for the first three thresholds,

$$\boxed{q_i^{(\mathrm{br}\,2DOF)} = q_i^{(\mathrm{bump}\,2DOF)}, \qquad i = 1,2,3.} \tag{23.43}$$

Proof. The fundamental threshold q_1 is the first admissible element of the return filter. Since the breather in strong overlap is read at the boundary as a single core, the admissible fundamental is the same as for the 2DOF bump:

$$q_1^{(\mathrm{br}\,2DOF)} = q_1^{(\mathrm{bump}\,2DOF)}.$$

The first isotropic threshold is identical by Lemma 23.3. The second weak anisotropic threshold is identical by Lemma 23.4. Since the admissible filter is defined by the normalized return map read at the boundary, and since this map is identical at LO, the equality holds for the LO admissible set, which gives (23.42). Restricting to the first three thresholds gives (23.43). □

23.3.5 Internal squared-mass gaps: canonical form at LO

In the LO regime of the neutral 2DOF breather (same core, strong overlap) and with the local EBK quantization introduced in §23.3.1, the squared-mass gaps are carried by the differences of the internal spectral function $\mathcal{S}(q)$.

For two admissible integers $q_b > q_a$, one defines

$$\Delta\mathcal{S}(q_b,q_a) := \mathcal{S}(q_b) - \mathcal{S}(q_a). \tag{23.44}$$

The factorized law of the gaps is then written

$$\boxed{\Delta m^2(q_b,q_a) = M_*^2\,\Delta\mathcal{S}(q_b,q_a) = M_*^2\,[\mathcal{S}(q_b) - \mathcal{S}(q_a)].} \tag{23.45}$$

The prefactor $M_*^2 > 0$ is common to all levels. The entire relative dependence on q is carried by $\mathcal{S}$.

Invariance under spectral translation. Only differences of $\mathcal{S}$ enter the squared-mass gaps. The transformation

$$\mathcal{S}(q) \longmapsto \mathcal{S}(q) + C$$

therefore does not modify any physical gap. For a family of three admissible thresholds

$$q_1 < q_2 < q_3,$$

one can introduce the spectral function anchored at the first threshold

$$\widehat{\mathcal{S}}_{q_1}(q) := \mathcal{S}(q) - \mathcal{S}(q_1). \tag{23.46}$$

It satisfies

$$\widehat{\mathcal{S}}_{q_1}(q_1) = 0, \qquad \widehat{\mathcal{S}}_{q_1}(q) > 0 \quad \text{for} \quad q > q_1.$$

Canonical form for three levels. With the anchoring (23.46), the two independent gaps are

$$\Delta m^2(q_2, q_1) = M_*^2\, \widehat{\mathcal{S}}_{q_1}(q_2), \qquad \Delta m^2(q_3, q_1) = M_*^2\, \widehat{\mathcal{S}}_{q_1}(q_3). \tag{23.47}$$

Their ratio is therefore

$$\boxed{\frac{\Delta m^2(q_3, q_1)}{\Delta m^2(q_2, q_1)} = \frac{\widehat{\mathcal{S}}_{q_1}(q_3)}{\widehat{\mathcal{S}}_{q_1}(q_2)} = \frac{\mathcal{S}(q_3) - \mathcal{S}(q_1)}{\mathcal{S}(q_2) - \mathcal{S}(q_1)}.} \tag{23.48}$$

This ratio is purely structural: it depends neither on M_*^2, nor on a common mass offset, nor on an additive translation of the spectral function.

Optional internal normalization. When $q_1 < q_2 < q_3$ are fixed, it is useful to normalize the fundamental gap to unity:

$$\mathscr{S}_{q_1,q_2}(q) := \frac{\mathcal{S}(q) - \mathcal{S}(q_1)}{\mathcal{S}(q_2) - \mathcal{S}(q_1)}. \tag{23.49}$$

Then

$$\mathscr{S}_{q_1,q_2}(q_1) = 0, \qquad \mathscr{S}_{q_1,q_2}(q_2) = 1,$$

and

$$\frac{\Delta m^2(q_3, q_1)}{\Delta m^2(q_2, q_1)} = \mathscr{S}_{q_1,q_2}(q_3). \tag{23.50}$$

This normalization does not fix any numerical value of the q_i in the theory. It only defines the invariant scale reading that will be consumed during an observational evaluation.

Shape constant of the 2DOF breather core: Γ_{shape}

Status. The constant Γ_{shape} does not measure the total quadratic energy of the breather core. It measures the shape residual that contributes to the squared-mass gaps after removal of the LO carrier. The carrier is determined by the EBK law

$$E_0(q) = h_{\text{eff}}\, \omega_0(q),$$

and must not be counted again in M_*^2.

Quadratic energy and carrier/residual separation. For the LO ansatz

$$u(\chi, t) = A\, a\!\left(\frac{\chi}{\Delta_*}\right) \cos(\omega t), \qquad a(\xi) = \operatorname{sech} \xi,$$

the quadratic contributions are written in the form

$$E_{\text{kin}} = \frac{\rho A^2 \omega^2}{2}\, \mathcal{B}_0,$$

$$E_{\text{tan}} = \frac{T A^2}{2} \left[C_\ell^{(0)} \mathcal{B}_0 + \Pi_{\text{cap}}^{(1)} \mathcal{B}_2 \right].$$

The first tangential term,

$$C_\ell^{(0)}\mathcal{B}_0,$$

fixes the carrier frequency ω_0. The term that contributes to the squared-mass gaps is therefore only the projected residual

$$\Pi_{\mathrm{cap}}^{(1)}\mathcal{B}_2.$$

Definition of the cap factor. The canonical conic gauge fixes the opening ψ_c by

$$\frac{2}{\pi}\left(\frac{\psi_c}{2}-\frac{\sin 2\psi_c}{4}\right)=\frac{1}{4},\qquad \psi_c = 1.1549407300050287\ldots.$$

The $\ell = 1$ cap projection factor is

$$\Pi_{\mathrm{cap}}^{(1)} = \sin^2\psi_c = 0.8368060146\ldots.$$

Radial shape ratio. With the profile $a(\xi) = \operatorname{sech}\xi$, we use

$$J_0 = \int_0^\infty 4\pi\xi^2 a(\xi)^2\,d\xi = \frac{\pi^3}{3},$$

$$J_2 = \int_0^\infty 4\pi\xi^2 a'(\xi)^2\,d\xi = 7.6339320582\ldots.$$

The quasi-planar reduction of the core gives

$$\frac{\mathcal{B}_2}{\mathcal{B}_0} = \frac{J_2}{J_0}\frac{1}{\Delta_*^2}.$$

Shape constant. With

$$\kappa_\omega = 54.9589563237,\qquad J_0 = 10.3354255601,\qquad J_2 = 7.6339320582,$$

we find

$$\Gamma_{\mathrm{shape}}^{(\mathrm{LO})} = 0.8368060146\,\frac{7.6339320582}{10.3354255601}\,\frac{1}{54.9589563237^2} = 2.04629\times10^{-4}.$$

We therefore retain

$$\boxed{\Gamma_{\mathrm{shape}}^{(\mathrm{LO})} = 2.0462927542\times10^{-4}.} \tag{23.51}$$

Guardrail. The formula

$$\frac{\mathcal{B}_2 + C_\ell^{(0)}\mathcal{B}_0}{\mathcal{B}_0}$$

describes the total quadratic ratio of the core. It must not be used for M_*^2, because it counts the carrier $C_\ell^{(0)}\mathcal{B}_0$ again. The prefactor of the squared-mass gaps uses only the projected residual $\Pi_{\mathrm{cap}}^{(1)}\mathcal{B}_2/\mathcal{B}_0$.

Core scale prefactor (LO) and shape factor

Purpose. The prefactor M_*^2 fixes the common absolute scale of the squared-mass gaps of the 2DOF breather. It factorizes into two blocks:

$$M_*^2 = \Gamma_{\mathrm{shape}}^{(\mathrm{LO})}\,\mathcal{B}_\nu^{(\mathrm{LO})}.$$

The first one, $\Gamma_{\mathrm{shape}}^{(\mathrm{LO})}$, is the shape residual of the breather core derived in (23.51). The second one, $\mathcal{B}_\nu^{(\mathrm{LO})}$, carries the absolute scale of the normalized mode.

Spectral normalization of the breather mode. We introduce

$$\boxed{\delta_\omega := \kappa_\omega^{-1}.} \tag{23.52}$$

The parameter δ_ω is the reduced spectral width associated with the closure of the mode width. In the local gauge of the 2DOF breather, an isotropic spectral cell of linear width δ_ω in the three reduced directions carries the normalized spectral volume

$$\boxed{\mathcal{V}_\omega(\delta_\omega) = \frac{\displaystyle\int_{\delta_\omega \mathcal{C}_1} d^3p}{\displaystyle\int_{\mathcal{C}_1} d^3p} = \delta_\omega^3,} \tag{23.53}$$

where $\mathcal{C}_1$ denotes the unit spectral cell in the reduced coordinates.

A quadratic insertion of the mode coefficient contains two mode factors contracted on the same spectral cell. It therefore carries the square of the normalized spectral volume:

$$\boxed{\mathcal{I}_\omega^{(2)} = \mathcal{V}_\omega(\delta_\omega)^2 = \delta_\omega^6.} \tag{23.54}$$

The squared-mass prefactor is bilinear in two quadratic insertions of this type. The total spectral normalization of the absolute block is therefore

$$\boxed{\mathcal{I}_\omega^{(2)}\,\mathcal{I}_\omega^{(2)} = \delta_\omega^6\,\delta_\omega^6 = \delta_\omega^{12}.} \tag{23.55}$$

These are not two independent physical paths, but the two quadratic contractions of the mode coefficient in the squared-mass invariant.

Common absolute block. We then define

$$\boxed{\mathcal{B}_\nu^{(\mathrm{LO})} := \Phi_{\mathrm{core}}^{(\mathrm{LO})}\,\delta_\omega^{12}\,\left(\frac{h_{\mathrm{eff}}\,c_t}{\ell_0}\right)^2.} \tag{23.56}$$

The purely geometric core factor is

$$\boxed{\Phi_{\mathrm{core}}^{(\mathrm{LO})} = \frac{C_\ell^{(0)}}{2\pi^2\,\big(b_0^{(2)}\big)^2}, \qquad C_\ell^{(0)} = 2.} \tag{23.57}$$

The spherical average on $\mathbb{S}^3$ and the plateau normalization are therefore applied only once, in $\Phi_{\mathrm{core}}^{(\mathrm{LO})}$. No residual factor $2\pi^2$, $1/(2\pi^2)$, or additional power of δ_ω must be added downstream.

Common prefactor of the gaps. The common mechanical prefactor of the squared-mass gaps is

$$\boxed{M_*^2 = \Gamma_{\mathrm{shape}}^{(\mathrm{LO})}\,\mathcal{B}_\nu^{(\mathrm{LO})}.} \tag{23.58}$$

In the LO canonical form used below,

$$\Delta m^2(q) = M_*^2\,\mathcal{S}(q),$$

the prefactor M_*^2 is common to all levels. All the spectral dependence in q is carried by $\mathcal{S}(q)$.

Structural evaluation of the prefactor. With

$$\kappa_\omega = 54.9589563237, \qquad \delta_\omega = 1.8195396472 \times 10^{-2},$$

we have

$$\delta_\omega^6 = 3.6288510209 \times 10^{-11}, \qquad \delta_\omega^{12} = 1.3168559732 \times 10^{-21}.$$

With

$$b_0^{(2)} = 1.809206964 \times 10^4, \qquad \Phi_{\mathrm{core}}^{(\mathrm{LO})} = 3.0954497129 \times 10^{-10},$$

we obtain the internal form

$$\mathcal{B}_\nu^{(\mathrm{LO})} = 4.0762614442 \times 10^{-31} \left(\frac{h_{\mathrm{eff}} c_t}{\ell_0} \right)^2.$$

Finally, with

$$\Gamma_{\mathrm{shape}}^{(\mathrm{LO})} = 2.0462927542 \times 10^{-4},$$

one obtains

$$\boxed{M_*^2 = 8.3412242574 \times 10^{-35} \left(\frac{h_{\mathrm{eff}} c_t}{\ell_0} \right)^2.} \tag{23.59}$$

This writing keeps h_{eff}, c_t, and ℓ_0 as internal quantities of the model. Any numerical conversion into external units is excluded from the present chapter.

Separation of the factors. The form (23.56) separates the geometric core factor $\Phi_{\mathrm{core}}^{(\mathrm{LO})}$, the absolute spectral normalization δ_ω^{12}, and the dimensional scale $(h_{\mathrm{eff}} c_t/\ell_0)^2$. The prefactor M_*^2 is thus completely factorized into a shape block $\Gamma_{\mathrm{shape}}^{(\mathrm{LO})}$ and an absolute scale block $\mathcal{B}_\nu^{(\mathrm{LO})}$.

Scope. In the LO canonical form used below,

$$\Delta m^2(q) = M_*^2 \, \mathcal{S}(q),$$

the prefactor M_*^2 is common to all levels. All the spectral dependence in q is carried by $\mathcal{S}(q)$.

Absolute masses: general form, LO case, and NLO status

Parametrization of the first three admissible levels. We denote by

$$q_1 < q_2 < q_3$$

the first three admissible thresholds of the 2DOF breather at LO. They are the first three elements of the set

$$\mathcal{Q}_{\mathrm{br}}^{(2D,\mathrm{LO})}.$$

By Proposition 23.5,

$$\mathcal{Q}_{\mathrm{br}}^{(2D,\mathrm{LO})} = \mathcal{Q}_{\mathrm{bump}}^{(2D,\mathrm{LO})},$$

and therefore

$$q_i^{(\mathrm{br},2DOF)} = q_i^{(\mathrm{bump},2DOF)}, \qquad i = 1, 2, 3. \tag{23.60}$$

The absolute masses of the first three levels are then parametrized by successively reinserting

$$q = q_1, \qquad q = q_2, \qquad q = q_3$$

into the canonical mass law of the 2DOF breather. Thus the three-level structure is preserved, without needing to freeze here the explicit numerical values of the admissible indices.

General form. Since the physical gaps depend only on differences of the spectral function, the absolute law is parametrized relative to the first admissible threshold q_1. We write

$$\boxed{m_i^2 = m_0^2 + M_*^2\,\widehat{\mathcal{S}}_{q_1}(q_i), \qquad \widehat{\mathcal{S}}_{q_1}(q_i) = \mathcal{S}(q_i) - \mathcal{S}(q_1), \qquad i = 1, 2, 3.} \tag{23.61}$$

Thus

$$\widehat{\mathcal{S}}_{q_1}(q_1) = 0,$$

and m_0^2 is the common offset of the spectral floor.

Strict LO case. At LO, the effective Hamiltonian of the 2DOF breather contains no common term independent of the level. Therefore

$$\boxed{m_0^2 = 0 \qquad \text{(LO)}.} \tag{23.62}$$

In this minimal case,

$$\boxed{m_1 = 0, \qquad m_2 = \sqrt{M_*^2\,\widehat{\mathcal{S}}_{q_1}(q_2)}, \qquad m_3 = \sqrt{M_*^2\,\widehat{\mathcal{S}}_{q_1}(q_3)}.} \tag{23.63}$$

Status of the common offset beyond LO. Beyond LO, a local isotropic anharmonicity of the core (the quartic u^4 block) can induce a common offset m_0^2. In the general theory of the 2DOF breather, this offset is not fixed numerically here: it depends on a later mesh specialization and on the corresponding quartic projection.

Common offset: general form beyond LO

Quartic projection quotient. We introduce the projection quotient

$$\chi_{\text{mesh}} := \frac{\displaystyle\int_{\mathbb{S}^3} d\Omega\, \langle (\nabla u)^4 \rangle_{\text{fen}}}{\displaystyle\int_{\mathbb{S}^3} d\Omega\, \langle u^4 \rangle_{\text{fen}}} \times \frac{I_2}{I_4}, \qquad I_2 = 2, \qquad I_4 = \frac{4}{3}. \tag{23.64}$$

This quotient depends on the effective projection retained for the considered mesh. No particular numerical value is imposed in the present chapter.

Effective quartic mesh coefficient. We then group the mesh-specialization data into a dimensionless coefficient

$$\boxed{\lambda_4^{(\text{mesh})} > 0,} \tag{23.65}$$

which will be fixed only when a candidate mesh is chosen and evaluated.

Common NLO offset. The common offset is then written in the general form

$$\boxed{m_0^2 = \lambda_4^{(\text{mesh})}\,\frac{I_4}{I_2}\,\mathcal{B}_\nu^{(\text{LO})}.} \tag{23.66}$$

Using $\dfrac{I_4}{I_2} = \dfrac{2}{3}$, one may also write

$$m_0^2 = \lambda_4^{(\text{mesh})}\left(\frac{2}{3}\right)\mathcal{B}_\nu^{(\text{LO})}.$$

Since $M_*^2 = \Gamma_{\rm shape}^{(\rm LO)} \mathcal{B}_\nu^{(\rm LO)}$, this form is equivalent to

$$\boxed{m_0^2 = \frac{\lambda_4^{(\rm mesh)}}{\Gamma_{\rm shape}^{(\rm LO)}} \frac{I_4}{I_2} M_*^2.} \tag{23.67}$$

This writing makes explicit that the common offset does not use $\Gamma_{\rm shape}^{(\rm LO)}$ again: it uses the common absolute block $\mathcal{B}_\nu^{(\rm LO)}$, while M_*^2 contains the breather shape residual.

Using $\dfrac{I_4}{I_2} = \dfrac{2}{3}$, one may also write

$$m_0^2 = \lambda_4^{(\rm mesh)} \left(\frac{2}{3}\right) \Phi_{\rm core}^{(\rm LO)} \frac{h_{\rm eff}^2}{\ell_0^2\, c_t^2}.$$

Absolute masses beyond LO. In any later mesh specialization, the masses of the first three admissible levels become

$$\boxed{m_1 = \sqrt{m_0^2}, \qquad m_2 = \sqrt{m_0^2 + M_*^2\, \widehat{\mathcal{S}}_{q_1}(q_2)}, \qquad m_3 = \sqrt{m_0^2 + M_*^2\, \widehat{\mathcal{S}}_{q_1}(q_3)}.} \tag{23.68}$$

Scope. The present chapter therefore establishes the general structure

$$\lambda_4^{(\rm mesh)} \;\longrightarrow\; m_0^2 \;\longrightarrow\; m_i^2 = m_0^2 + M_*^2\, \widehat{\mathcal{S}}_{q_1}(q_i), \qquad i = 1, 2, 3.$$

without imposing here that one particular mesh is the only possible one. A particular specialization, for example dodecahedral, may be used in a dedicated numerical evaluation, but it is not part of the general theoretical closure of the present chapter.

Chapter 24

Six-bump 2DOF composite in strong overlap: effective 3×2 reduction

The object studied in this chapter is not posited as a triplet of independent 2DOF breathers. It is constructed as a composite of six 2DOF bump constituents in strong overlap. The 3×2 structure then appears as an effective reduction: the six sub-lobes regroup into three coherent pairs, and the internal projection then selects three collective breather-like witnesses.

The terms "breather-like" and "3×2 reduction" therefore designate projected collective directions of the composite, not elementary objects introduced at the outset.

Mechanical idea of the composite. The composite $\mathcal{C}_{3\times2}$ is a reduced mechanical solution constructed from six 2DOF bump constituents in strong overlap. These six constituents organize into three coherent pairs,

$$(b_1^+, b_1^-), \qquad (b_2^+, b_2^-), \qquad (b_3^+, b_3^-),$$

but the resulting pairs are not posited as three independent 2DOF breathers. They first define a collective organization of the composite, and the internal projection then selects three effective breather-like witnesses.

The role of the present chapter is to construct the mechanical object associated with these six 2DOF bumps in strong overlap, and then to describe the effective 3×2 reduction allowed by the internal coupling. The three collective directions will be denoted

$$T_{q_1}, \qquad T_{q_2}, \qquad T_{q_3},$$

or, when the breather-like reading is used,

$$B^{\mathrm{eff}}_{q_1}, \qquad B^{\mathrm{eff}}_{q_2}, \qquad B^{\mathrm{eff}}_{q_3}.$$

These objects are projected witnesses of the composite, not elementary constituents introduced at the outset.

The preclusion, fine-stiffness, and projected EM-like channel couplings act on these collective witnesses and define a reduced internal-equilibrium operator. The eigenmodes of this operator describe the coherent configurations of the composite: they give the ways in which the six sub-lobes can regroup and balance in the branch under consideration.

The indices q_1, q_2, q_3 are admissible EBK indices left symbolic in the present chapter.

24.1 Reduced LO existence of the composite

Mechanical idea. The composite $\mathcal{C}_{3\times2}$ is a reduced mechanical solution formed from three neutral 2DOF breathers coupled together. Each 2DOF breather is itself a bound structure with

two correlated sub-lobes in strong overlap. The composite therefore contains three internal pairs,

$$(b_1^+, b_1^-), \qquad (b_2^+, b_2^-), \qquad (b_3^+, b_3^-),$$

but these six sub-lobes do not constitute six independent objects: they are organized into three admissible 2DOF breathers.

Admissible spectral triplet. We consider three admissible EBK indices

$$(q_1, q_2, q_3) \in \mathcal{Q}^3_{\mathrm{adm}}, \qquad q_1 < q_2 < q_3.$$

To each index q_i is associated an admissible neutral 2DOF breather, denoted

$$B_{q_i} \equiv (b_i^+, b_i^-).$$

The associated composite is denoted

$$\boxed{\mathcal{C}_{3\times2}(q_1, q_2, q_3).} \tag{24.1}$$

Mechanical definition of the composite. We define

$$\boxed{\mathcal{C}_{3\times2}(q_1, q_2, q_3) := \big[(b_1^+, b_1^-), (b_2^+, b_2^-), (b_3^+, b_3^-)\big].} \tag{24.2}$$

This notation designates a composite solution with three 2DOF breathers, not a collection of six free sub-lobes. The three pairs (b_i^+, b_i^-) remain bound as neutral breathers.

LO collective subspace. The reduced description of the composite is made on the three collective coordinates

$$B_{q_1}, \qquad B_{q_2}, \qquad B_{q_3}.$$

We denote

$$\mathscr{V}_{\mathcal{C}} := \mathrm{span}\{B_{q_1}, B_{q_2}, B_{q_3}\}. \tag{24.3}$$

The reduced part of the composite is written

$$\Psi_{3\times2} = \sum_{i=1}^{3} a_i \, B_{q_i}(\Omega_i), \tag{24.4}$$

where a_i are the reduced amplitudes of the three branches and Ω_i denotes their internal orientation.

Reduced existence criterion. At LO, the internal preclusion and EM-like interactions between the three branches define a real quadratic form on $\mathscr{V}_{\mathcal{C}}$. This quadratic form contains diagonal terms, associated with the reduced detunings of the branches, and off-diagonal terms, associated with inter-branch couplings.

The restriction of this quadratic form to $\mathscr{V}_{\mathcal{C}}$ is a real symmetric operator of rank three. It therefore admits three real eigenmodes. These eigenmodes are the coherent organizations in which the three 2DOF breathers can associate and balance inside the composite.

Status of the solution. The composite $\mathcal{C}_{3\times2}$ is thus a reduced composite solution at LO: its mechanical constituents are the three admissible 2DOF breathers, and its eigenmodes are determined by the internal-equilibrium operator constructed in the following sections. The closure presented here establishes the reduced existence of the composite and of its internal modes; it is not an absolute mass closure of the eigenmodes.

24.2 Minimal spinorial branch of the composite

Spin of the mechanical constituents. Each branch B_{q_i} of the composite is an admissible neutral 2DOF breather. According to the construction of the 2DOF breather, recalled in §23.3.2, such a branch carries a half-integer spinorial structure. We therefore write

$$s(B_{q_i}) = \frac{1}{2}, \qquad i = 1, 2, 3.$$

The composite $\mathcal{C}_{3\times 2}$ associates three spinorial objects. It therefore retains a half-integer branch under a complete rotation.

Minimal sector. The light neutral branch studied in this chapter is the minimal spinorial branch of the composite. It corresponds to the internal coupling

$$\boxed{\left(\frac{1}{2} \otimes \frac{1}{2}\right)_0 \otimes \frac{1}{2} = \frac{1}{2}.} \tag{24.5}$$

In this reading, two 2DOF breathers of the composite form a compensated sector, whereas the third carries the residual spinorial lift of the global composite. The branch under consideration therefore satisfies

$$\boxed{s_{\mathcal{C}}^{\min} = \frac{1}{2}.} \tag{24.6}$$

Distinct half-integer sector. The coupling of three spinorial branches also admits a symmetric sector with total spin

$$s = \frac{3}{2}.$$

This sector belongs to a distinct composite branch. It does not enter the minimal spinorial branch (24.6), which is the light neutral branch considered in the rest of the chapter.

Independence of the internal diagonalization. The reduced operator $H_{\mathcal{C}}^{\mathrm{LO}}$ constructed below acts on the three collective coordinates

$$B_{q_1}, \qquad B_{q_2}, \qquad B_{q_3},$$

in the minimal spinorial branch. It describes the internal equilibria and the eigenmodes of the composite in this sector, but does not re-diagonalize the full space of spin couplings.

24.3 Projective six-sub-lobe closure of the composite

Purpose. The minimal spinorial branch (24.6) is interpreted at the fine scale as an organization of six 2DOF bump sub-lobes grouped into three coherent pairs. The notation $\mathcal{C}_{3\times 2}$ denotes this effective reduction: it does not mean that three independent 2DOF breathers are introduced as primitive constituents.

We denote the three admissible EBK scales by

$$q_1 < q_2 < q_3.$$

The closure below remains symbolic in the q_i. Possible numerical specializations belong to the evaluation sections.

24.3.1 Projective half-cycle and bottom–fine preclusion

Lemma 24.1 (Projective half-cycle of the composite channel)**.** *In an internal bilobe channel, the opposite directions of the same pair of sub-lobes represent the same channel axis. We therefore identify*

$$n \sim -n.$$

The angular space of the channel is the projective quotient

$$S^1/\{\pm 1\},$$

represented by the canonical half-cycle

$$\theta \in [0, \pi], \qquad d\mu_{\mathrm{proj}} = \frac{d\theta}{\pi}.$$

Corollary 24.2 (Projective bottom–fine width)**.** *For a fine core with reduced longitudinal profile* $\operatorname{sech}^2(qz)$*,*

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2 u \, du = 2.$$

The raw width is therefore $2/q$*. Referred to the projective half-cycle* (24.1)*, the effective width is*

$$\boxed{\ell_q^{\mathrm{eff}} = \frac{2}{\pi q}.} \tag{24.7}$$

Consequently, the projective bottom–fine preclusion satisfies

$$\boxed{P_{1,q}^{\mathrm{proj}} = q\frac{\pi}{2} P_{1,q}^{\mathrm{vol}}.} \tag{24.8}$$

24.3.2 Inward branch and fine axial separation

Fine collective center. In the strong-overlap branch, the two fine scales q_2, q_3 are placed in the same lobe of the q_1 branch. We write

$$Z = \eta z_\star,$$

where $z_\star$ denotes the axial maximum of the q_1 branch. The projective bottom–fine preclusion (24.8) balances the bottom restoring term and selects an inward branch

$$\boxed{Z_\star = \eta_\star z_\star, \qquad \eta_\star < 1.} \tag{24.9}$$

Fine axial separation. We write

$$R_{q_2} = Z_\star + \frac{\Delta R_\star}{2}, \qquad R_{q_3} = Z_\star - \frac{\Delta R_\star}{2}.$$

The reduced separation is

$$y_\star := q_2 \Delta R_\star.$$

It is fixed by the stationary condition

$$\boxed{\Lambda_\Delta(Z_\star) y_\star = 2\operatorname{sech}^2(y_\star)\tanh(y_\star).} \tag{24.10}$$

Thus

$$\boxed{\Delta R_\star = \frac{y_\star}{q_2}.} \tag{24.11}$$

The parameter $\Lambda_\Delta(Z_\star)$ is the reduced stiffness induced by the axial curvature of the q_1 bottom.

24.3.3 Projected internal witnesses

Wide witness. The q_1-like witness is the envelope mode of the wide branch, corrected by the collective displacement of the fine packet:

$$T_1 \simeq \partial_{u_1} S + \alpha_1 \partial_Z S.$$

Lemma 24.3 (Internal q_2-like projector)**.** *Let*

$$A_{q_2} := \partial_{\Delta R} S, \qquad D_q := \partial_z B_q.$$

In the inward strong-overlap branch,

$$A_{q_2} = -\frac{1}{2} D_{q_2} + \frac{1}{2} D_{q_3} + O(e^{-q_2 \Delta R_\star}).$$

The component D_{q_3} is a parasitic fine translation. We define

$$\mathcal{C}_{q_2} = \mathrm{span}\{D_{q_3}, \partial_Z S\},$$

and the metric projector

$$\Pi_{\perp \mathcal{C}} = I - C_\mu (G_C^{-1})^{\mu\nu} \langle C_\nu, \cdot \rangle_G, \qquad (G_C)_{\mu\nu} = \langle C_\mu, C_\nu \rangle_G.$$

The internal witness is

$$T_{q_2} := \Pi_{\perp \mathcal{C}} A_{q_2}.$$

Then

$$\boxed{T_{q_2} = \partial_{u_{q_2}} S + O(e^{-q_2 \Delta R_\star}) + O\left(\frac{q_2}{q_3}\right).} \tag{24.12}$$

Consequently,

$$\boxed{G_{22}^{\mathrm{br2}} = \langle T_{q_2}, T_{q_2} \rangle_G = \frac{G_{u_1 u_1}}{q_2} \left[1 + O\left(\frac{q_2}{q_3}\right)\right].} \tag{24.13}$$

24.3.4 Angular co-rotation of the q_3-like witness

We parameterize the fine cross by a global rotation ψ and a relative deformation χ. The internal orientations are

$$\theta_{q_2} = \psi - \frac{\chi}{2}, \qquad \theta_{q_3} = \psi + \frac{\chi}{2}.$$

Lemma 24.4 (Co-rotation of the q_3-like mode)**.** *The minimal angular block is*

$$E_{\mathrm{ang}} = \frac{1}{2} k_{q_2} \theta_{q_2}^2 + \frac{1}{2} k_{q_3} \theta_{q_3}^2 + \frac{1}{2} k_{\mathrm{rel}} \chi^2.$$

Quasi-static minimization in ψ gives a collective mode

$$T_{q_3} = A_\chi + \gamma A_\psi,$$

with

$$\boxed{\gamma = \frac{k_{q_2} - k_{q_3}}{2(k_{q_2} + k_{q_3})}.} \tag{24.14}$$

In the crossed q_2/q_3 locking, the local stiffness grows like the fineness q, then is saturated by the width compatibility

$$\Xi_{q_2 q_3} = \frac{4 q_2 q_3}{(q_2 + q_3)^2}.$$

Thus

$$\boxed{\frac{k_{q_3}}{k_{q_2}} \simeq \frac{q_3}{q_2} \Xi_{q_2 q_3}.} \tag{24.15}$$

The metric of the co-rotating mode is

$$\boxed{G_{33}^{\mathrm{br2}}(\gamma) = G_{\chi\chi} + 2\gamma G_{\chi\psi} + \gamma^2 G_{\psi\psi}.} \tag{24.16}$$

24.3.5 Fine diagonal correction and quasi-resonance

Lemma 24.5 (Fine diagonal stiffness correction)**.** *In the inward branch, the mode T_{q_2} is the internal axial component of the fine q_2/q_3 separation, whereas the mode T_{q_3} is a collective transverse mode. The fine overlap therefore raises the axial diagonal $H_{22}^{\mathrm{br}2}$ through the width compatibility*

$$\Xi_{q_2q_3} = \frac{4q_2q_3}{(q_2+q_3)^2}.$$

After removing the parasitic component carried by q_3, the effective fraction is

$$1 - \frac{q_2}{q_3}.$$

Thus

$$\boxed{\mathcal{C}_K^{\mathrm{fine}} = 1 + \Xi_{q_2q_3}\left(1 - \frac{q_2}{q_3}\right) + O(e^{-q_2\Delta R_\star}).} \tag{24.17}$$

The quasi-resonance of the fine block is then described by

$$\boxed{\epsilon = \frac{r-1}{r+1}, \qquad r = \frac{|B_{q_1,zz}(Z_\star)|}{|B_{q_1,\perp\perp}(Z_\star)|}\frac{G_{33}^{\mathrm{br}2}}{G_{22}^{\mathrm{br}2}}\mathcal{C}_K^{\mathrm{fine}}.} \tag{24.18}$$

24.3.6 Reduced fine scale and matrix form

We define the common scale of the fine block by

$$\boxed{M_{\mathrm{fine}} := \frac{m_2+m_3}{2}.} \tag{24.19}$$

The reduced matrix of the projective sector is

$$\boxed{\mathcal{M}_{\mathrm{red}} = \begin{pmatrix} \mu_1 & \alpha & 0 \\ \alpha & 1+\epsilon & \beta \\ 0 & \beta & 1-\epsilon \end{pmatrix}.} \tag{24.20}$$

The ratios are

$$\boxed{\alpha = \frac{a_{\mathrm{geom}}}{M_{\mathrm{fine}}}, \qquad \beta = \frac{b_{\mathrm{EM}}}{M_{\mathrm{fine}}}.} \tag{24.21}$$

The wide–fine geometric coupling is

$$\boxed{a_{\mathrm{geom}} \simeq -2I_3 B'''_{q_1}(Z_\star)\Delta R_\star.} \tag{24.22}$$

The projected EM-like coupling of the fine block is imported from the field–solution coupling:

$$\boxed{b_{\mathrm{EM}} = \frac{\Gamma_{\mathrm{EM}}\,\Xi_{q_2q_3}\,(2/\pi)}{\sqrt{G_{22}^{\mathrm{br}2}G_{33}^{\mathrm{br}2}}}.} \tag{24.23}$$

The crossed closure of the fine scale requires that the two routes

$$M_{\mathrm{fine}}^{(\alpha)} = \frac{a_{\mathrm{geom}}}{\alpha}, \qquad M_{\mathrm{fine}}^{(\beta)} = \frac{b_{\mathrm{EM}}}{\beta}$$

coincide at the LO order of the reduced sector.

24.4 EBK windows and spectral centers

Common spectral parameters. We set

$$\boxed{\Delta_* := \frac{1}{\kappa_\omega}, \qquad r_\Delta := \sqrt{\frac{\Delta_*}{2}}.} \tag{24.24}$$

The constant J_2 is that of the canonical profile $a(\xi) = \operatorname{sech}\xi$, defined in the preceding chapters.

Base spectral center. The first resolved spectral center of the 2DOF breather is defined by

$$\boxed{\ell_{\mathrm{b}} := 2\,\mathrm{round}\left(\frac{J_2}{2r_\Delta}\right).} \tag{24.25}$$

This definition projects the resolution scale J_2/r_Δ onto the coherent even sector of the 2DOF composite.

LO closure of the centers. At LO, the spectral centers of the q_i branches are defined by

$$\boxed{\ell_q = \ell_{\mathrm{b}} + 4\pi\left(1 - \frac{r_\Delta}{3}\right)\left(q^{1/2+\Delta_*} - 1\right).} \tag{24.26}$$

The associated EBK width is

$$\boxed{\sigma_q = \ell_q\, r_\Delta.} \tag{24.27}$$

EBK window. The normalized spectral window associated with q is denoted by

$$\Pi_q(\ell) = \frac{\exp\left[-\dfrac{(\ell-\ell_q)^2}{2\sigma_q^2}\right]}{\left(\sum_{\ell'\geq 0}\exp\left[-\dfrac{(\ell'-\ell_q)^2}{\sigma_q^2}\right]\right)^{1/2}}, \qquad \sum_{\ell\geq 0}\Pi_q(\ell)^2 = 1. \tag{24.28}$$

Centers of the triplet. For an admissible triplet (q_1, q_2, q_3), we write

$$\ell_i := \ell_{q_i}, \qquad \sigma_i := \sigma_{q_i}, \qquad \Pi_i(\ell) := \Pi_{q_i}(\ell). \tag{24.29}$$

24.5 Coherent inter-band mediators

NOGO of diagonal overlap. The windows Π_i may be centered on separated spectral bands. A strictly diagonal coupling in ℓ,

$$\sum_\ell \Pi_i(\ell)\Pi_j(\ell)S_\ell,$$

is in general not sufficient to carry the main internal coupling of the composite. The composite $\mathcal{C}_{3\times 2}$ requires inter-band mediation.

Central spectral gaps. We define

$$L^0_{ij} := |\ell_j - \ell_i|. \tag{24.30}$$

Coherent even preclusion sector. Preclusion compares two complete 2DOF objects, hence two pairs of sub-lobes. The odd component of the spectral mediator cancels by exchange of the two sub-lobes, whereas the even component remains coherent. We therefore introduce the projector

$$\boxed{\mathcal{P}_{\mathrm{even}}(L) = \frac{1+(-1)^L}{2}.} \tag{24.31}$$

The preclusion kernel effectively read by the composite is the coherent even envelope

$$\boxed{\widehat{G}_L^{(P,\mathrm{coh})} = \mathcal{P}_{\mathrm{even}}(L)\,\left|\widehat{G}_L^{(P)}\right|.} \tag{24.32}$$

Spectral closure of the triplet. The three coherent mediators are not independent. They satisfy the triad closure

$$\boxed{L_{12}^{\mathrm{coh}} + L_{23}^{\mathrm{coh}} = L_{13}^{\mathrm{coh}}.} \tag{24.33}$$

The pair 12, with the smallest spectral gap in the order $q_1 < q_2 < q_3$, locks the first upper even mediator:

$$\boxed{L_{12}^{\mathrm{coh}} = 2\left\lceil \frac{L_{12}^0}{2} \right\rceil.} \tag{24.34}$$

We then set

$$\boxed{L_{13}^{\mathrm{coh}} = 2\,\mathrm{round}\left(\frac{L_{13}^0}{2}\right),} \tag{24.35}$$

and then

$$\boxed{L_{23}^{\mathrm{coh}} = L_{13}^{\mathrm{coh}} - L_{12}^{\mathrm{coh}}.} \tag{24.36}$$

24.6 Internal orientation of the composite

Reduced orientations. With each branch B_{q_i}, we associate a unit internal orientation

$$\mathbf{n}_i, \qquad \|\mathbf{n}_i\| = 1.$$

The relational coupling of the composite depends on the scalar products $\mathbf{n}_i \cdot \mathbf{n}_j$.

Variational closure of the orientations. The internal orientations of the composite are fixed by cancellation of the projected vector residual in the reduced channel. The three branches carry the phase signs

$$\sigma_1 = +1, \qquad \sigma_2 = +1, \qquad \sigma_3 = -1,$$

which gives the oriented resultant

$$\boxed{\mathbf{R} := \sigma_1\mathbf{n}_1 + \sigma_2\mathbf{n}_2 + \sigma_3\mathbf{n}_3 = \mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3.} \tag{24.37}$$

In the Galerkin reduction, the LO orientation energy is the positive quadratic term associated with this resultant:

$$\boxed{\mathcal{Q}_{\mathrm{orient}} = \|\mathbf{R}\|^2 = \|\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3\|^2, \qquad \|\mathbf{n}_i\| = 1.} \tag{24.38}$$

This form does not rely on a separately imposed length relation: it is the quadratic residual of the vector projection. Any nonzero resultant leaves an uncompensated oriented charge in the reduced channel; the LO orientation closure therefore consists in minimizing $\mathcal{Q}_{\mathrm{orient}}$.

Since $\mathcal{Q}_{\text{orient}} \geq 0$, its global minimum is reached as soon as the residual vanishes. The variational condition is therefore

$$\boxed{\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3 = 0.} \tag{24.39}$$

The closure (24.39) imposes

$$\mathbf{n}_3 = \mathbf{n}_1 + \mathbf{n}_2.$$

Since the three orientations are unit vectors,

$$1 = \|\mathbf{n}_3\|^2 = \|\mathbf{n}_1 + \mathbf{n}_2\|^2 = 2 + 2\,\mathbf{n}_1 \cdot \mathbf{n}_2,$$

hence

$$\boxed{\mathbf{n}_1 \cdot \mathbf{n}_2 = -\frac{1}{2}.} \tag{24.40}$$

Contracting $\mathbf{n}_3 = \mathbf{n}_1 + \mathbf{n}_2$ with $\mathbf{n}_1$ and $\mathbf{n}_2$ then gives

$$\mathbf{n}_1 \cdot \mathbf{n}_3 = \mathbf{n}_1 \cdot (\mathbf{n}_1 + \mathbf{n}_2) = 1 - \frac{1}{2} = \frac{1}{2},$$

and similarly

$$\mathbf{n}_2 \cdot \mathbf{n}_3 = \frac{1}{2}.$$

Thus

$$\boxed{\mathbf{n}_1 \cdot \mathbf{n}_3 = +\frac{1}{2}, \qquad \mathbf{n}_2 \cdot \mathbf{n}_3 = +\frac{1}{2}.} \tag{24.41}$$

The configuration is therefore fixed, up to a global rotation, by cancellation of the projected vector residual:

$$\boxed{(\mathbf{n}_i \cdot \mathbf{n}_j)_{12,13,23} = \left(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right).} \tag{24.42}$$

Stability of the orientation closure. The previous conclusion does not depend on the particular choice of the Euclidean norm as long as the orientation functional is a positive quadratic form of the projected residual. More generally, for any symmetric positive definite operator $\mathcal{M}$,

$$\mathcal{Q}_{\mathcal{M}} = \langle \mathbf{R}, \mathcal{M}\mathbf{R} \rangle$$

admits the same minimum when the realizable condition $\mathbf{R} = 0$ is satisfied. Likewise, replacing $\mathcal{Q}_{\text{orient}}$ by a positive monotone function of $\|\mathbf{R}\|^2$ does not change the LO minimizer. The triplet

$$\left(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)$$

is therefore stable in the class of positive isotropic closures that penalize the projected vector resultant. Anisotropic or weighted corrections belong to a higher order and are not consumed in the LO closure of the composite.

24.7 Reduced internal-equilibrium operator of the composite

Reduced form. In the collective basis

$$\{B_{q_1}, B_{q_2}, B_{q_3}\},$$

we associate a reduced internal-equilibrium operator with the composite. We keep the notation $H_{\mathcal{C}}^{\text{LO}}$, but here this notation denotes a dimensionless internal-stability matrix, not an absolute energy or a squared mass.

At LO, this reduced operator is

$$\boxed{H_{\mathcal{C}}^{\mathrm{LO}}(q_1,q_2,q_3)=\begin{pmatrix} E_1 & V_{12} & V_{13} \\ V_{12} & E_2 & V_{23} \\ V_{13} & V_{23} & E_3 \end{pmatrix}.} \tag{24.43}$$

The diagonal coefficients E_i are reduced internal detunings, defined up to a common constant. The off-diagonal coefficients V_{ij} measure the reduced couplings between branches of the composite. The whole set fixes the possible coherent organizations of the tri-breather system, but does not by itself fix a mass scale.

The freedom to add a common constant to the detunings makes it possible to choose

$$\boxed{E_1 = 0.} \tag{24.44}$$

This choice fixes the internal origin of the detunings; it is not a zero-mass condition.

Common scale of the reduced matrix. The reduced matrix of the six-bump 2DOF composite is defined, at LO, up to a common scalar factor. This factor fixes the absolute scale of the entries of H_C^{LO}, but it does not modify the internal angles arising from the real diagonalization, because these are invariant under the transformation

$$H_C^{\mathrm{LO}} \longmapsto \alpha\, H_C^{\mathrm{LO}}, \qquad \alpha > 0.$$

We choose as the central value of the common normalization

$$\boxed{A_0 = \pi^{3/2}\kappa_\omega^2\left(1-\frac{\Delta_*}{2}\right).} \tag{24.45}$$

This expression groups three LO normalization factors:

$$A_0 = \underbrace{\pi^{3/2}}_{\text{projective cell}}\ \underbrace{\kappa_\omega^2}_{\text{spectral plateau}}\ \underbrace{\left(1-\frac{\Delta_*}{2}\right)}_{\text{boundary matching}}.$$

The factor $\pi^{3/2}$ is the projective-cell normalization used for the reduced 3×2 operator. The factor κ_ω^2 comes from the two insertions of plateau spectral resolution, with $\kappa_\omega = \Delta_*^{-1}$. The factor $(1-\Delta_*/2)$ is the first-order affine closure of the boundary matching between the two sub-lobes; it retains the linear loss of overlap over a reduced half-width.

Stability of the common scale. The matching factor

$$\mathcal{R}_\Delta^{\mathrm{aff}} := 1-\frac{\Delta_*}{2}$$

is an LO boundary closure. Neighboring regularizations with the same first-order expansion,

$$\mathcal{R}_\Delta^{\exp} := \exp\!\left(-\frac{\Delta_*}{2}\right), \qquad \mathcal{R}_\Delta^{\mathrm{rat}} := \frac{1}{1+\Delta_*/2},$$

differ from $\mathcal{R}_\Delta^{\mathrm{aff}}$ only at order $O(\Delta_*^2)$. With $\Delta_* \simeq 1.82\times 10^{-2}$, this dispersion is below the percent level on the common scale.

Thus, (24.45) must be read as an LO-regularized central normalization of H_C^{LO}. It does not affect the internal angles of the projective reduction, but any absolute prediction that consumes the global scale of H_C^{LO} inherits this normalization uncertainty.

Reduction of the channel amplitudes. The 2DOF breather is represented, in the internal space of the two sub-lobes, by the reduced weights

$$c_+ = +1, \qquad c_- = -1.$$

The EM-like channel reads the neutral dipolar component of the doublet. With the projective normalization of this channel, the reduced amplitude is

$$A_{\mathrm{EM}} = A_0 \, \frac{c_+ - c_-}{2} = A_0.$$

The preclusion channel, in contrast, reads the even contribution, quadratic in each sub-lobe. The two sub-lobe contributions add with the opposite sign proper to the preclusion channel:

$$A_P = -A_0 \, (c_+^2 + c_-^2) = -2A_0.$$

We obtain

$$\boxed{A_P = -2A_0, \qquad A_{\mathrm{EM}} = A_0.} \tag{24.46}$$

Spectral channel coefficients. The inter-branch couplings consume the reduced zonal coefficients of the EM-like and preclusion channels. For any harmonic mediator L, we set

$$\widehat{G}_L^{(\mathrm{EM})} := \widehat{G}_\ell^{(\mathrm{EM})}\Big|_{\ell=L}, \tag{24.47}$$

and

$$\widehat{G}_L^{(P)} := \widehat{G}_\ell^{(P)}\Big|_{\ell=L}. \tag{24.48}$$

The coefficients $\widehat{G}_\ell^{(\mathrm{EM})}$ and $\widehat{G}_\ell^{(P)}$ are the zonal coefficients of the channel kernels defined by (20.12)–(20.13). For the tri-breather composite, they are evaluated at the coherent mediator $L = L_{ij}^{\mathrm{coh}}$, defined by (24.34)–(24.36).

Off-diagonal entries. The internal inter-branch couplings are

$$\boxed{V_{ij}^{\mathrm{LO}} = (\mathbf{n}_i \cdot \mathbf{n}_j) A_0 \left[-2 \left| \widehat{G}^{(P)}_{L_{ij}^{\mathrm{coh}}} \right| + \widehat{G}^{(\mathrm{EM})}_{L_{ij}^{\mathrm{coh}}} \right].} \tag{24.49}$$

The notation P denotes the preclusion channel here.

Saturating diagonals. We define

$$\boxed{\ell_c = \frac{\ell_1}{1 + \frac{3}{2} r_\Delta}, \qquad E_\infty = 4 - r_\Delta \log 6.} \tag{24.50}$$

For $i = 2, 3$, the internal detunings are

$$\boxed{E_i = E_\infty \left[1 - \exp\left(-\frac{\ell_i - \ell_1}{\ell_c} \right) \right].} \tag{24.51}$$

This closure expresses the loss of spectral coherence of branch i relative to the base branch q_1. The factor $6 = 3 \times 2$ corresponds to the number of sub-lobes of the composite $\mathcal{C}_{3\times 2}$.

Dimensional status. The coefficients E_i and V_{ij} are dimensionless. The eigenvalues of $H_{\mathcal{C}}^{\mathrm{LO}}$ are therefore reduced internal levels of the composite. They are used to determine the eigenmodes and the internal transition matrix, but they are not identified at this stage with masses or squared masses.

A mass closure of the eigenmodes would require an additional dimensioned scale and a common floor of the composite. These quantities are not introduced in the LO closure of the present chapter.

24.8 Internal eigenmodes

Diagonalization. The internal eigenmodes of the composite are defined by

$$\boxed{H_{\mathcal{C}}^{\mathrm{LO}}(q_1, q_2, q_3)\, U_{\mathcal{C}} = U_{\mathcal{C}} \Lambda_{\mathcal{C}}.} \tag{24.52}$$

The columns of $U_{\mathcal{C}}$ describe the normal modes of the composite, that is, the coherent organizations of the three coupled 2DOF breathers. In each eigenmode, the three branches $B_{q_1}, B_{q_2}, B_{q_3}$ participate with relative amplitudes determined by the corresponding eigenvector.

The matrix $U_{\mathcal{C}}$ is therefore the transition matrix between the collective basis $\{B_{q_i}\}$ and the basis of the internal eigenmodes of the composite. The eigenvalues gathered in $\Lambda_{\mathcal{C}}$ are reduced levels of internal stability; they are not dimensioned masses at this stage.

Internal angles. When $U_{\mathcal{C}}$ is parameterized by three real rotations at LO, we denote the internal angles by

$$\vartheta_{12}(q_1, q_2, q_3), \qquad \vartheta_{13}(q_1, q_2, q_3), \qquad \vartheta_{23}(q_1, q_2, q_3).$$

These angles characterize only the diagonalization of the composite $\mathcal{C}_{3\times 2}(q_1, q_2, q_3)$. Their interpretation in an external reading basis belongs to a separate correspondence chapter.

Scope. The present chapter fixes no numerical triplet (q_1, q_2, q_3). It defines the parametric construction of the composite $\mathcal{C}_{3\times 2}$, its minimal spinorial branch, its reduced internal-equilibrium operator and its eigenmodes. The evaluations obtained after inserting a numerical triplet belong to a separate evaluation section.

The closure presented here determines the reduced LO existence of the composite, its minimal spinorial sector, the internal structure of the coupling between the three 2DOF breathers, and the transition matrix toward its eigenmodes. It does not constitute an absolute mass closure of the eigenmodes of the composite.

Chapter 25

Bi–soliton of quasi-c_t 2DOF: prototype of the quasi-c_t branch

In addition to the one-lobe solitonic solution (8.6), the focusing envelope equation admits *two bound-center* configurations (bi–solitons), in which two distinct localized centers belong to the same global composite solution on $\mathbb{S}^3$ and form a bound branch.

25.1 Bi–solitonic ansatz and binding

A symmetric comoving configuration is written

$$A_{2\ell}(S,T) = A_0\Big[\operatorname{sech}\Big(\tfrac{S-S_0-VT}{\lambda_{\rm env}}\Big)\, e^{+i\frac{\phi}{2}} + \operatorname{sech}\Big(\tfrac{S+S_0+VT}{\lambda_{\rm env}}\Big)\, e^{-i\frac{\phi}{2}}\Big]\,, \tag{25.1}$$

where ϕ is the relative phase between the two localized centers and $2S_0$ their slow separation.

The overlap of the tails induces an effective binding potential $U_{\rm int}(2S_0)$ which selects the separation and locks the composite branch. In the bound branch, the retained configuration satisfies

$$\phi \simeq 0,$$

whereas $\phi \simeq \pi$ corresponds to an anti-bound configuration.

Geometric closure on $\mathbb{S}^3$. On the compact sphere, the natural geometry of a two-distinct-center configuration carried by the same great geodesic is the antipodal closure. We therefore introduce the geometric half–separation s_0 by

$$\boxed{2s_0 \simeq \pi R_0, \qquad S_0 = \varepsilon\, s_0\,.} \tag{25.2}$$

This relation must be read as the *candidate geometry of global closure* of the two-center branch: it expresses antipodal compatibility on $\mathbb{S}^3$, without yet prejudging a complete demonstration of the dynamical locking of the separation.

The bi–soliton is then described as a two-center collective branch, parametrized by a common amplitude, relative phase, and carrier frequency (A_0, ϕ, ω_0). In this reading, the coherence of the branch may persist even when the separation of the two centers becomes large compared with their own width.

25.2 Complete two-center mechanical solution

The corresponding complete HSM solution is written

$$\psi(x,t) \;\approx\; \varepsilon\, A_{2\ell}(S,T)\, e^{i(qs-\omega_0 t)}, \qquad \begin{cases} u_1 = A_{2\ell}\cos(qs-\omega_0 t), \\ u_2 = A_{2\ell}\sin(qs-\omega_0 t), \end{cases} \tag{25.3}$$

with $q \simeq \omega_0/c_t$.

This writing explicitly shows the masses rotating at the carrier $e^{-i\omega_0 t}$ under a two-center envelope. The slow dynamics is described by a small number of collective parameters (A_0, ϕ, S_0, V), so that the two centers constitute a **two-center composite branch** rather than two independent objects.

25.3 Global spectral link: dipolar $\ell = 1$ structure on $\mathbb{S}^3$

Independently of the local quasi-propagative description, the spectral geometry of $\mathbb{S}^3$ makes the lowest nontrivial structure with two opposite poles appear at rank $\ell = 1$. The associated eigenvalue is

$$\lambda_1 = \frac{3}{R_0^2}.$$

Restricted to a great geodesic parametrized by $s = R_0\psi$, the natural geometric dependence of a dipole is proportional to

$$\cos\psi = \cos\Big(\frac{s}{R_0}\Big),$$

which exhibits two extrema of opposite sign at the antipodal points.

The robust conclusion useful here is therefore the following: a configuration with two localized centers, separated by approximately a half-circumference, is geometrically compatible with the global dipolar $\ell = 1$ structure of $\mathbb{S}^3$. This compatibility fixes the *global reading* of the solution, without imposing at this stage a stronger one-dimensional identification.

We thus obtain two complementary readings of the same mechanical object on $\mathbb{S}^3$:

- a **local reading**: a composite solution with two localized centers;
- a **global reading**: a dipolar closure of rank $\ell = 1$ with two antipodal anti-nodes.

Conclusion. The quasi-c_t 2DOF bi–soliton thus constitutes a **two-center mechanical support** compatible with the global geometry of $\mathbb{S}^3$. This chapter establishes the coherence of this two-lobe structure; the construction of coherent states on the internal H/V doublet, and the correlated interpretation that follows from it, belong to a distinct step built next on this support.

25.4 Structural result: quasi-c_t two-center support

Objective. The purpose of the present chapter is not to construct directly the correlated state of type H/V, but to establish the *mechanical support* on which such a construction can then rest. We therefore isolate here the result useful for what follows: the existence of a coherent two-center reading for the quasi-c_t 2DOF branch.

Proposition 25.1 (Two-center support for the quasi-c_t branch)**.** *In the quasi-c_t 2DOF sector, the structure defined by the bi–localized ansatz* (25.1) *and its mechanical reconstruction* (25.3) *provides a* ***localized two-center support*** *compatible with the antipodal closure* (25.2) *and with the global dipolar $\ell = 1$ reading on $\mathbb{S}^3$.*

In particular, this structure:

1. *describes two distinct localized centers belonging to the same collective branch;*
2. *admits a unique mechanical reconstruction with common carrier ω_0;*
3. *is compatible with a global reading with two antipodal anti-nodes;*

4. *therefore provides the minimal geometric and mechanical support of a two-lobe quasi-c_t branch.*

Justification. The ansatz (25.1) explicitly exhibits two distinct localized profiles, with slow separation $2S_0$, sharing the same branch amplitude, the same carrier, and the same relative phase. The reconstruction (25.3) shows that these two centers are not two independent objects, but the two spatial components of the same HSM solution with a slow envelope.

The antipodal closure (25.2) provides the natural geometry of this branch on $\mathbb{S}^3$. Finally, the spectral reading of Section 25.3 establishes that this configuration is compatible with the global dipolar $\ell = 1$ structure, that is, with the lowest nontrivial structure with two opposite poles on $\mathbb{S}^3$. The whole therefore fixes the status of the solution as a coherent two-center support. □

Scope for what follows. Proposition 25.1 does not yet construct the correlated state itself, but it fixes exactly the mechanical ingredients required in the following chapter.

First, the two spatial arms A and B are the two lobes of a *same* HSM solution, and not two independent packets prepared separately. Second, this same two-lobe support admits, in the 2DOF tangent plane, two degenerate orthogonal internal orientations, denoted H and V, carried by the same bi–localized envelope and the same carrier ω_0. One can therefore define, on a single two-lobe spatial support, the two configurations $|HH\rangle$ and $|VV\rangle$.

The following chapter can then construct the coherent superposition

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + e^{i\Delta}|VV\rangle),$$

and then study the action of local filters on each lobe. The CHSH-type violation will therefore not rest on two independent objects, but on the internal H/V coherence of a *single* quasi-c_t mode with two mechanically locked lobes.

25.5 Theoretical CHSH experiment on the quasi-c_t 2DOF bi–soliton: preparation

Objective. We consider the quasi-c_t 2DOF bi–soliton as a **single composite solution** with two centers on $\mathbb{S}^3$. The goal is to define, from this solution, a CHSH-type theoretical experiment based on **local binary projections** performed on each of the two centers.

External references. The CHSH terminology refers to the inequality of Clauser–Horne–Shimony–Holt [6]. The bound $2\sqrt{2}$ used as a comparison benchmark is the Cirel'son–Tsirelson bound [7]. In the present section, these results are not postulated as dynamical ingredients of HSM: they serve only as comparison language for the correlation calculated from the quasi-c_t bi–solitonic solution.

Guiding idea. In the HSM framework, the two centers of the bi–soliton are not two independent objects put in interaction, but two localized maxima of a **same global solution**. The expected theoretical correlations must therefore be read as the consequence of this global coherence, and not as the result of a delayed exchange between two separated subsystems.

25.5.1 Support state: two-center bi–soliton

We start from the complete mechanical solution of the quasi-c_t 2DOF bi–soliton:

$$\psi(x,t) \;\approx\; \varepsilon\, A_{2\ell}(S,T)\, e^{i(qs-\omega_0 t)}, \qquad \begin{cases} u_1 = A_{2\ell}\cos(qs-\omega_0 t), \\ u_2 = A_{2\ell}\sin(qs-\omega_0 t), \end{cases} \tag{25.4}$$

where the envelope $A_{2\ell}$ has two localized centers along a great geodesic.

We denote by x_A and x_B the two centers of the bi–soliton. They define the two **theoretical measurement arms** of the CHSH experiment:

$$A \equiv x_A, \qquad B \equiv x_B.$$

25.5.2 Local internal structure and polarization basis

In the neighborhood of each of the two centers, the 2DOF solution lives in a real internal plane generated by the two tangential components (u_1, u_2). In accordance with the two-component structure of the quasi-c_t 2DOF sector, we consider in this plane an orthonormal doublet of degenerate internal orientations

$$\mathbf{e}_H, \qquad \mathbf{e}_V, \qquad \mathbf{e}_H \cdot \mathbf{e}_V = 0,$$

interpreted as an **internal polarization basis**. The two states H and V then correspond to the same bi–localized envelope carried by two orthogonal internal orientations of the same mechanical support.

The composite bi–soliton thus defines the same global internal state, locally readable on each of the two centers in this (H, V) basis.

25.5.3 Local settings, filters, and binary observables

For each arm, we introduce an angular setting:

$$a \text{ at arm } A, \qquad b \text{ at arm } B.$$

At angle a, the local directions are defined by

$$|a,+\rangle = \cos a\,|H\rangle + \sin a\,|V\rangle, \qquad |a,-\rangle = -\sin a\,|H\rangle + \cos a\,|V\rangle. \tag{25.5}$$

Similarly, at angle b,

$$|b,+\rangle = \cos b\,|H\rangle + \sin b\,|V\rangle, \qquad |b,-\rangle = -\sin b\,|H\rangle + \cos b\,|V\rangle. \tag{25.6}$$

Physically, a local filter oriented along a (resp. b) separates the local internal component into two orthogonal channels $(+)$ and $(-)$. If $|\eta_A\rangle$ denotes the normalized internal state at arm A, the projected intensities are

$$I_{A,\pm}(a) = |\langle a,\pm|\eta_A\rangle|^2, \qquad I_{A,+}(a) + I_{A,-}(a) = 1,$$

and similarly at arm B,

$$I_{B,\pm}(b) = |\langle b,\pm|\eta_B\rangle|^2, \qquad I_{B,+}(b) + I_{B,-}(b) = 1.$$

The ideal binary reading then consists in assigning the value $+1$ to channel $(+)$ and the value -1 to channel $(-)$. In the corresponding operator writing, we introduce the projectors

$$\Pi_{a,\pm} = |a,\pm\rangle\langle a,\pm|, \qquad \Pi_{b,\pm} = |b,\pm\rangle\langle b,\pm|,$$

and the binary observables

$$A(a) := \Pi_{a,+} - \Pi_{a,-}, \qquad B(b) := \Pi_{b,+} - \Pi_{b,-}. \tag{25.7}$$

By construction,

$$A(a)^2 = B(b)^2 = \mathbf{1},$$

and the elementary measurement outcomes are indeed ± 1.

25.5.4 Reduced reference correlated state

The bi–soliton first provides a single mechanical support: the two centers A and B are two lobes of the same global solution, and not two states prepared independently. In the 2DOF internal plane, the degeneracy of the two orthogonal orientations H and V allows the local reading to be restricted to the two-level subspace generated by

$$|H\rangle, \qquad |V\rangle.$$

In this reduction, the two configurations

$$|H\rangle_A|H\rangle_B, \qquad |V\rangle_A|V\rangle_B$$

represent two coherent internal readings of the same bi–localized support.

Ideal reduced modal configuration. For the CHSH theoretical experiment, we consider the ideal reduced component of the bi–solitonic solution in the modal subspace $(H,V)_A \otimes (H,V)_B$:

$$\Psi_{\rm red} = \frac{1}{\sqrt{2}}\Big(|H\rangle_A|H\rangle_B + e^{i\Delta}|V\rangle_A|V\rangle_B\Big), \tag{25.8}$$

where Δ is the global relative phase between the two internal readings.

This writing is not used here as a postulated quantum state of a system with two independent particles. It is the reduced modal coordinate of a global HSM solution with two centers, projected on the two degenerate internal orientations H and V. The factor $1/\sqrt{2}$ expresses the ideal symmetric configuration between these two degenerate orientations.

The reading rule used in this mathematical experiment is the Born rule applied to the reduced modal component: the probability of a filter outcome is the ratio between the quadratic norm of the projected component and the total quadratic norm of the reduced component. In this section, this rule is an effective reading postulate. It is not derived from the local classical dynamics of the HSM field. This rule is not a local hidden-variable factorization $P_{\sigma\tau}(a,b) = P_\sigma(a,\lambda)P_\tau(b,\lambda)$; it interrogates a nonfactorizable global solution.

In the ideal coherent case, the phase reference is chosen so that

$$\Delta = 0. \tag{25.9}$$

Deviations from this ideal coherence are treated below by the visibility factor, cf. (25.21).

Status of the CHSH calculation. The calculation that follows establishes a conditional statement: if the HSM bi–solitonic solution is represented by the nonfactorizable reduced component $\Psi_{\rm red}$, and if this component is read by the effective Born rule (25.10), then the local binary projections produce the correlation calculated below.

The result must therefore not be read as a violation of CHSH by a local classical field endowed with a classical local readout. In such a readout, the binary outcomes would be local functions of the field and of a common cause, and the classical Bell bound $S \le 2$ would apply. The value $2\sqrt{2}$ obtained below is that of the Born reading on the nonfactorizable modal component; it is not a consequence of a local factorization of the field.

25.5.5 Theoretical correlation

For outcomes $\sigma, \tau \in \{+,-\}$, we define the local modal projectors

$$\Pi^A_\sigma(a), \qquad \Pi^B_\tau(b),$$

associated with the two orthogonal outputs of the local filters. The coincidence probabilities are defined by the projected HSM quadratic norm:

$$P_{\sigma\tau}(a,b) := \frac{\left\| \Pi_\sigma^A(a)\Pi_\tau^B(b)\Psi_{\mathrm{red}} \right\|_{\mathrm{HSM}}^2}{\|\Psi_{\mathrm{red}}\|_{\mathrm{HSM}}^2}. \tag{25.10}$$

This equation is the Born rule for the reduced modal component. It defines the readout used in the mathematical experiment of this section. It does not describe a local classical readout based on the intensity or on the pointwise value of the field at the two centers.

The binary correlation associated with settings a and b is then

$$E(a,b) := P_{++}(a,b) + P_{--}(a,b) - P_{+-}(a,b) - P_{-+}(a,b). \tag{25.11}$$

This quantity is the building block of the theoretical CHSH experiment.

Once $E(a,b)$ has been calculated, four settings will be introduced,

$$a, \quad a', \quad b, \quad b'$$

and the combination

$$S = E(a,b) - E(a,b') + E(a',b) + E(a',b'). \tag{25.12}$$

25.5.6 Scope of the construction

The present section fixes only the **preparation** of the CHSH theoretical experiment:

- choice of the support solution;
- identification of the two measurement arms;
- definition of the local settings;
- definition of the binary observables;
- definition of the correlation $E(a,b)$.

The explicit calculation of $E(a,b)$, then of the combination S, will be carried out in the next section.

25.6 Theoretical CHSH experiment on the quasi-c_t 2DOF bi–soliton: correlation and inequality

Objective. The purpose of this section is to calculate explicitly the correlation associated with local binary projections on the two centers of the quasi-c_t 2DOF bi–soliton. We start from the reduced modal component Ψ_{red}, then the correlation

$$E(a,b) = P_{++}(a,b) + P_{--}(a,b) - P_{+-}(a,b) - P_{-+}(a,b).$$

We then deduce the CHSH combination.

25.6.1 Local observables in the (H, V) basis

In the $(|H\rangle, |V\rangle)$ basis, the local states

$$|a, +\rangle = \cos a\,|H\rangle + \sin a\,|V\rangle, \qquad |a, -\rangle = -\sin a\,|H\rangle + \cos a\,|V\rangle$$

define the two output modal projectors

$$\Pi^A_+(a) = |a, +\rangle\langle a, +|, \qquad \Pi^A_-(a) = |a, -\rangle\langle a, -|.$$

Similarly, at arm B,

$$\Pi^B_+(b) = |b, +\rangle\langle b, +|, \qquad \Pi^B_-(b) = |b, -\rangle\langle b, -|.$$

These projectors are not introduced as fundamental quantum observables: they represent the two orthogonal mechanical outputs of an internal polarization filter in the (H, V) plane.

25.6.2 Global correlated state

We recall the reduced modal component introduced in (25.8):

$$\Psi_{\text{red}} = \frac{1}{\sqrt{2}}\Big(|H\rangle_A|H\rangle_B + e^{i\Delta}|V\rangle_A|V\rangle_B\Big). \tag{25.13}$$

The ideal coherent case corresponds to

$$\Delta = 0. \tag{25.14}$$

25.6.3 Explicit calculation of the correlation

The projected modal amplitudes are

$$C_{\sigma\tau}(a, b) := \langle a, \sigma|_A \langle b, \tau|_B\, \Psi_{\text{red}}\rangle.$$

They are

$$C_{++} = \frac{1}{\sqrt{2}}\left(\cos a \cos b + e^{i\Delta} \sin a \sin b\right),$$

$$C_{--} = \frac{1}{\sqrt{2}}\left(\sin a \sin b + e^{i\Delta} \cos a \cos b\right),$$

$$C_{+-} = \frac{1}{\sqrt{2}}\left(-\cos a \sin b + e^{i\Delta} \sin a \cos b\right),$$

$$C_{-+} = \frac{1}{\sqrt{2}}\left(-\sin a \cos b + e^{i\Delta} \cos a \sin b\right).$$

The quadratic reading rule gives

$$P_{\sigma\tau}(a, b) = |C_{\sigma\tau}(a, b)|^2.$$

We then obtain

$$\boxed{E(a, b) = \cos(2a)\cos(2b) + \cos\Delta\, \sin(2a)\sin(2b).} \tag{25.15}$$

Direct calculation. By summing the projected probabilities with the binary signs,

$$E(a, b) = P_{++} + P_{--} - P_{+-} - P_{-+},$$

the norm terms give $\cos(2a)\cos(2b)$, whereas the interference terms between the two global modal components give $\cos\Delta\, \sin(2a)\sin(2b)$. This gives (25.15). □

Ideal coherent case. When $\Delta = 0$, the correlation simplifies to

$$\boxed{E(a,b) = \cos\big(2(a-b)\big).} \tag{25.16}$$

Associated probabilities. In this ideal case,

$$P_{\text{same}}(a,b) = \cos^2(a-b), \qquad P_{\text{diff}}(a,b) = \sin^2(a-b). \tag{25.17}$$

25.6.4 CHSH combination

For four settings a, a', b, b', we define

$$S = E(a,b) - E(a,b') + E(a',b) + E(a',b'). \tag{25.18}$$

Standard optimal choice. With the choice

$$a = 0, \qquad a' = \frac{\pi}{4}, \qquad b = \frac{\pi}{8}, \qquad b' = \frac{3\pi}{8}, \tag{25.19}$$

and the ideal correlation (25.16), we obtain

$$\boxed{|S| = 2\sqrt{2}.} \tag{25.20}$$

Calculation. We have successively

$$E(a,b) = \cos\Big(2\Big(0 - \frac{\pi}{8}\Big)\Big) = \cos\Big(-\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2},$$

$$E(a,b') = \cos\Big(2\Big(0 - \frac{3\pi}{8}\Big)\Big) = \cos\Big(-\frac{3\pi}{4}\Big) = -\frac{\sqrt{2}}{2},$$

$$E(a',b) = \cos\Big(2\Big(\frac{\pi}{4} - \frac{\pi}{8}\Big)\Big) = \cos\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2},$$

$$E(a',b') = \cos\Big(2\Big(\frac{\pi}{4} - \frac{3\pi}{8}\Big)\Big) = \cos\Big(-\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}.$$

Thus

$$S = \frac{\sqrt{2}}{2} - \Big(-\frac{\sqrt{2}}{2}\Big) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2\sqrt{2}.$$

□

25.6.5 Relative dephasing, visibility, and effect on CHSH

If the global relative phase Δ fluctuates from one preparation to another, we introduce the coherence factor

$$\Gamma := \langle e^{i\Delta} \rangle. \tag{25.21}$$

In the case where the distribution of Δ is symmetric and centered, the phase reference can be chosen so that Γ is real and positive; we then define

$$\Gamma = V \in [0,1].$$

The averaged correlation becomes

$$E(a,b) = \cos(2a)\cos(2b) + V\ \sin(2a)\sin(2b). \tag{25.22}$$

Important remark. Expression (25.22) describes a *relative dephasing* between the components $|HH\rangle$ and $|VV\rangle$. It must not be confused with an isotropic reduced-contrast model of the form

$$E_{\mathrm{iso}}(a,b) = V\,\cos\big(2(a-b)\big),$$

which corresponds to a stronger hypothesis of uniform attenuation of the whole correlation.

CHSH value for the standard angles. Keeping the settings (25.19), we obtain directly

$$S_{\mathrm{std}}(V) = \sqrt{2}\,(1+V). \tag{25.23}$$

In particular,

$$S_{\mathrm{std}}(1) = 2\sqrt{2}, \qquad S_{\mathrm{std}}(0) = \sqrt{2}.$$

With this standard choice of angles, the CHSH violation, in the sense of the effective Born reading defined in (25.10), therefore persists as soon as

$$V > \sqrt{2} - 1. \tag{25.24}$$

Case of the isotropic model. If one adopts instead the simplified model

$$E_{\mathrm{iso}}(a,b) = V\,\cos\big(2(a-b)\big),$$

then the corresponding CHSH bound is

$$|S|_{\max}^{(\mathrm{iso})} = 2\sqrt{2}\,V, \tag{25.25}$$

and the CHSH violation, in the sense of this effective Born reading, persists if and only if

$$V > \frac{1}{\sqrt{2}}. \tag{25.26}$$

This latter writing must therefore be used only for this *isotropic model* and not for the exact relative dephasing (25.22).

25.6.6 Local margins and non-signaling

Whatever the value of Δ, the marginal probabilities obtained by summing the projected coincidences are

$$P_A(+|a) = P_{++}(a,b) + P_{+-}(a,b) = \frac{1}{2}, \qquad P_A(-|a) = \frac{1}{2},$$

and

$$P_B(+|b) = P_{++}(a,b) + P_{-+}(a,b) = \frac{1}{2}, \qquad P_B(-|b) = \frac{1}{2}.$$

They therefore do not depend on the distant setting. The correlations of the two-lobe bi–soliton modify the coincidences, but not the local margins.

25.6.7 HSM interpretation

Physical reading in HSM. In the present framework, the theoretical violation of CHSH does not come from two independent objects that would exchange a signal or a delayed field. Nor does it come from a local hidden-variable model attaching a predefined outcome to each center. It results from the quadratic modal projection of the same global HSM solution with two centers.

The two centers x_A and x_B are spatially separated, but they are not treated here as two subsystems prepared independently. They are two local readings of the same global modal component Ψ_{red}. The local filters select two output bases in the internal (H,V) plane, and the coincidences are then calculated by projecting the global solution onto these outputs.

Conceptual consequence. The theoretical CHSH experiment of the quasi-c_t 2DOF bi–soliton therefore shows that a nonfactorizable reduced modal component, associated with a global HSM solution with two centers, reproduces Bell-type correlations when it is read by the effective Born rule. It does not demonstrate that a local classical readout of the HSM field violates Bell inequalities. This result must not be read as a violation of CHSH by a local common-cause model; it rests on the nonfactorization of the reduced solution and on the quadratic projection rule for the modes.

Conclusion. In the ideal coherent case $\Delta = 0$, the reduced component of the quasi-c_t 2DOF bi–soliton, read by the effective Born rule (25.10), leads to the correlation law

$$E(a, b) = \cos 2(a - b)$$

and reaches the value

$$|S| = 2\sqrt{2}.$$

The status of this result is therefore conditional: it establishes that the HSM two-center solution can carry the modal algebraic structure of quantum correlations if it is read according to the Born rule. It does not constitute, as it stands, a derivation of Bell violation from a local classical readout of the HSM field. A local classical readout of the same support would belong to the Bell common-cause framework and would be bounded by $S \leq 2$.

Chapter 26

Common-wave 2DOF bi–bump: prototype of the composite bump branch

Problem statement. We seek here an object distinct from the quasi-c_t 2DOF bi–soliton. The *common-wave 2DOF bi–bump* is conceived as a **single composite solution** of the fundamental HSM equation, possessing two localized, spatially distinct centers, but carried by the same global spectral structure on $\mathbb{S}^3$.

In other words, we seek neither a simple superposition

$$\Psi_{B,1} + \Psi_{B,2}$$

of two independent bumps, nor a retarded interaction between two separated objects, but a **global two-center solution**, whose two bumps are two maxima of the same field closed on $\mathbb{S}^3$.

26.1 Spectral preamble: parity constraint for the common wave on $\mathbb{S}^3$

Starting point. The unit 2DOF bump has a local radial core and a weak exterior defined on $\mathbb{S}^3$. In the study of the bi–bump, the new point is therefore not the presence of a spectral exterior, but the possibility that the **same global component** simultaneously supports two localized centers.

Mode $\ell = 1$ and antipodal parity. The fundamental nontrivial zonal mode

$$Y_{100}(\psi) \;\propto\; C_1^{(1)}(\cos\psi) \;\propto\; \cos\psi, \tag{26.1}$$

satisfies

$$-\Delta_{\mathbb{S}^3} Y_{100} = \frac{3}{R_0^2}\, Y_{100}. \tag{26.2}$$

It has two opposite antinodes on $\mathbb{S}^3$, but it is *odd* under antipody: if $\iota : x \mapsto \bar{x}$ denotes the antipodal involution, then

$$Y_{100} \circ \iota = -\, Y_{100}.$$

Consequence for the bi–bump. In a branch with two antipodal centers, compatibility with Y_{100} therefore depends on the *parity of the combination of the two cores*:

- the **symmetric** combination

$$\Phi_{\mathrm{sym}}(x) := \Phi_B(r_+(x)) + \Phi_B(r_-(x))$$

  is antipodally *even*;

- the **antisymmetric** combination

$$\Phi_{\mathrm{odd}}(x) := \Phi_B(r_+(x)) - \Phi_B(r_-(x))$$

  is antipodally *odd*.

Thus, Y_{100} cannot serve as a common wave for a *symmetric* branch with two same-sign bumps, but is instead the natural candidate for an *antisymmetric* two-center branch.

Retained reading. The correct guiding idea therefore becomes:

two local bump-type cores	+	a global mode Y_{100} of the same parity	$\Longrightarrow$	two-center compos

The sought bi–bump is therefore not understood as two independent bumps related by a field transport, but as two localized maxima of the **same global spectral structure**, provided that the parity of the common mode is compatible with the branch considered.

26.2 Definition of the common-wave bi–bump

We call **common-wave 2DOF bi–bump** any stationary or quasi-stationary solution Ψ_{2B} of the fundamental HSM equation such that:

1. Ψ_{2B} has **two localized centers** x_+ and x_- on $\mathbb{S}^3$;
2. near each of these centers, the local profile is of 2DOF bump type;
3. Ψ_{2B} is a **single global solution** and not the juxtaposition of two independent solutions;
4. the two centers share the **same global spectral component**, compatible with the parity of the composite branch considered on $\mathbb{S}^3$.

This last property is essential: on $\mathbb{S}^3$, modes are closed and can simultaneously support several localized maxima of the same solution, provided that their parity structure is compatible with the composite branch considered.

26.3 Doctrinal note: local radial core and common global wave

Principle. In the bump branch, two description levels must be distinguished:

1. the **local core**, which remains bump-like, centered, and radial in the neighborhood of a given point of $\mathbb{S}^3$;
2. the **weak exterior**, which develops on the global modes of $\mathbb{S}^3$ and can therefore carry a spectral structure common to several centers.

In other words, the radiality of the bump holds for the *local core* and must not be confused with the complete global structure of the solution on $\mathbb{S}^3$.

Consequence. The *no-go* concerning a *purely radial* bump does not forbid the existence of a **common-wave bi–bump**. It only means that such an object cannot be obtained by remaining within the radial core branch alone. To obtain two localized centers in the same solution, one must add to the bump core a common global spectral component on $\mathbb{S}^3$, compatible with the parity of the composite branch considered.

Retained reading. The doctrine adopted below is therefore the following:

local radial bump core + shared global mode on $\mathbb{S}^3$, of compatible parity $\Longrightarrow$ two-center co

In this reading, the two centers of a bi–bump are not two independent objects that influence each other at a distance, but two localized maxima of the **same global solution** closed on $\mathbb{S}^3$.

Scope. The present section fixes the correct existence problem of the common-wave 2DOF bi–bump: it is a global solution with two antipodal centers coupled to a common wave, and not a simple superposition of two independent bumps. It therefore fixes the correct doctrine: the bi–bump, if it exists, is neither a purely radial bump, nor a simple sum of two independent bumps, but a **global composite solution** whose coherence is carried by a common wave on $\mathbb{S}^3$.

26.3.1 Kinematic compatibility of the 2DOF bump bi-soliton

Two massive heads. The 2DOF bump bi-soliton consists of two localized 2DOF bump cores. Each head has its own comoving frame, proper energy, and longitudinal momentum. Their centers are denoted

$$s_1(t), \qquad s_2(t),$$

and the collective coordinates are introduced as

$$S(t) = \frac{s_1(t) + s_2(t)}{2}, \qquad d(t) = s_2(t) - s_1(t).$$

The variable S describes the global motion of the bi-soliton, whereas d describes the internal separation of the two heads.

Kinematics of the individual heads. Each 2DOF bump head falls within the framework of localized massive solutions of §7.1. For head i, when its longitudinal dynamics is reduced to the comoving massive core, one has

$$E_i(u_i) = \Gamma_i E_{i,0}, \qquad P_i(u_i) = \Gamma_i \frac{u_i}{c_t^2} E_{i,0}, \qquad \Gamma_i = \frac{1}{\sqrt{1 - u_i^2/c_t^2}},$$

with

$$u_i = \dot{s}_i, \qquad i = 1, 2.$$

This writing applies to the two heads separately. It does not assume that the complete bi-soliton already has a single comoving frame.

Locked composite subregime. When the internal separation is fixed or adiabatic,

$$d(t) = d_0 \qquad \text{or} \qquad |\dot{d}| \ll |\dot{S}|,$$

the bi-soliton can be described, at the rank considered, by a global motion of center $S(t)$. In this subregime, the composite has an effective comoving frame and a total proper energy

$$E_{\mathrm{bi},0} = E_{1,0} + E_{2,0} + E_{\mathrm{int}}(d_0),$$

where $E_{\rm int}(d_0)$ denotes the internal binding or interaction energy at given separation. The boost law of the composite is then written

$$E_{\rm bi}(U) = \Gamma_U E_{\rm bi,0}, \qquad P_{\rm bi}(U) = \Gamma_U \frac{U}{c_t^2} E_{\rm bi,0}, \qquad \Gamma_U = \frac{1}{\sqrt{1 - U^2/c_t^2}},$$

with

$$U = \dot{S}.$$

Regime with two independent motions. When the two heads have independent velocities,

$$u_1 \neq u_2,$$

or when the separation $d(t)$ becomes a non-negligible dynamical variable, the bi-soliton does not reduce to a single massive core. Its kinematics is then that of a two-body system:

$$E_{\rm bi} = E_1(u_1) + E_2(u_2) + E_{\rm int}(d, \dot{d}, \ldots),$$

and not that of a single particle of type

$$E_{\rm bi} = \Gamma E_{\rm bi,0}.$$

The boost law of the kinematics chapter then applies to the individual heads, whereas the relative dynamics must be treated through the collective variables (S, d).

Radial compatibility. Each 2DOF bump head carries a localized radial source. The comoving sheath criterion (7.3) therefore applies separately to each head:

$$|u_i| < c_r, \qquad c_r^2 = \kappa_r c_t^2, \qquad i = 1, 2.$$

In the locked subregime, the total source of the composite is a finite sum of localized sources at fixed separation. It remains localized in the comoving coordinate of the center S, and the strictly comoving global sheath is localized for

$$|U| < c_r.$$

In the regime with two independent motions, there is in general no single sheath of the form $R(s - Ut)$. The radial response must then be read as the possibly coupled superposition of the responses associated with the two moving sources.

Scope. The 2DOF bump bi-soliton is therefore not automatically a single massive particle. It becomes one only in the composite subregime where the two heads share a global motion and a controlled internal separation. Outside this subregime, the correct kinematics is that of two coupled massive solitons.

26.4 Two-center ansatz and common wave

We denote by $\Psi_B(x; x_0)$ a 2DOF bump localized around the center x_0. The problem is not to posit

$$\Psi_{2B}(x) = \Psi_B(x; x_+) + \Psi_B(x; x_-)$$

as a free sum, but to seek a composite ansatz of the form

$$\Psi_{2B}(x) = a_+(x)\, \Psi_B(x; x_+) + a_-(x)\, \Psi_B(x; x_-) + \eta(x), \tag{26.3}$$

where:

- x_+ and x_- are the two geodesic centers;
- $a_\pm(x)$ represent the local weights of the two cores;
- $\eta(x)$ is the **global common component**, responsible for the coherence of the whole.

The term η is not a negligible correction: it carries precisely the common wave which makes the bi–bump a global solution and not a state with two independent objects.

26.5 Formulation of the existence problem

Let the fundamental HSM equation be written, in symbolic form, as

$$\mathcal{F}[\Psi; \omega_0, \kappa_\mu, \kappa_r, \ldots] = 0. \tag{26.4}$$

The existence problem for the bi–bump consists in establishing that there exist parameters

$$(x_+, x_-, \omega_0, \ldots)$$

and a global correction η such that the ansatz (26.3) satisfies equation (26.4).

This question may be reformulated in variational terms: if $E[\Psi]$ denotes the HSM energy functional, one seeks a critical branch of E in the class of two-center fields,

$$\delta E[\Psi_{2B}] = 0,$$

with local stability with respect to variations in separation, phase, and spectral content.

26.6 Role of the geometry of $\mathbb{S}^3$

The key point is that, on $\mathbb{S}^3$, waves are **closed**. The compactness of the geometry therefore allows the same global spectral structure to carry two localized maxima, without needing to introduce a field transport between two independent objects.

The sought bi–bump must thus be understood as

$$\text{local bump-type cores} \quad + \quad \text{shared global mode on } \mathbb{S}^3.$$

In this reading, the two centers may be far from one another, including at the antipode scale, while remaining two components of the same global solution.

26.7 Guiding criterion

The guiding existence criterion for a common-wave bi–bump is therefore the following:

> *There exists a common-wave 2DOF bi–bump if the fundamental HSM equation admits a two-center global solution whose each center is locally of bump type, and whose global coherence is ensured by a common spectral component on $\mathbb{S}^3$, compatible with the parity of the considered composite branch.*

This criterion fixes the admissible branch of the lifting problem: only the antisymmetric branch is compatible with the common mode Y_{100}. In contrast, it fixes the right mathematical problem and clearly distinguishes the common-wave bi–bump:

- from a simple sum of two independent bumps;
- from a composite state governed only by an external interaction;
- from a quasi-c_t bi–soliton, whose construction proceeds through an envelope reduction.

26.8 Derivation program

To effectively establish the existence of a common-wave bi–bump, it will be necessary to:

1. identify the spectral decomposition of the 2DOF bump on $\mathbb{S}^3$;
2. determine which global components, of compatible parity, can be common to two localized centers;
3. construct the corrected ansatz (26.3);
4. inject this ansatz into the fundamental equation or into the energy functional;
5. show the existence of a stable two-center critical branch.

Status. The present section therefore fixes the **exact framework of the existence problem**. It establishes the correct notion of a bi-bump in HSM: a branch with two antipodal centers coupled to an odd common wave.

26.9 Framework criterion for the composite bump branch

Objective. We seek to formulate a theoretical existence criterion for composite solutions of the bump branch, of which the common-wave 2DOF bi–bump is the first case. It is therefore not a universality theorem on all HSM families, but a framework criterion specific to the composite bump branch.

Proposition 26.1 (Framework criterion for a composite bump branch)**.** *Let there be a family of HSM bumps. Suppose that there exists:*

1. ***A reference local bump.*** *A localized solution $\Psi_B(x; x_0)$ centered at x_0, of bump type.*
2. ***A compatible global spectral decomposition.*** *A decomposition of Ψ_B on the eigenmodes of $\mathbb{S}^3$, such that there exists a nontrivial global spectral subspace able to support simultaneously two localized maxima at two distinct centers x_+ and x_-.*
3. ***A two-center ansatz with a common wave.*** *A field of the form*

$$\Psi_{2B}(x) = a_+(x)\,\Psi_B(x; x_+) + a_-(x)\,\Psi_B(x; x_-) + \eta(x), \tag{26.5}$$

   where $\eta(x)$ is a common global component, not reducible to the simple sum of two independent bumps.
4. ***A closure by the fundamental equation.*** *There exist geometric and spectral parameters*

$$(x_+, x_-, \omega_0, \ldots)$$

   such that Ψ_{2B} satisfies the HSM fundamental equation

$$\mathcal{F}[\Psi_{2B}; \omega_0, \kappa_\mu, \kappa_r, \ldots] = 0. \tag{26.6}$$

5. ***A local two-center stability.*** *In the class of two-center fields, Ψ_{2B} is a stable critical point of the HSM energy functional: there exists a geodesic distance $d_* = d(x_+, x_-)$ for which the corresponding branch is locally stable.*

*Then there exists a branch of **common-wave composite bump solutions**, consisting of two localized bump-type centers, but forming a **single global solution** on $\mathbb{S}^3$. The 2DOF bi–bump is the first representative of this branch.*

Justification. The local bump Ψ_B provides the localized elementary building block around each center. The global spectral decomposition on $\mathbb{S}^3$ makes it possible to identify a subspace of closed modes able to carry two localized maxima. The ansatz (26.5) then explicitly introduces the two centers (x_+, x_-) as well as the common component η, responsible for the global coherence of the solution.

Condition (26.6) ensures that we are not treating a simple approximate superposition of two independent bumps, but indeed an effective solution of the fundamental equation. Finally, local stability in the two-center class selects a robust composite branch. The two centers are then two localized maxima of the same global solution, and not two autonomous objects put into interaction. □

Interpretation. Proposition 26.1 does not say that every bump automatically generates a bi–bump. It fixes the framework criterion: as soon as a *global common wave* can support two bump-type centers and the corresponding ansatz closes on the fundamental equation, one obtains a **single two-center composite solution**.

Key geometric point. The role of $\mathbb{S}^3$ is essential: the compactness of the geometry imposes closed modes, so that the same global spectral structure can simultaneously carry two localized maxima. The link therefore does not have to be interpreted as delayed field transport between two independent objects, but as the coherence of the same global wave.

Scope. The preceding proposition is a framework criterion. It fixes the proper notion of a common-wave composite bump solution, and the proper proof program: identify the shared spectral subspace, construct η, then close the two-center ansatz on the HSM fundamental equation.

Corollary 26.2 (Effective derivation program)**.** *To effectively establish the existence of a common-wave 2DOF bi–bump, it is sufficient to:*

1. *determine the spectral decomposition of the 2DOF bump on* $\mathbb{S}^3$*;*
2. *identify the common global components compatible with two centers;*
3. *explicitly construct the correction* η *in* (26.5)*;*
4. *verify the closure on* (26.6)*;*
5. *establish the local stability of the two-center branch.*

26.10 Spectral ansatz of the common-wave 2DOF bi–bump

Local cores. Let $\Phi_B(r)$ be the local profile of the unit 2DOF bump, with $r_\pm(x) = d(x, x_\pm)$ for two antipodal centers x_+ and x_- on $\mathbb{S}^3$.

Two natural combinations. We introduce the two superpositions

$$\Phi_{\mathrm{sym}}(x) := \Phi_B(r_+(x)) + \Phi_B(r_-(x)), \qquad \Phi_{\mathrm{odd}}(x) := \Phi_B(r_+(x)) - \Phi_B(r_-(x)). \tag{26.7}$$

Under the antipodal map $\iota : x \mapsto \bar{x}$, one has

$$\Phi_{\mathrm{sym}} \circ \iota = \Phi_{\mathrm{sym}}, \qquad \Phi_{\mathrm{odd}} \circ \iota = -\Phi_{\mathrm{odd}}.$$

Exact obstruction for the symmetric branch. Since the global mode Y_{100} is antipodally odd, one immediately obtains

$$\boxed{\langle \Phi_{\mathrm{sym}}, Y_{100} \rangle_{\mathbb{S}^3} = 0.} \tag{26.8}$$

Thus, the *symmetric* branch with two bumps of the same sign cannot couple to Y_{100}: it is therefore not the correct branch for a common wave of type $\ell = 1$.

Retained branch. Instead, we retain the **antisymmetric branch**

$$\Psi_{2B}(x,t) = e^{-i\omega_0 t}\Big[A_B\,\Phi_{\mathrm{odd}}(x) + A_{\mathrm{ext}}\,Y_{100}(x)\Big]. \tag{26.9}$$

This branch is the most natural one for a coupling to Y_{100}, since Φ_{odd} and Y_{100} have the same parity.

26.11 Reduced projection on the antisymmetric branch

Reduced stationary problem. We denote by $\mathcal{L}_{\omega_0}$ the linearized stationary operator associated with the frequency ω_0, and by $\mathcal{N}$ the nonlinear / closure terms. The fundamental equation is written symbolically as

$$\mathcal{L}_{\omega_0}\Psi + \mathcal{N}[\Psi] = 0. \tag{26.10}$$

We project this problem onto the reduced subspace

$$\mathcal{B}^{(-)}_{\mathrm{red}} := \mathrm{span}\{\Phi_{\mathrm{odd}}, Y_{100}\}.$$

Reduced system. We obtain the system

$$\begin{pmatrix} \mathcal{M}_{BB}(\omega_0) & \mathcal{M}_{BY}(\omega_0) \\ \mathcal{M}_{YB}(\omega_0) & \mathcal{M}_{YY}(\omega_0) \end{pmatrix} \begin{pmatrix} A_B \\ A_{\mathrm{ext}} \end{pmatrix} + \begin{pmatrix} \mathcal{N}_B(A_B, A_{\mathrm{ext}}) \\ \mathcal{N}_Y(A_B, A_{\mathrm{ext}}) \end{pmatrix} = 0, \tag{26.11}$$

with

$$\mathcal{M}_{YB} = \mathcal{M}_{BY}, \qquad \mathcal{N}_B, \mathcal{N}_Y = \mathcal{O}(\|(A_B, A_{\mathrm{ext}})\|^2).$$

Global block. The mode Y_{100} satisfies

$$-\Delta_{\mathbb{S}^3} Y_{100} = \frac{3}{R_0^2} Y_{100},$$

therefore, in the linearized exterior part,

$$\mathcal{M}_{YY}(\omega_0) = \Big(\frac{3c_t^2}{R_0^2} - \omega_0^2\Big)\,\|Y_{100}\|^2_{\mathbb{S}^3}. \tag{26.12}$$

We introduce the exterior critical frequency

$$\omega_{\mathrm{ext}} := \frac{\sqrt{3}\,c_t}{R_0}. \tag{26.13}$$

26.12 Effective coupling of the antisymmetric branch to the mode Y_{100}

Nonzero overlap. In the antisymmetric branch, one has

$$\mathcal{M}_{BY}(\omega_0) = \langle \Phi_{\text{odd}},\, \mathcal{L}_{\omega_0} Y_{100} \rangle_{\mathbb{S}^3} = \Big(\frac{3c_t^2}{R_0^2} - \omega_0^2 \Big) \langle \Phi_{\text{odd}}, Y_{100} \rangle_{\mathbb{S}^3}. \tag{26.14}$$

Proposition 26.3 (Non-vanishing of the antisymmetric overlap)**.** *If the bump core Φ_B is strongly localized around each of the two antipodal centers, then*

$$\boxed{\langle \Phi_{\text{odd}}, Y_{100} \rangle_{\mathbb{S}^3} > 0} \tag{26.15}$$

for the phase choice where $Y_{100}(x_+) = +1$ and $Y_{100}(x_-) = -1$.

Proof. Near x_+, the mode Y_{100} is

$$Y_{100}(x) = Y_{100}(x_+) + \mathcal{O}(r_+^2/R_0^2) = 1 + \mathcal{O}(r_+^2/R_0^2),$$

whereas near x_-,

$$Y_{100}(x) = Y_{100}(x_-) + \mathcal{O}(r_-^2/R_0^2) = -1 + \mathcal{O}(r_-^2/R_0^2).$$

Since $\Phi_B \geq 0$ and is localized near each of the centers, we define

$$I_B := \int_{\mathbb{S}^3} \Phi_B(r_+)\, dV > 0.$$

By antipodal symmetry, one then has

$$\int_{\mathbb{S}^3} \Phi_B(r_-)\, dV = I_B,$$

and we obtain

$$\langle \Phi_{\text{odd}}, Y_{100} \rangle_{\mathbb{S}^3} = 2I_B + \mathcal{O}\Big(I_B\, \frac{\ell_B^2}{R_0^2} \Big).$$

In particular,

$$\langle \Phi_{\text{odd}}, Y_{100} \rangle_{\mathbb{S}^3} > 0 \qquad \text{for } \ell_B \ll R_0,$$

where ℓ_B denotes the localization scale of the bump core.

□

Expansion near the threshold. By setting

$$\delta := \omega_0 - \omega_{\text{ext}},$$

one has

$$\mathcal{M}_{YY}(\omega_0) = -2\omega_{\text{ext}}\, \|Y_{100}\|_{\mathbb{S}^3}^2\, \delta + \mathcal{O}(\delta^2), \tag{26.16}$$

and

$$\mathcal{M}_{BY}(\omega_0) = -2\omega_{\text{ext}}\, \langle \Phi_{\text{odd}}, Y_{100} \rangle_{\mathbb{S}^3}\, \delta + \mathcal{O}(\delta^2). \tag{26.17}$$

26.13 Local bump block and effective coefficient

Unit block of the bump. The local profile Φ_B of the 2DOF bump satisfies the local stationary equation already established above for the bump branch. If $\mathcal{L}_B$ denotes the local linearization near Φ_B, one has

$$\mathcal{L}_B \Phi_B = 2\, \beta_{\text{loc}}\, \Phi_B^3$$

with $\beta_{\text{loc}} > 0$. Consequently,

$$\mathcal{M}_B^{(\text{unit})} := \langle \Phi_B, \mathcal{L}_B \Phi_B \rangle = 2\beta_{\text{loc}} \int \Phi_B^4\, dV > 0. \tag{26.18}$$

Local block of the antisymmetric branch. It follows that

$$\mathcal{M}_{BB}(\omega_0) = 2\,\mathcal{M}_B^{(\mathrm{unit})}(\omega_0) - 2\,\mathcal{M}_{\mathrm{ov}}(\omega_0), \tag{26.19}$$

where $\mathcal{M}_{\mathrm{ov}}$ is the overlap between the two cores. In the regime of strong localization and antipodal separation,

$$\mathcal{M}_{\mathrm{ov}}(\omega_0) = \mathcal{O}(e^{-\pi R_0/\ell_B}),$$

therefore

$$\boxed{\mathcal{M}_{BB}(\omega_{\mathrm{ext}}) \neq 0.} \tag{26.20}$$

Effective coefficient. One can then eliminate A_B locally and introduce

$$\Lambda_{\mathrm{eff}}(\omega_0) := \mathcal{M}_{YY}(\omega_0) - \frac{\mathcal{M}_{YB}(\omega_0)\mathcal{M}_{BY}(\omega_0)}{\mathcal{M}_{BB}(\omega_0)}. \tag{26.21}$$

Since $\mathcal{M}_{YY} = \mathcal{O}(\delta)$ and $\mathcal{M}_{BY}\mathcal{M}_{YB} = \mathcal{O}(\delta^2)$, one obtains

$$\Lambda_{\mathrm{eff}}(\omega_0) = -2\omega_{\mathrm{ext}}\,\|Y_{100}\|_{\mathbb{S}^3}^2\,\delta + \mathcal{O}(\delta^2). \tag{26.22}$$

The effective coefficient therefore indeed changes sign near ω_{ext}.

26.14 Local nondegeneracy lemma for the 2DOF bump

Role in the proof. This section provides the only additional local building block needed for the full lifting of the antisymmetric branch of the common-wave 2DOF bi–bump. It does not concern the unit bump as an isolated solution, but only the control of the *local bump block* in the two-center reduction.

Local setting. In the quasi-flat tangential cap, the local stationary profile of the 2DOF bump satisfies the equation

$$-\Phi_B''(s) + \Omega_0^2\,\Phi_B(s) - \beta_{\mathrm{loc}}\,\Phi_B(s)^3 = 0, \qquad \Omega_0 > 0, \quad \beta_{\mathrm{loc}} > 0, \tag{26.23}$$

with solution

$$\Phi_B(s) = A_0\,\mathrm{sech}(\Omega_0 s), \qquad A_0^2 = \frac{2\Omega_0^2}{\beta_{\mathrm{loc}}}. \tag{26.24}$$

Real and phase linearizations. We introduce the operators

$$L_+ := -\partial_s^2 + \Omega_0^2 - 3\beta_{\mathrm{loc}}\,\Phi_B^2, \tag{26.25}$$

$$L_- := -\partial_s^2 + \Omega_0^2 - \beta_{\mathrm{loc}}\,\Phi_B^2. \tag{26.26}$$

Lemma 26.4 (Local nondegeneracy of the bump core)**.** *The operators L_+ and L_- satisfy:*

1.

$$\ker L_- = \mathrm{span}\{\Phi_B\}; \tag{26.27}$$

2.

$$\ker L_+ = \mathrm{span}\{\Phi_B'\}; \tag{26.28}$$

3. L_- is strictly positive on the orthogonal of Φ_B in $L^2(\mathbb{R})$;

4. L_+ has a unique simple negative direction, a simple kernel generated by Φ_B', and is strictly positive on

$$X_+ := \Big\{ u \in H^1(\mathbb{R}) \,;\, \langle u, \Phi_B \rangle_{L^2} = 0,\ \langle u, \Phi_B' \rangle_{L^2} = 0 \Big\}. \tag{26.29}$$

In particular, there exist $c_-, c_+ > 0$ such that

$$\langle v, L_- v \rangle_{L^2} \geq c_- \, \|v\|_{H^1}^2 \qquad \forall v \in H^1(\mathbb{R}),\ \langle v, \Phi_B \rangle_{L^2} = 0, \tag{26.30}$$

and

$$\langle u, L_+ u \rangle_{L^2} \geq c_+ \, \|u\|_{H^1}^2 \qquad \forall u \in X_+. \tag{26.31}$$

Proof. We introduce the reduced variable $y = \Omega_0 s$. With

$$\Phi_B(s) = A_0 \operatorname{sech}(\Omega_0 s), \qquad A_0^2 = \frac{2\Omega_0^2}{\beta_{\mathrm{loc}}},$$

one obtains

$$L_- = \Omega_0^2 \Big(-\partial_y^2 + 1 - 2 \operatorname{sech}^2 y \Big), \tag{26.32}$$

$$L_+ = \Omega_0^2 \Big(-\partial_y^2 + 1 - 6 \operatorname{sech}^2 y \Big). \tag{26.33}$$

These are the standard Pöschl–Teller operators [8]. The spectral properties needed for the lifting are recalled below.

For L_-, the discrete spectrum reduces to the eigenvalue 0, with eigenfunction $\operatorname{sech} y$, which gives $\ker L_- = \operatorname{span}\{\Phi_B\}$, and then strict positivity on its orthogonal.

For L_+, the translation invariance of the profile imposes

$$L_+ \Phi_B' = 0,$$

therefore $\Phi_B' \in \ker L_+$. The exact spectrum of $-\partial_y^2 + 1 - 6\operatorname{sech}^2 y$ moreover has a unique simple negative eigenvalue, and no other zero eigenvalue. Thus, $\ker L_+$ is exactly generated by Φ_B', and the positive subspace has codimension 2.

Finally, the unique negative eigenfunction of L_+ is even (and proportional to $\operatorname{sech}^2(\Omega_0 s)$), so it has nonzero overlap with $\Phi_B(s) \propto \operatorname{sech}(\Omega_0 s)$. It therefore cannot belong to X_+. The coercive estimates (26.30)–(26.31) then follow from the spectral gap and the ellipticity of $L_\pm$. □

Corollary 26.5 (Coercive complex local block modulo phase and translation)**.** *For a complex perturbation $\eta = u + i\, v$, define*

$$\mathcal{Q}_B[\eta] := \langle u, L_+ u \rangle_{L^2} + \langle v, L_- v \rangle_{L^2}. \tag{26.34}$$

Then there exists $c_B > 0$ such that

$$\mathcal{Q}_B[\eta] \geq c_B \Big(\|u\|_{H^1}^2 + \|v\|_{H^1}^2 \Big) \tag{26.35}$$

for every perturbation satisfying

$$\langle u, \Phi_B \rangle_{L^2} = 0, \qquad \langle u, \Phi_B' \rangle_{L^2} = 0, \qquad \langle v, \Phi_B \rangle_{L^2} = 0. \tag{26.36}$$

Proof. Apply lemma 26.4 separately to the real part and to the imaginary part. □

Use for the bi–bump. In the antisymmetric branch with fixed antipodal centers, the translation and phase modes of the local core are precisely the neutral modes to be eliminated in the Lyapunov–Schmidt reduction [9]. The preceding corollary therefore provides the required local coercivity on the bump block, before taking into account the weak coupling between the two cores.

26.15 Local supercritical bifurcation at the level of the reduced system

Even structure of the reduced energy. In the 2DOF bump branch, the reduced energy is even in (A_B, A_{ext}), because it comes from quadratic gradient / elasticity terms and from a local closure depending on $|\Psi|^2$. After local elimination of A_B, the effective energy therefore reads

$$E_{\mathrm{eff}}(A_{\mathrm{ext}};\omega_0) = \frac{1}{2}\,\Lambda_{\mathrm{eff}}(\omega_0)\,A_{\mathrm{ext}}^2 + \frac{1}{4}\,\Gamma_4(\omega_0)\,A_{\mathrm{ext}}^4 + \mathcal{O}(A_{\mathrm{ext}}^6). \tag{26.37}$$

Positivity of the reduced quartic. The coefficient Γ_4 receives:

- a contribution $\Gamma_4^{(\mathrm{loc})} > 0$ from the local convexity of the bump closure ($V''(0) > 0$) ;
- a contribution $\Gamma_4^{(\mathrm{grad})} \geq 0$ from the minimal gradient / preclusion cost.

Thus

$$\boxed{\Gamma_4(\omega_0) > 0} \tag{26.38}$$

near the threshold.

Reduced scalar equation. The stationarity condition $\partial_{A_{\mathrm{ext}}} E_{\mathrm{eff}} = 0$ gives

$$\Lambda_{\mathrm{eff}}(\omega_0)\,A_{\mathrm{ext}} + \Gamma_4(\omega_0)\,A_{\mathrm{ext}}^3 + \mathcal{O}(A_{\mathrm{ext}}^5) = 0. \tag{26.39}$$

Proposition 26.6 (Local supercritical bifurcation of the antisymmetric branch)**.** *Under the preceding hypotheses, the reduced system admits a local nontrivial branch of antisymmetric two-center solutions, bifurcating near*

$$\omega_{\mathrm{ext}} = \frac{\sqrt{3}\,c_t}{R_0}.$$

This bifurcation is supercritical*:*

$$|A_{\mathrm{ext}}(\omega_0)| \asymp |\omega_0 - \omega_{\mathrm{ext}}|^{1/2}, \qquad |A_B(\omega_0)| \asymp |\omega_0 - \omega_{\mathrm{ext}}|^{1/2}, \tag{26.40}$$

on the side of the threshold where $\Lambda_{\mathrm{eff}}(\omega_0) < 0$, that is, locally for

$$\omega_0 > \omega_{\mathrm{ext}}$$

at leading rank.

Proof. The expansion (26.22) gives a transverse sign change of Λ_{eff}. The positivity (26.38) then gives the supercritical normal form of (26.39). There is therefore a nontrivial branch of reduced solutions, with amplitude

$$A_{\mathrm{ext}}^2 = -\frac{\Lambda_{\mathrm{eff}}(\omega_0)}{\Gamma_4(\omega_0)} + \mathcal{O}(\Lambda_{\mathrm{eff}}^2),$$

which implies the scaling law (26.40). □

26.16 Full lifting of the antisymmetric branch

Objective. We now want to lift the local branch obtained at the level of the reduced system into an effective local branch of solutions of the fundamental HSM equation, in the antisymmetric class with fixed antipodal centers.

Reduced field. Write

$$\Psi_{\mathrm{red}}(x,t) = e^{-i\omega_0 t}\Big[A_B(\omega_0)\,\Phi_{\mathrm{odd}}(x) + A_{\mathrm{ext}}(\omega_0)\,Y_{100}(x)\Big], \tag{26.41}$$

where $(A_B(\omega_0), A_{\mathrm{ext}}(\omega_0))$ is the local reduced branch given by proposition 26.6.

Antisymmetric decomposition. We work in the subspace

$$L^2_{\mathrm{odd}}(\mathbb{S}^3) := \{u \in L^2(\mathbb{S}^3)\,;\, u\circ\iota = -u\},$$

and introduce

$$\mathcal{E}^{(-)}_{\mathrm{red}} := \mathrm{span}\{\Phi_{\mathrm{odd}},\, Y_{100}\}, \qquad L^2_{\mathrm{odd}}(\mathbb{S}^3) = \mathcal{E}^{(-)}_{\mathrm{red}} \oplus \mathcal{E}^{(-)}_{\mathrm{comp}}. \tag{26.42}$$

We seek an exact solution in the form

$$\Psi(x,t) = \Psi_{\mathrm{red}}(x,t) + \xi(x,t), \qquad \xi \in \mathcal{E}^{(-)}_{\mathrm{comp}}. \tag{26.43}$$

Local bump block. The local coercivity needed for the lifting is provided by corollary 26.5. In other words, once the center and phase of each local core have been fixed, the local quadratic form of the 2DOF bump is strictly positive on the complement of the neutral modes. This property will be used below to obtain the invertibility of the complementary block after adding the weak coupling between the two cores and the global mode Y_{100}.

Exterior spectral gap. In the antisymmetric subspace, the first critical global mode is Y_{100}, with eigenvalue

$$\lambda_1 = \frac{3}{R_0^2}.$$

The next odd mode corresponds to $\ell = 3$, hence

$$\lambda_3 = \frac{15}{R_0^2}, \qquad \lambda_3 - \lambda_1 = \frac{12}{R_0^2}.$$

There therefore exists $c_{\mathrm{ext}} > 0$ such that, for ω_0 close enough to ω_{ext}, the exterior part of the linearized operator is coercive on the antisymmetric complement of Y_{100}:

$$\langle u, \mathcal{L}_{\omega_0} u\rangle \geq c_{\mathrm{ext}}\|u\|^2_{L^2} \qquad \text{on } \mathcal{E}^{(-)}_{\mathrm{comp}} \cap \{Y_{100}\}^{\perp}. \tag{26.44}$$

Smallness of the inter-core overlap. Since the two cores are antipodal and strongly localized, the direct overlap of their tails is exponentially small:

$$\mathcal{M}_{\mathrm{ov}} = \mathcal{O}\big(e^{-\pi R_0/\ell_B}\big). \tag{26.45}$$

Consequently, the complete linearization in the antisymmetric class is a small perturbation of the diagonal block "local core + global mode".

Complementary operator. Write

$$\mathcal{L}_{\mathrm{comp}} := \Pi^{(-)}_{\mathrm{comp}} D\mathcal{F}(\Psi_{\mathrm{red}})\big|_{\mathcal{E}^{(-)}_{\mathrm{comp}}}. \tag{26.46}$$

The local coercivity of the bump, the exterior spectral gap, and the smallness of the overlap then imply the existence of a constant $c_{\mathrm{comp}} > 0$ such that

$$\|\mathcal{L}^{-1}_{\mathrm{comp}}\| \leq c^{-1}_{\mathrm{comp}} \tag{26.47}$$

in a neighborhood of the threshold.

Residual of the reduced field. Since Ψ_{red} satisfies the system projected onto $\mathcal{E}_{\mathrm{red}}^{(-)}$, the residual

$$\mathfrak{R}_{\mathrm{red}} := \mathcal{F}[\Psi_{\mathrm{red}}]$$

satisfies

$$\Pi_{\mathrm{red}}^{(-)}\mathfrak{R}_{\mathrm{red}} = 0.$$

Its complementary component is of higher order:

$$\|\Pi_{\mathrm{comp}}^{(-)}\mathfrak{R}_{\mathrm{red}}\| \leq C_{\mathrm{red}}\,|A_{\mathrm{ext}}|^3. \tag{26.48}$$

The terms arising from the direct overlap of the two cores are absorbed into this bound, thanks to (26.45).

Fixed-point equation. Write

$$\mathcal{F}[\Psi_{\mathrm{red}} + \xi] = \mathfrak{R}_{\mathrm{red}} + D\mathcal{F}(\Psi_{\mathrm{red}})[\xi] + \mathcal{R}(\xi),$$

where $\mathcal{R}(\xi) = \mathcal{O}(\|\xi\|^2)$. After projection onto $\mathcal{E}_{\mathrm{comp}}^{(-)}$, one obtains

$$\xi = -\mathcal{L}_{\mathrm{comp}}^{-1}\Pi_{\mathrm{comp}}^{(-)}\mathfrak{R}_{\mathrm{red}} - \mathcal{L}_{\mathrm{comp}}^{-1}\Pi_{\mathrm{comp}}^{(-)}\mathcal{R}(\xi). \tag{26.49}$$

For $|A_{\mathrm{ext}}|$ small enough, this map is contractive on a small ball of $\mathcal{E}_{\mathrm{comp}}^{(-)}$.

Theorem 26.7 (Full local lifting of the antisymmetric branch)**.** *Under the following hypotheses:*

1. *the local coercivity of the bump core given by corollary 26.5 ;*
2. *the strong localization of the two bump cores ;*
3. *the validity of the local reduced branch given by proposition 26.6 ;*
4. *the local regularity of the HSM nonlinearity ;*

there exists a neighborhood of ω_{ext} in which the antisymmetric reduced branch lifts to an effective local branch of solutions of the fundamental HSM equation, of the form

$$\Psi_*(x,t) = \Psi_{\mathrm{red}}(x,t) + \xi_*(x,t), \qquad \xi_* \in \mathcal{E}_{\mathrm{comp}}^{(-)}, \tag{26.50}$$

with the estimate

$$\|\xi_*\| = \mathcal{O}(|A_{\mathrm{ext}}|^3). \tag{26.51}$$

In particular, the antisymmetric two-center branch coupled to Y_{100} defines a ***complete local branch*** *of HSM solutions.*

Proof. The reduced branch provides Ψ_{red}. The corollary 26.5 gives the local coercivity on the bump block, while the exterior spectral gap provides coercivity on the complement of Y_{100}. The smallness of the inter-core overlap therefore preserves the invertibility of $\mathcal{L}_{\mathrm{comp}}$ on $\mathcal{E}_{\mathrm{comp}}^{(-)}$.

The residual of the reduced field satisfies

$$\|\Pi_{\mathrm{comp}}^{(-)}\mathfrak{R}_{\mathrm{red}}\| \leq C_{\mathrm{red}}|A_{\mathrm{ext}}|^3,$$

by (26.48). The nonlinear remainder is quadratic in ξ. The equation projected onto the complement is therefore written as the fixed-point equation (26.49). For $|A_{\mathrm{ext}}|$ small enough, the ball

$$\|\xi\| \leq C|A_{\mathrm{ext}}|^3$$

is stable under this map, and the Lipschitz constant of the quadratic term is strictly less than 1. Banach's fixed-point theorem then gives a unique $\xi_* \in \mathcal{E}^{(-)}_{\text{comp}}$.

By construction,

$$\Pi^{(-)}_{\text{comp}} \mathcal{F}[\Psi_{\text{red}} + \xi_*] = 0,$$

whereas the reduced part is canceled by the reduced branch obtained from proposition 26.6. Thus one obtains a solution of the complete equation in the antisymmetric class under consideration:

$$\Psi_* = \Psi_{\text{red}} + \xi_*.$$

Finally, the stability of the ball directly gives

$$\|\xi_*\| = \mathcal{O}(|A_{\text{ext}}|^3),$$

which is the estimate (26.51). □

Complete local conclusion. In the antisymmetric class with fixed antipodal centers, the possibility of the common-wave 2DOF bi–bump is therefore no longer only a reduced result: it lifts to a **complete local branch** of HSM solutions.

Status of the result. The preceding result closes the local existence proof in the antisymmetric class. It is a local result near the threshold: the nonperturbative analysis far from the threshold and the global stability belong to a regime distinct from the theorem proved here.

Scope. The preceding proposition provides the local reduced system. The lifting theorem 26.7 then establishes that this reduced branch lifts to an effective local branch of solutions of the fundamental HSM equation, with control of the spectral complement and of the reduction residual.

Conclusion. The theoretical point now acquired is the following:

- the antipodal **symmetric** branch with two same-sign bumps cannot be carried by Y_{100} ;
- the **antisymmetric** branch does not suffer from this obstruction ;
- at the level of the reduced system, this antisymmetric branch has a **local supercritical bifurcation** near ω_{ext}.

The common-wave 2DOF bi-bump thus has a complete local closure near the threshold, on the antisymmetric branch compatible with the common mode Y_{100}. This closure is local in amplitude and frequency; it does not claim a global classification of the branches far from the threshold.

Chapter 27

Composite 3DOF solutions: foundation, LO closures, and checks

Acronym convention. In this chapter, we call *CB3* the canonical **tri-lobe** geometry with three centers $\{-\delta, 0, +\delta\}$, used as the support of 3DOF composite reductions. We denote by *Br3* the **3DOF breather-like** branch built on this geometry, and we distinguish in particular:

$$\text{neutral Br3: } \widehat{w} \propto (1, -2, 1), \qquad \text{charged-like Br3: } \widehat{w} \propto (1, -1, 1),$$

according to the monopolar label $n_{\mathrm{EM}}(\widehat{w})$ and the symmetry class considered.

27.1 Prerequisites and notation (leading rank, 3DOF)

Positive mechanical constants: volumic density ρ, tension T, radial stiffness μ_2. Tangential speed $c_t^2 = T/\rho$, local ratio

$$\kappa_\mu := \frac{\mu_2}{T} > 0.$$

Global geodesic radius R_0. Core scale $\lambda = c_t/\omega_0$ with $\lambda \ll R_0$. Canonical core profile $a(\xi) = \operatorname{sech}\xi$, $\xi = s/\lambda$, measure $4\pi\,\xi^2\,d\xi$. Action (one ring):

$$h_{\mathrm{eff}} = \frac{\pi}{2}\rho c_t\,\mathcal{C}_*(u^*, \varepsilon), \qquad \varepsilon = \frac{\lambda}{R_0} \ll 1,$$

where $u^* \in [1.7, 2.1]$ is the MSE matching window (cf. §27.3).

Shape constants (profile $a = \operatorname{sech}$).

$$J_0 = \int_0^\infty 4\pi\xi^2 a^2\,d\xi = \frac{\pi^3}{3}, \qquad J_2 = \int_0^\infty 4\pi\xi^2 (a')^2\,d\xi \simeq 7.6339320582. \tag{27.1}$$

We shall use

$$K_{\mathrm{bump}} = (J_0 + J_2)J_0, \qquad K_0^{(Z)} = J_0^2, \qquad K_2^{(Z)} = J_0 J_2.$$

27.2 Derived constants (3DOF families)

Common LO normalization. In the HSM convention (profile $a(\xi) = \operatorname{sech}\xi$, measure $4\pi\,\xi^2\,d\xi$), check (ii) fixes

$$A_0^2\,\lambda^2 = \frac{2\,h_{\mathrm{eff}}}{\rho\,c_t}\,\frac{1}{J_0 + J_2}, \qquad J_0 = \frac{\pi^3}{3}, \quad J_2 \simeq 7.6339320582, \quad c_t^2 = \frac{T}{\rho}. \tag{27.2}$$

Amplitude convention. Throughout this chapter we keep the same amplitude convention A_0 as in the previous chapters of the 3DOF branch; in particular, the combination $A_0^2\lambda^2$ is the normalized quantity fixed by the action through (27.2).

Moment $J_{6,2}$ (profile sech**, same measure).** We define the ξ^6-weighted moment used in the quartic closures by

$$J_{6,2} \;=\; \int_0^{+\infty} 4\pi\,\xi^6\,a(\xi)^2\,d\xi \quad \text{with } \; a(\xi)=\operatorname{sech}\xi \quad \Rightarrow \quad J_{6,2} \;\approx\; 2.786580061\times 10^2. \tag{27.3}$$

Scale prefactors (common) and channel coefficients. It is useful to distinguish:

- *common prefactors* induced by the action normalization (27.2), which serve as scale checks and must not be confused with the pure channel coefficients;
- *pure channel coefficients*, namely ζ_{3D} for the strong-like channel and Γ_{P} for the Preclusion channel, defined separately by the corresponding channel closures (cf. §27.5.1 for ζ_{3D} and (20.60) for Γ_{P}).

Common prefactors (check). By factoring the natural denominator $J_0+\frac{1}{\kappa_\mu}J_2$, we introduce the common combinations

$$\boxed{\widetilde{\zeta}_{3D} \;=\; \left(A_0^2\lambda^2\right)\,\frac{J_{6,2}}{J_0+\frac{1}{\kappa_\mu}J_2}}, \qquad \boxed{\widetilde{\gamma}_{\mathrm{P}} \;=\; \left(A_0^2\lambda^2\right)\,\frac{J_2}{J_0+\frac{1}{\kappa_\mu}J_2}}. \tag{27.4}$$

Remark: $\widetilde{\zeta}_{3D}$ and $\widetilde{\gamma}_{\mathrm{P}}$ are *scale prefactors*; they must not be confused with the pure channel coefficients. In particular,

$$\widetilde{\zeta}_{3D}\neq\zeta_{3D}, \qquad \widetilde{\gamma}_{\mathrm{P}}\neq\Gamma_{\mathrm{P}}.$$

They serve only as scale checks in the closures, while the purely channel part is defined separately in (27.26).

Status. The prefactors $\widetilde{\zeta}_{3D}$ and $\widetilde{\gamma}_{\mathrm{P}}$ are defined here in symbolic form from the action normalization (27.2), the canonical profile, and the moments J_p. Their role is to separate scale prefactors from pure channel coefficients.

Theoretical traceability. The quantities (27.2)–(27.4) follow from the same profile measure as the moments J_p. They introduce no additional degree of freedom.

27.3 External modulus on $\mathbb{S}^3$ and layer support

External zonal equation (leading rank, isotropic) for the envelope $A(\psi)$:

$$-\frac{T}{R_0^2}\left(\sin^2\psi\,A'(\psi)\right)' \;+\; \frac{T}{R_0^2}S^2\,A(\psi) \;=\; \rho\,\omega_0^2\,\sin^2\psi\,A(\psi), \qquad S^2=\ell(\ell{+}2). \tag{27.5}$$

Langer reduction $y(\psi)=\sqrt{\sin\psi}\,A(\psi)$:

$$y''+\Big[\frac{\rho}{T}\omega_0^2-U(\psi)\Big]y=0, \qquad U(\psi)=\frac{S^2+\frac{1}{4}}{\sin^2\psi}-\frac{1}{4}.$$

At the boundary $\psi_c=u^*\varepsilon$:

$$k_*^2=\frac{\rho}{T}\omega_0^2-U(\psi_c), \qquad g_*\simeq\frac{1}{u^*\,\varepsilon}, \qquad \varepsilon=\lambda/R_0. \tag{27.6}$$

Layer support (Robin condition):

$$\boxed{\chi = \frac{(2k_*)^2}{(2k_*)^2 + g_*^2} \in (0,1)}, \qquad \widehat{\beta}_0 \equiv \chi. \tag{27.7}$$

Universal MSE:

$$\boxed{y\cot y = 1 - u^* \tanh u^*, \quad y = (n{+}1)u^*\,\varepsilon, \quad u^* \in [1.7, 2.1].} \tag{27.8}$$

The strict decrease of $y\cot y$ between its poles ensures at most one solution per band.

Regularized LO closure of the 3DOF cutoff. The MSE window

$$u_* \in [1.7, 2.1]$$

defines the admissible domain of core–sheath matching at LO. It represents the matching window in which the boundary layer remains regular and in which the thin reduction used in (27.8) applies. In this window, the cutoff u_* is not fixed by a unique root of the local equation: it must be chosen by a matching-gauge closure.

The central closure used here is the multiplicative minimax of the reduced core constant. This choice is adapted to a positive constant entering as a multiplicative factor: it balances the maximum relative error at the two edges of the admissible window.

We write

$$k := \frac{\psi_{\mathrm{cut}}}{\psi_c} > 1, \qquad \psi_{\mathrm{cut}} = k\,\psi_c,$$

where ψ_c is the core boundary and ψ_{cut} the outer boundary of the matching annulus. The reduced radial problem associated with the annular capacity is the harmonic problem

$$\big(\sin^2\psi\, W'(\psi)\big)' = 0, \qquad W(\psi_c) = 1, \qquad W(\psi_{\mathrm{cut}}) = 0.$$

Its closed-form solution is

$$\boxed{W(\psi) = \frac{\cot\psi - \cot\psi_{\mathrm{cut}}}{\cot\psi_c - \cot\psi_{\mathrm{cut}}}.} \tag{27.9}$$

The associated radial flux is therefore controlled by the DtN jump

$$\Lambda_0(\psi_c, \psi_{\mathrm{cut}}) = \frac{1}{\cot\psi_c - \cot\psi_{\mathrm{cut}}}.$$

The reduced core constant is obtained by relating this flux to the boundary normalization factor of the annulus. We write

$$C_\ell(0, 3\mathrm{D}) = \frac{\cot\psi_c - \cot\psi_{\mathrm{cut}}}{\mathcal{B}(\psi_c, \psi_{\mathrm{cut}})^2},$$

where $\mathcal{B}(\psi_c, \psi_{\mathrm{cut}})$ is the radial normalization factor associated with the harmonic profile W. In the thin regime

$$\psi_c \ll 1, \qquad \psi_{\mathrm{cut}} = k\psi_c, \qquad k > 1,$$

the elementary expansions of $\cot\psi$ give

$$\cot\psi_c - \cot(k\psi_c) = \frac{k-1}{k\,\psi_c} + O(\psi_c),$$

and the radial normalization factor expands as

$$\mathcal{B}(\psi_c, k\psi_c) = \frac{(k^3 - 3k + 2)}{6k}\,\psi_c^2 + O(\psi_c^4).$$

It follows that

$$C_\ell(0,3\mathrm{D}) = \psi_c^{-5}\frac{36\,k(k-1)}{(k^3-3k+2)^2} + O(\psi_c^{-3}).$$

The reduced radial-capacity constant is therefore

$$\boxed{\mathcal{C}_{3D}^{(0)}(k) = \frac{36\,k(k-1)}{(k^3-3k+2)^2}.} \tag{27.10}$$

Canonical profile normalization. The function $\mathcal{C}_{3D}^{(0)}(k)$ encodes the reduced radial capacity. It remains to fix the normalization of the canonical transverse profile. The symmetric reference point is

$$k = 2.$$

The canonical core constant associated with the normalized transverse profile is the one-dimensional integral of the sech core:

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2 u\, du = 2.$$

We therefore impose

$$\boxed{\mathsf{C}_{\mathrm{prof}}^{(3)}(2) = 2.} \tag{27.11}$$

Now

$$\mathcal{C}_{3D}^{(0)}(2) = \frac{36\cdot 2\cdot(2-1)}{(2^3-3\cdot 2+2)^2} = \frac{72}{16} = \frac{9}{2}.$$

We therefore define

$$\boxed{\Xi_{\mathrm{prof}} := \frac{\mathsf{C}_{\mathrm{prof}}^{(3)}(2)}{\mathcal{C}_{3D}^{(0)}(2)} = \frac{2}{9/2} = \frac{4}{9}.} \tag{27.12}$$

The transverse profile constant actually compared on the window is then

$$\boxed{\mathsf{C}_{\mathrm{prof}}^{(3)}(k) = \Xi_{\mathrm{prof}}\,\mathcal{C}_{3D}^{(0)}(k).} \tag{27.13}$$

Lemma 27.1 (Multiplicative metric for positive constants)**.** *For a positive constant with no natural additive scale, the comparison criterion invariant under a multiplicative change of unit is the logarithmic criterion*

$$d(C_1,C_2) = \left|\log\frac{C_1}{C_2}\right|.$$

Proof. The constant $\mathsf{C}_{\mathrm{prof}}^{(3)}(k)$ is positive and enters as a multiplicative core factor. An admissible comparison must therefore not depend on a global renormalization $C \mapsto aC$, with $a > 0$. The ratio C_1/C_2 is invariant under this transformation, whereas the difference $C_1 - C_2$ is not. The symmetry between overestimation and underestimation also imposes

$$d(C_1,C_2) = d(C_2,C_1),$$

which selects the absolute value of the logarithm of the ratio:

$$d(C_1,C_2) = \left|\log\frac{C_1}{C_2}\right|.$$

□

Proposition 27.2 (Regularized LO central cutoff)**.** *The central cutoff $u_*^\star$ is defined as the point of the admissible window $[1.7, 2.1]$ that minimizes the maximum multiplicative deviation from the two endpoints:*

$$u_*^\star = \arg\min_{u\in[1.7,2.1]} \max\left(\left|\log\frac{\mathsf{C}^{(3)}_{\mathrm{prof}}(u)}{\mathsf{C}^{(3)}_{\mathrm{prof}}(1.7)}\right|, \left|\log\frac{\mathsf{C}^{(3)}_{\mathrm{prof}}(u)}{\mathsf{C}^{(3)}_{\mathrm{prof}}(2.1)}\right|\right). \tag{27.14}$$

It therefore satisfies

$$\boxed{\log \mathsf{C}^{(3)}_{\mathrm{prof}}(u_*^\star) = \frac{1}{2}\left[\log \mathsf{C}^{(3)}_{\mathrm{prof}}(1.7) + \log \mathsf{C}^{(3)}_{\mathrm{prof}}(2.1)\right], \qquad u_*^\star \in [1.7, 2.1].} \tag{27.15}$$

Proof. On the admissible window, the function

$$\mathcal{C}^{(0)}_{3D}(k) = \frac{36\,k(k-1)}{(k^3-3k+2)^2}$$

is strictly decreasing. Indeed, for $k > 1$,

$$k^3 - 3k + 2 = (k-1)^2(k+2),$$

hence

$$\mathcal{C}^{(0)}_{3D}(k) = \frac{36k}{(k-1)^3(k+2)^2},$$

and

$$\frac{d}{dk}\log \mathcal{C}^{(0)}_{3D}(k) = \frac{1}{k} - \frac{3}{k-1} - \frac{2}{k+2} < 0.$$

Since Ξ_{prof} is constant on the window, the monotonicity also holds for $\mathsf{C}^{(3)}_{\mathrm{prof}}(k)$.

The multiplicative minimax consists in minimizing the larger of the two logarithmic deviations from the endpoints. For a positive strictly monotone function, this maximum is minimized when the two deviations are equal:

$$\log \mathsf{C}^{(3)}_{\mathrm{prof}}(1.7) - \log \mathsf{C}^{(3)}_{\mathrm{prof}}(u_*^\star) = \log \mathsf{C}^{(3)}_{\mathrm{prof}}(u_*^\star) - \log \mathsf{C}^{(3)}_{\mathrm{prof}}(2.1).$$

We therefore obtain

$$\log \mathsf{C}^{(3)}_{\mathrm{prof}}(u_*^\star) = \frac{1}{2}\left[\log \mathsf{C}^{(3)}_{\mathrm{prof}}(1.7) + \log \mathsf{C}^{(3)}_{\mathrm{prof}}(2.1)\right].$$

Existence and uniqueness follow from the continuity and strict monotonicity of $\mathsf{C}^{(3)}_{\mathrm{prof}}$ on $[1.7, 2.1]$. □

The scalar solution of (27.15) gives

$$\boxed{u_*^\star = 1.8782600799\ldots} \tag{27.16}$$

The high-overlap upper bound for a tri-lobe geometry is therefore

$$\boxed{\delta_{\max} = \frac{u_*^\star}{2} = 0.93913003995\ldots.} \tag{27.17}$$

Stability of the center choice. The multiplicative minimax criterion fixes the central value used in the LO calculations. Alternative centers of the same window give nearby values:

center criterion	u_*
multiplicative minimax	1.8782600799
harmonic midpoint of $[1.7, 2.1]$	1.8789473684
geometric midpoint of $[1.7, 2.1]$	1.8894443628
arithmetic midpoint of $[1.7, 2.1]$	1.9000000000

The relative dispersion remains of order one percent. Quantities that consume $u_*^\star$, in particular $\delta_{\max} = u_*^\star/2$, must therefore be read as LO central values associated with this center closure.

27.4 Core cutoff, EBK, and normalizations

Time average $\langle\cos^2\rangle = \frac{1}{2}$. Energy–frequency rule (one ring): $E_0 = h_{\text{eff}}\,\omega_{\text{br}}$ with, in conical 3DOF,

$$\omega_{\text{br}} = \sigma\,\omega_0, \qquad \sigma = \sin^2\alpha \in \Big\{\frac{1}{3}, \frac{2}{3}\Big\}.$$

The action normalization consumed here takes up the canonical EBK closure of the core, cf. (20.54).

$$\boxed{A_0^2\,\lambda^2 = \frac{2\,h_{\text{eff}}}{\rho c_t}\,\frac{1}{J_0 + J_2}, \qquad \lambda = \frac{c_t}{\omega_0}.} \tag{27.18}$$

At LO, λ (and therefore ω_0) is determined by the core+χ closure.

27.5 Weighted multi-lobe 3DOF cores: general framework (LO)

Status of the section. This section fixes the general combinatorial framework of the multi-lobe 3DOF motifs: positions, weights, shape sums, pair-by-pair quartic reduction, and discrete labels. The channel closures written below are generic operational forms. Specialized sectors may introduce their own reduced response parameter. In particular, in the Z-like 3DOF branch, the active closure uses the reduced parameter $\eta = \eta_\mu^{(Z)}(\lambda_*)$ defined in (28.98), and not a direct reading of the symbol κ_μ as a constant reduced parameter.

Positions and weights (discrete motif + scale normalization). We consider a quasi-flat core (profile $a(\xi) = \operatorname{sech}\xi$, $\xi = s/\lambda$) composed of N lobes placed at positions $\{\xi_i\}_{i=1}^N$ (for example $\{-\delta, 0, +\delta\}$) and described by a *discrete motif* $\widehat{w} = (\widehat{w}_1, \ldots, \widehat{w}_N) \in \mathbb{Z}^N$. *Remark.* In a fixed N–lobe geometry, in practice one assumes $\widehat{w}_i \neq 0$; the case $\widehat{w}_i = 0$ corresponds to the *reduction* toward an $(N-1)$-lobe problem, treated separately.

The internal field is written

$$A_{\text{in}}(\xi) \;\propto\; \sum_{i=1}^{N} w_i\,a(\xi - \xi_i), \qquad a(\xi) = \operatorname{sech}\xi,$$

where $w \in \mathbb{R}^N$ is the vector of *real* weights effectively injected into the functionals. The global scale can be fixed by a normalization of the type $S_0[w;\delta] = 1$. However, in the canonical outputs used later in this chapter, we work directly with the *primitive motifs* $\widehat{w}$, and the forms $S_0, S_2, S_{4,\text{pair}}$ that appear in the closures are the raw forms associated with these motifs. Thus, two collinear motifs (e.g. $(1,-1,1)$ and $(2,-2,2)$) describe the same internal direction and do not define two distinct states.

Cross-overlaps. The cross-overlaps $J_p^{\times}(\Delta)$ are defined by the 3D translational convolutions in (28.33). This convention is the one used in all the multi-lobe closures below.

Unit quartic moment. We also set

$$\boxed{J_4 := J_4^{\times}(0) = \int_0^{\infty} 4\pi\, \xi^2\, a(\xi)^4\, d\xi, \qquad a(\xi) = \operatorname{sech} \xi.} \tag{27.19}$$

Shape sums (exact). Writing $\Delta_{ij} := |\xi_i - \xi_j|$, one has

$$\boxed{\begin{aligned} S_0[\widehat{w};\delta] &= J_0 \sum_i \widehat{w}_i^2 \;+\; 2\sum_{i<j} \widehat{w}_i \widehat{w}_j\, J_0^{\times}(\Delta_{ij}), \\ S_2[\widehat{w};\delta] &= J_2 \sum_i \widehat{w}_i^2 \;+\; 2\sum_{i<j} \widehat{w}_i \widehat{w}_j\, J_2^{\times}(\Delta_{ij}), \end{aligned} \qquad \Delta_{ij} := |\xi_i - \xi_j|\ .} \tag{27.20}$$

Pair-by-pair quartic reduction. For the Preclusion channel, the LO form used is

$$\boxed{S_{4,\mathrm{pair}}[\widehat{w};\delta] := 6 \sum_{i<j} (\widehat{w}_i \widehat{w}_j)^2\, J_4^{\times}(\Delta_{ij}).} \tag{27.21}$$

Remark on the pair-by-pair reduction. The form $S_{4,\mathrm{pair}}$ does not claim to represent the full quartic expansion of $\left(\sum_i \widehat{w}_i a_i\right)^4$. It retains only the *pair-by-pair* part used at leading rank to model the local-overlap cost of the Preclusion channel. The purely self-local terms and the higher-order quartic structures in overlap are either absorbed into the unit normalization, or treated as subdominant at LO in the composite reduction retained here.

EM charge (LO): discrete label. At leading rank (linear coupling), the *monopole label* is carried by the unnormalized integer motif $\bar{w}$, before shape normalization:

$$\boxed{n_{\mathrm{EM}}(\bar{w}) := \sum_{i=1}^{N} \bar{w}_i \in \mathbb{Z}.} \tag{27.22}$$

The branch is neutral if $n_{\mathrm{EM}} = 0$, and charged if $n_{\mathrm{EM}} \neq 0$. When one then works with a normalized representative $w(\delta)$ or $\widehat{w}(\delta)$, the discrete label remains that of the unnormalized integer motif $\bar{w}$.

In strong overlap, the effective amplitude of the monopole may be reduced by a form factor depending on $\widehat{w}$ and on δ, as well as, in conical 3DOF, by the projection $\sigma = \sin^2 \alpha$. The label n_{EM} remains a discrete channel marker; it does not imply direct additivity of the effective source (cf. §28.6.7).

Repulsive Preclusion (short-range) – pair-by-pair quartic form. The Preclusion channel is modeled at LO as a local cost of quartic overlap of the composite field. The reduced repulsive contribution is then written

$$\boxed{\beta^{(\mathsf{P})}[\widehat{w};\delta] = \kappa_\mu\, \Gamma_{\mathrm{P}}\, \frac{S_{4,\mathrm{pair}}[\widehat{w};\delta]}{\Big(S_0[\widehat{w};\delta] + \kappa_\mu^{-1} S_2[\widehat{w};\delta]\Big)^2}, \qquad \Gamma_{\mathrm{P}} = \frac{4J_4}{J_0^2}.} \tag{27.23}$$

Thus, the Preclusion channel involves no constant specific to the composite family: its global strength is carried by the universal constant Γ_{P} of the channel, while the dependence on internal geometry is entirely contained in $S_{4,\mathrm{pair}}[\widehat{w};\delta]$.

Attractive strong-like (quartic, pair-by-pair) – weight form. Similarly, the attractive quartic contribution is written

$$\boxed{\beta^{(\mathrm{F})}[\widehat{w};\delta] = \zeta_{3D}\,\frac{\displaystyle\sum_{i<j} c_{ij}^{(\mathrm{F})}(\widehat{w})\,J_4^{\times}(\Delta_{ij})}{S_0[\widehat{w};\delta]\;+\;\dfrac{1}{\kappa_\mu}S_2[\widehat{w};\delta]}\,,\qquad \zeta_{3D}>0\,.} \tag{27.24}$$

Status of the combinatorial coefficients of the strong-like channel. In this framework section, the coefficients $c_{ij}^{(\mathrm{F})}(\widehat{w})$ are left symbolic: they are fixed explicitly family by family in the specialization subsections (Br-like, Z-like, etc.). The present framework assumes only their positivity and their combinatorial dependence on the primitive motif $\widehat{w}$.

Effective rule and closure. We set $\beta^{(\mathrm{eff})} = \beta^{(\mathrm{F})} - \beta^{(\mathrm{P})}$ (when both channels coexist). The LO closure is then written in the canonical form

$$\boxed{\lambda_*^4[\widehat{w}] = \frac{2\,h_{\mathrm{eff}}\sqrt{\rho\,T}}{\kappa_\mu\Big(K[\widehat{w}] - \beta^{(\mathrm{eff})}[\widehat{w};\delta]K_0[\widehat{w}]\Big)\;-\;\beta^{(\mathrm{eff})}[\widehat{w};\delta]K_2[\widehat{w}]}\,,\quad \begin{cases} K[\widehat{w}] = (S_0[\widehat{w}]+S_2[\widehat{w}])\,S_0[\widehat{w}], \\ K_0[\widehat{w}] = (S_0[\widehat{w}])^2, \qquad K_2[\widehat{w}] = S_0[\widehat{w}]S_2[\widehat{w}], \end{cases}} \tag{27.25}$$

where the forms S_0, S_2 are evaluated on the primitive motif $\widehat{w}$ at fixed geometry δ. The projected branch frequency is $\omega_{\mathrm{br}} = \sigma\,\omega_0 = \sigma\,c_t/\lambda_*$ (conical 3DOF), and the reduced mechanical mass associated with an EBK level q is written

$$m(q,\sigma) = \frac{E_0}{c_t^2} = \frac{q\,h_{\mathrm{eff}}\,\omega_{\mathrm{br}}}{c_t^2} = \frac{q\,h_{\mathrm{eff}}\,\sigma}{c_t\,\lambda_*}.$$

In the present LO part, the EBK filter imposes $q_{\mathrm{adm}} = 1$ (cf. §28.6.4 for the Z-like case).

Proposition 27.3 (LO value of the boundary constant $\widehat{\beta}_0$ (Z-like 3DOF))**.** *In the thin window $\psi_c = u^*\varepsilon$ with $\varepsilon = \lambda/R_0 \ll 1$, one has $g_* \simeq (u^*\varepsilon)^{-1} \gg 1$ and $k_* = \mathcal{O}(1)$. It follows that*

$$\widehat{\beta}_0 \equiv \chi = \frac{(2k_*)^2}{(2k_*)^2 + g_*^2} = \mathcal{O}((u^*\varepsilon)^2) \xrightarrow[\varepsilon\to 0]{} 0.$$

At leading order (LO), we therefore adopt

$$\boxed{\widehat{\beta}_0 = 0}.$$

Remark. *This LO value concerns the boundary factor of the 3DOF reduction in the thin window $\psi_c = u^*\varepsilon$, and must not be confused with the homonymous quantity introduced in the 2DOF Z-like reduction, which belongs to another effective closure.*

27.5.1 Channel constants used in pure theory

Reminder. The coefficient ζ_{3D} of the strong-like channel is a pure number fixed by the HSM normalization of the attractive 3DOF channel. In contrast, the Preclusion channel is no longer parameterized here by a proper constant ζ_{P} specific to the composite 3DOF families: it is written directly in terms of the universal constant Γ_{P} of the Preclusion channel, defined in the corresponding chapter.

Reduced constants consumed by the closures. The 3DOF closures consume the attractive coefficient ζ_{3D}, derived in §27.5.2, and the universal constant of the Preclusion channel. We group these two constants in the form

$$\boxed{\zeta_{3D} = \frac{15}{8}, \qquad \Gamma_{\mathrm{P}} = \frac{4J_4}{J_0^2}.} \tag{27.26}$$

Consequence. In the 3DOF closures of this chapter, the attractive strong-like channel is carried by ζ_{3D}, while the Preclusion channel is written in the form

$$\beta^{(\mathsf{P})}[\widehat{w};\delta] = \kappa_\mu\,\Gamma_{\mathrm{P}}\,\frac{S_{4,\mathrm{pair}}[\widehat{w};\delta]}{\big(S_0[\widehat{w};\delta] + \kappa_\mu^{-1} S_2[\widehat{w};\delta]\big)^2},$$

without introducing an additional constant specific to the composite families. The present section fixes no dimensional identification: it defines only the reduced coefficients needed for the theoretical closures.

27.5.2 Derivation of the coefficient ζ_{3D}

Status of the derivation. The present subsection fixes the value of the *pure channel coefficient* ζ_{3D} from the reference family retained for the attractive quartic 3DOF closure. The notation $\beta_{\mathrm{br}}^{(\mathrm{th})}$ denotes here the theoretical attractive closure of this 3DOF breather-like reference family.

Starting form. We consider the attractive quartic closure of the 3DOF channel in the form

$$\beta_{\mathrm{br}}^{(\mathrm{th})}(\delta,\varepsilon;\kappa_\mu) = \frac{3}{1 + C_{\tan}^{J} + \chi^{J}/2}\,\frac{J_4^{\times}(\delta)}{S_0(\delta,\varepsilon) + \frac{1}{\kappa_\mu} S_2(\delta,\varepsilon)}. \tag{27.27}$$

The factor 3 is the number of unordered pairs in a tri-lobe family:

$$\#\{(i,j)\,:\,1 \le i < j \le 3\} = \binom{3}{2} = 3.$$

It is therefore the LO combinatorial factor of pair–to–pair overlap associated with the 3DOF reference family used to define the attractive channel in the present HSM normalization.

By definition, the pure channel coefficient ζ_{3D} is the prefactor multiplying the quartic overlap in (27.27), namely

$$\boxed{\zeta_{3D} = \frac{3}{1 + C_{\tan}^{J} + \chi^{J}/2}.} \tag{27.28}$$

Isotropic decomposition of the Jacobi channel. The Jacobi channel that corrects the attractive quartic closure is carried by the tangential symmetric trace-free quadrupolar part. In three dimensions, the space of symmetric trace-free tensors has dimension

$$\dim \mathrm{STF}_2(\mathbb{R}^3) = 5.$$

We decompose this space isotropically into a direction aligned with the attractive matching channel and its orthogonal complement:

$$\mathrm{STF}_2(\mathbb{R}^3) = \mathcal{J}_{\parallel} \oplus \mathcal{J}_{\perp}, \qquad \dim \mathcal{J}_{\parallel} = 1, \qquad \dim \mathcal{J}_{\perp} = 4.$$

Lemma 27.4 (Isotropic weights of the Jacobi channel)**.** *With the isotropic trace normalization, the geometric weights associated with the two Jacobi subspaces are*

$$\boxed{C_{\tan}^{J} = \frac{1}{5}, \qquad \chi^{J} = \frac{4}{5}.} \tag{27.29}$$

Proof. The isotropic normalization on $\mathrm{STF}_2(\mathbb{R}^3)$ assigns to every subspace a weight equal to its rank divided by the total rank. Since

$$\dim \mathcal{J}_{\parallel} = 1, \qquad \dim \mathcal{J}_{\perp} = 4, \qquad \dim \mathrm{STF}_2(\mathbb{R}^3) = 5,$$

we obtain

$$C_{\tan}^{J} = \frac{\dim \mathcal{J}_{\parallel}}{\dim \mathrm{STF}_2(\mathbb{R}^3)} = \frac{1}{5}, \qquad \chi^{J} = \frac{\dim \mathcal{J}_{\perp}}{\dim \mathrm{STF}_2(\mathbb{R}^3)} = \frac{4}{5}.$$

□

Lemma 27.5 (Half-weight of the transverse complement)**.** *In the attractive quartic 3DOF closure, the transverse complement $\mathcal{J}_{\perp}$ enters with the half-weight*

$$\boxed{\mathcal{W}_{\perp}^{J} = \frac{1}{2}\chi^{J}.} \tag{27.30}$$

Proof. The aligned channel $\mathcal{J}_{\parallel}$ is collinear with the attractive pair–to–pair matching. Its contribution is therefore read only once in the normalized quartic projection and carries the full weight $C_{\tan}^{J}$.

The complement $\mathcal{J}_{\perp}$ is not aligned with the attractive matching. It enters only through its symmetrized even component in the quartic overlap. For an unordered pair (i, j), the two transverse readings

$$i \to j, \qquad j \to i$$

represent the same geometric overlap. The non-oriented projection is therefore the symmetrized average

$$\Pi_{\mathrm{pair}}^{\mathrm{sym}} = \frac{1}{2}(\Pi_{i\to j} + \Pi_{j\to i}).$$

The total transverse weight χ^{J} is thus shared by this symmetrized projection, which gives

$$\mathcal{W}_{\perp}^{J} = \frac{1}{2}\chi^{J}.$$

This factor $1/2$ does not come from the dimension of $\mathrm{STF}_2(\mathbb{R}^3)$, already counted in $\chi^{J} = 4/5$; it comes from the non-oriented symmetrization of the quartic pair–to–pair overlap. □

We therefore obtain

$$\boxed{C_{\tan}^{J} = \frac{1}{5}, \qquad \chi^{J} = \frac{4}{5}.} \tag{27.31}$$

The denominator of the channel coefficient becomes

$$1 + C_{\tan}^{J} + \frac{1}{2}\chi^{J} = 1 + \frac{1}{5} + \frac{1}{2}\cdot\frac{4}{5} = 1 + \frac{1}{5} + \frac{2}{5} = \frac{8}{5}. \tag{27.32}$$

Final value. Substituting (27.32) into (27.28), we obtain

$$\boxed{\zeta_{3D} = \frac{3}{8/5} = \frac{15}{8} = 1.875.} \tag{27.33}$$

Consequence. The attractive quartic closure is then written

$$\boxed{\beta_{\mathrm{br}}^{(\mathrm{th})}(\delta,\varepsilon;\kappa_\mu)=\zeta_{3D}\,\frac{J_4^{\times}(\delta)}{S_0(\delta,\varepsilon)+\frac{1}{\kappa_\mu}S_2(\delta,\varepsilon)},\qquad \zeta_{3D}=1.875.} \tag{27.34}$$

27.5.3 Tri-lobe 3DOF (CB3): class reduction and discrete selection of motifs $\widehat{w}$

Tri-lobe geometry. We consider the canonical configuration $\{-\delta,0,+\delta\}$ and an internal field

$$A_{\mathrm{in}}(\xi)\ \propto\ \sum_{i=1}^{3}\widehat{w}_i\,a(\xi-\xi_i),\qquad a(\xi)=\operatorname{sech}\xi,\qquad (\xi_1,\xi_2,\xi_3)=(-\delta,0,+\delta),$$

where $\widehat{w}\in\mathbb{Z}^3$ denotes here the primitive discrete motif effectively used for the canonical closure of the 3DOF tri-lobe class.

Principle: selection by minimization (not an arbitrary choice). In the present chapter, δ is the continuous variable that controls the *overlap* (equilibrium), whereas $\widehat{w}$ indexes a *discrete mode*. As for the breather branch, $\widehat{w}$ is not set arbitrarily: it is selected by minimization of the denominator *within an admissible class*. In some classes (notably neutral + symmetry), the constraints are sufficient to reduce the class to a single primitive motif, which then appears as a "locking".

Monopole neutrality (minimal version). The monopole label is $n_{\mathrm{EM}}(\widehat{w})=\sum_i\widehat{w}_i$ (cf. (27.22)). In the *neutral* sector (neutral Br3), we therefore impose

$$\boxed{\widehat{w}_1+\widehat{w}_2+\widehat{w}_3=0} \tag{27.35}$$

(channel constraint, independent of δ).

Tri-lobe symmetry. The configuration $\{-\delta,0,+\delta\}$ is symmetric under the reflection $\xi\mapsto-\xi$. A canonical state of the symmetric branch imposes

$$\boxed{\widehat{w}_1=\widehat{w}_3} \tag{27.36}$$

(the outer lobes play the same role).

Primitive integer solution: neutral Br3. Combining (27.35) and (27.36), we obtain $\widehat{w}_2=-2\widehat{w}_1$. The *primitive* integer triplet (with no common factor) is therefore

$$\boxed{\widehat{w}_{\mathrm{Br3}}^{(0)}=(1,-2,1)} \tag{27.37}$$

and any other choice in the same class is a global integer multiple, $\widehat{w}=k(1,-2,1)$. These multiples describe the *same internal direction* and are therefore redundant in the canonical selection.

Conclusion. For the neutral Br3 branch in the symmetric tri-lobe CB3 class, the admissible class reduces to $\widehat{w}\propto(1,-2,1)$: the only remaining equilibrium parameter is δ, then determined by minimization of the denominator on the admissible window $[\delta_{\min},\delta_{\max}]$.

Chapter 28

3DOF bump composites (internal channel s): resolved CB3 and breather branches

28.1 Purpose, LO assumptions, whiteness, and filters

Purpose. We study the existence of bound states formed from $N \in \{2, 3\}$ *3DOF bump* objects (localized solutions of the 3DOF sector), coupled by the *strong-like* channel (Chap. 18) and subject to a universal core repulsion of Preclusion type (Preclusion chapter). No electromagnetic or gravitational coupling is considered in this chapter.

LO assumptions. Thin-tube regime $\lambda/R_0 \ll 1$, quasi-flat cap, relative rest of the constituents. Each 3DOF constituent carries an EBK level

$$q_i \in \{q_1, q_2, q_3\},$$

where $q_1 = 1$ denotes the fundamental level. The resolved CB3 branch therefore does not assume that all constituents are at the fundamental level: it constructs a color–white state from three 3DOF bumps that may belong to different families.

The HSM filters (MSE/sheath, Gram, EBK, scale hierarchy) bound the admissible set without introducing any new free constant. The dynamical radial contributions controlled by κ_r belong to the radial NLO and are not included in the LO CB3 closure.

Internal naming convention. In this chapter, the letters

$$\mathsf{u}, \qquad \mathsf{d}, \qquad \mathsf{Q}_\uparrow$$

are used as internal HSM labels for 3DOF bump families. They do not denote Standard Model flavors. In the text, we refer to the u-like, d-like, and $\mathsf{Q}_\uparrow$-like branches. In subscripts, to keep formulas readable, we use the bare letters

$$\mathsf{u}, \qquad \mathsf{d}, \qquad \mathsf{Q}_\uparrow.$$

Thus, a subscript such as

$$\mathsf{ud}\mathsf{Q}_\uparrow$$

denotes an internal HSM motif of CB3 type, and not an external phenomenological identification.

Color channel and whiteness condition. Each 3DOF bump carries a strong-like charge I_{s} defined via the projection on the channel projector $\boldsymbol{\pi}^{(\mathsf{col})}$ (Chap. 18). At LO rank, a net attractive bond requires a *white* combination in this channel: without whiteness, the strong-like contribution is zero or non-attractive, and no bound state is expected. Whiteness is therefore an **existence filter**.

Two physical regimes (without assuming experimental identification). We distinguish two branches, according to the degree of overlap of the cores:

- **Resolved CB3 branch.** The bumps remain identifiable. The main internal geometric variable is a reduced separation δ. For CB2, the pair-interaction writing provides the LO closure. For CB3, the mass closure is collective: the strong-like channel is evaluated on the total white source, in accordance with (21.32).
- **Breather branch (strong overlap).** The lobes are no longer independent constituents: the composite is a *unitary multi-lobe solution*. In addition to δ, there is an internal *shape* degree of freedom (mixing) denoted $\widehat{w}$, which must be **determined** (and not postulated).

Equilibrium principle. In the resolved branch, the existence of a white composite results from the collective strong-like matching, the whiteness constraint, and the geometric filtering of admissible separations. Preclusion enters as a core cost when the profiles enter strong overlap; it does not replace the collective closure of the white source. An admissible solution corresponds to a stationary configuration in the filtered interval

$$\delta \in [\delta_{\min}, \delta_{\max}].$$

.

28.2 Geometric stationarity energy of the resolved branches

Purpose. Before the mass closure of a white composite, one must determine the admissible geometric configurations. At this level, the resolved centers are described by their positions ξ_i, their effective strong-like sources, and the Preclusion core cost. This geometric energy is used to select the stationary separations; for CB3, it is not the final mass closure.

LO geometric form. For a set of N localized centers, the geometric stationarity energy is written at LO rank in the form

$$E_N^{\mathrm{geom}} = \frac{1}{2} \sum_{i \neq j} I_{\mathsf{s}}^{(i)} I_{\mathsf{s}}^{(j)} \, \mathcal{G}_{\mathsf{s}}(\xi_i, \xi_j) + \sum_{i<j} V_{\mathsf{P}}^{(\lambda)}(|\xi_i - \xi_j|), \tag{28.1}$$

where $\mathcal{G}_{\mathsf{s}}$ is the strong-like Green kernel normalized in the HSM conventions, and $\mathcal{V}_{\mathsf{P}}$ is the short-range cost of the Preclusion channel.

Geometric reduction. In the canonical CB2 and CB3 geometries (Sec. 28.3), all inter-center distances are expressed as functions of a single parameter δ, and the stationarity energy reduces to a function

$$E_N^{\mathrm{geom}}(\delta) = \mathcal{A}_N \, \mathcal{G}_{\mathsf{s}}(\delta) + \mathcal{B}_N \, \mathcal{G}_{\mathsf{s}}(2\delta) + V_{\mathsf{P}}^{(\lambda,N)}(\delta), \tag{28.2}$$

where $\mathcal{A}_N, \mathcal{B}_N$ are fixed by the geometry and by the retained white combination.

Separation between stationarity and mass closure. For CB2, the pair-interaction description may be sufficient to close the resolved branch at LO rank. For CB3, geometric stationarity is not sufficient: the mass closure is collective and must be constructed from the total white source defined in (21.32). The value δ_* selected by stationarity is therefore subsequently consumed by the color–white closure of CB3.

Definition of the stationary separation. An admissible resolved configuration is defined by a stationary separation in the filtered interval:

$$\boxed{\delta_* = \arg \min_{\delta \in [\delta_{\min}, \delta_{\max}]} E_N^{\mathrm{geom}}(\delta).} \tag{28.3}$$

The resolved or fused nature of the configuration is then controlled by the degree of overlap of the profiles.

28.3 CB2 and CB3 geometries: reduction to one variable

28.3.1 CB2 configuration ($N = 2$)

Geometry. Two constituents at the positions $\xi_1 = -\delta$ and $\xi_2 = +\delta$:

$$d_{12} = 2\delta.$$

Effective energy.

$$E_2(\delta) = I_{\mathsf{s}}^{(1)} I_{\mathsf{s}}^{(2)}\, \mathcal{G}_{\mathsf{s}}(2\delta) + \mathcal{V}_{\mathsf{P}}(2\delta). \tag{28.4}$$

28.3.2 CB3 configuration ($N = 3$)

Geometry. Three constituents at the positions

$$\xi_1 = -\delta, \qquad \xi_2 = 0, \qquad \xi_3 = +\delta,$$

therefore

$$d_{12} = d_{23} = \delta, \qquad d_{13} = 2\delta.$$

Effective energy.

$$E_3(\delta) = \sum_{i<j} I_{\mathsf{s}}^{(i)} I_{\mathsf{s}}^{(j)}\, \mathcal{G}_{\mathsf{s}}(d_{ij}) + \sum_{i<j} V_{\mathsf{P}}^{(\lambda)}(d_{ij}). \tag{28.5}$$

White symmetric case (useful form). In a white symmetric combination, the two pairs at distance δ carry the same strong-like coefficient, while the pair at distance 2δ carries a coefficient fixed by the same white projection. We may therefore write

$$E_3(\delta) = \mathcal{A}_3\, \mathcal{G}_{\mathsf{s}}(\delta) + \mathcal{B}_3\, \mathcal{G}_{\mathsf{s}}(2\delta) + 2\, V_{\mathsf{P}}^{(\lambda)}(\delta) + V_{\mathsf{P}}^{(\lambda)}(2\delta), \tag{28.6}$$

where $\mathcal{A}_3$ and $\mathcal{B}_3$ are entirely determined by the retained white projection, with no new free constant.

28.4 Resolved CB3 branch: optimum in δ and mass

Pair geometric stationarity. The pair forms inherited from the channel chapters provide a stationarity condition for the reduced separation δ. We write

$$V_{\mathsf{s}}(d) = -\mathcal{C}_{\mathsf{s}}\,\frac{I_{\mathsf{s}}^{(1)} I_{\mathsf{s}}^{(2)}}{4\pi}\,\frac{e^{-d/\lambda_{\mathsf{s}}}}{d}, \qquad V_{\mathsf{P}}^{(\lambda)}(d) = +\frac{C_{\mathrm{P}}(\lambda)}{\lambda}\,\frac{e^{-d/\widehat{\xi}_{\mathrm{P}}}}{d}, \qquad \widehat{\xi}_{\mathrm{P}} := \frac{\xi_{\mathrm{P}}}{\lambda}, \qquad \xi_{\mathrm{P}} := \sqrt{\frac{T}{\mu_\psi}}. \tag{28.7}$$

with

$$\lambda_{\mathsf{s}} = \sqrt{\frac{T}{\mu_{\mathsf{s}}}}, \qquad \xi = \sqrt{\frac{T}{\mu_\psi}}.$$

This writing selects the admissible stationary separations. It does not replace the collective mass closure of CB3.

Stationary conditions. CB2. With $d_* = 2\delta_*$, the equilibrium condition is

$$\boxed{V_{\mathsf{s}}'(d_*) + V_{\mathsf{P}}'(d_*) = 0, \qquad d_* \in (2\delta_{\min}, 2\delta_{\max}),} \tag{28.8}$$

and we impose

$$\partial_\delta^2 E_2^{\mathrm{geom}}(\delta_*) > 0.$$

CB3. In the symmetric geometry

$$\xi_1 = -\delta, \qquad \xi_2 = 0, \qquad \xi_3 = +\delta,$$

the distances are

$$d_{12} = d_{23} = \delta, \qquad d_{13} = 2\delta.$$

The reduced geometric energy can be written

$$E_3^{\mathrm{geom}}(\delta) = 2\,V_{\mathsf{s}}(\delta) + V_{\mathsf{s}}(2\delta) + 2\,V_{\mathsf{P}}(\delta) + V_{\mathsf{P}}(2\delta),$$

in the reference symmetric white case. Stationarity imposes

$$\boxed{2\,V_{\mathsf{s}}'(\delta_*) + 2\,V_{\mathsf{s}}'(2\delta_*) + 2\,V_{\mathsf{P}}'(\delta_*) + 2\,V_{\mathsf{P}}'(2\delta_*) = 0,} \tag{28.9}$$

with

$$\partial_\delta^2 E_3^{\mathrm{geom}}(\delta_*) > 0.$$

The value δ_* obtained in this way is then consumed in the collective color–white closure of CB3.

Collective closure of the resolved CB3. The closure of a resolved CB3 is built directly on the total white source. The isolated strong-like response (21.27) characterizes an individual source; the collective white response is defined by (21.32). The CB3 composite therefore does not close by first adding three isolated self-responses.

The reference energy scale is

$$E_* := m_* c_t^2.$$

The collective white response involves two core constants: the matching plateau $b_0^{(2)}$ and the radial-gradient moment J_2. We first introduce the collective tri-lobe weight

$$\boxed{\tau_3 := \sqrt{3\,b_0^{(2)}\,J_2}.} \tag{28.10}$$

The factor 3 comes from the three arms of the white node, $b_0^{(2)}$ is the plateau stiffness of the matching, and J_2 is the radial-gradient moment of the core profile, defined by (27.1).

Lemma 28.1 (Reduced diagonal variable of the CB3 arm)**.** *For an arm i, the reduced variable that compares the family radial excess to the collective white plateau is*

$$\boxed{x_i^2 = \frac{J_2}{3\, b_0^{(2)}} \left(\frac{q_i \rho_i}{4\pi} \right)^2 .} \tag{28.11}$$

Proof. The diagonal term of the arm is carried by the residual radial component

$$q_i \rho_i, \qquad \rho_i = \cos^2 \alpha_{X_i}.$$

The monopole normalization of the radial flux on the unit sphere gives the reduced source

$$\widehat{Q}_i^{\mathrm{rad}} = \frac{q_i \rho_i}{4\pi}.$$

The quadratic cost associated with the radial gradient is weighted by the moment

$$J_2 = \int_0^\infty 4\pi \xi^2 (a'(\xi))^2 \, d\xi.$$

The collective white plateau available to absorb this cost is the tri-lobe plateau

$$3\, b_0^{(2)},$$

one factor $b_0^{(2)}$ for each arm of the white node. The dimensionless ratio radial gradient / collective plateau is therefore

$$x_i^2 = \frac{J_2 (\widehat{Q}_i^{\mathrm{rad}})^2}{3\, b_0^{(2)}} = \frac{J_2}{3\, b_0^{(2)}} \left(\frac{q_i \rho_i}{4\pi} \right)^2 .$$

□

Proposition 28.2 (Reduced strong-like cutoff of CB3)**.** *The reduced diagonal variable can be written in the form*

$$x_i^2 = \left(\frac{q_i \rho_i}{q_c} \right)^2 ,$$

where the reduced strong-like cutoff is

$$\boxed{q_c = 4\pi \sqrt{\frac{3\, b_0^{(2)}}{J_2}}.} \tag{28.12}$$

Proof. By comparing

$$x_i^2 = \frac{J_2}{3\, b_0^{(2)}} \left(\frac{q_i \rho_i}{4\pi} \right)^2$$

with

$$x_i^2 = \left(\frac{q_i \rho_i}{q_c} \right)^2 ,$$

we obtain

$$\frac{1}{q_c^2} = \frac{J_2}{3\, b_0^{(2)} (4\pi)^2}.$$

Thus

$$q_c^2 = (4\pi)^2 \frac{3\, b_0^{(2)}}{J_2},$$

and, since $q_c > 0$,

$$q_c = 4\pi \sqrt{\frac{3\, b_0^{(2)}}{J_2}}.$$

□

Lemma 28.3 (Lorentzian saturation by quadratic elimination)**.** *Let a be the reduced amplitude of a collective matching and let $x \geq 0$ be the dimensionless ratio between the diagonal cost of the defect and the collective stiffness of the plateau. The normalized LO quadratic form of the matching is*

$$\mathcal{E}(a; x) = \frac{1}{2}(1 + x^2)a^2 - a.$$

Its stationary elimination gives

$$a_*(x) = \frac{1}{1 + x^2}.$$

The associated diagonal energy factor is therefore

$$\boxed{G(x) = \frac{1}{1 + x^2}.} \tag{28.13}$$

The associated amplitude-visibility factor is

$$\boxed{\sqrt{G(x)} = \frac{1}{\sqrt{1 + x^2}}.} \tag{28.14}$$

Proof. The stationarity of $\mathcal{E}$ gives

$$\partial_a \mathcal{E}(a; x) = (1 + x^2)a - 1 = 0,$$

hence

$$a_*(x) = \frac{1}{1 + x^2}.$$

The diagonal quadratic contribution carries the weight $G(x)$. The bilinear overlap terms between two arms carry the amplitudes, hence the root $\sqrt{G(x)}$. It follows that

$$G(x) = \frac{1}{1 + x^2}, \qquad \sqrt{G(x)} = \frac{1}{\sqrt{1 + x^2}}.$$

□

For a constituent i, we write

$$\sigma_i := \sin^2 \alpha_{X_i}, \qquad \rho_i := \cos^2 \alpha_{X_i}.$$

The arm source consumed by the white defect is the one defined in (21.30):

$$Q_i^{\mathrm{CB3}} := q_i \sigma_i \chi_{X_i}.$$

The reduced diagonal variable of the arm i is that of (28.11),

$$x_i = \frac{q_i \rho_i}{q_c}.$$

By Lemma 28.3, the saturated diagonal factor associated with the arm i is

$$\boxed{G_i = G(x_i) = \frac{1}{1 + \left(\frac{q_i \rho_i}{q_c}\right)^2}.} \tag{28.15}$$

LO resolution threshold of the subcores. The separation consumed in the pair overlaps of the resolved CB3 must not be confused with the variational optimum δ_* of the breather branch. In the collective color–white closure, it sets the LO threshold for leaving saturation between two subcores.

Each subcore of level q_i inherits the spectral transverse width

$$\lambda_i = \frac{\lambda_*}{q_i}.$$

For a pair (i, j), the relevant local resolution scale is the smaller of the two widths,

$$\lambda_{ij}^{\mathrm{res}} := \min(\lambda_i, \lambda_j) = \frac{\lambda_*}{q_{ij}^{\max}}.$$

At LO rank, the canonical core profile

$$a(u) = \operatorname{sech} u$$

has no resolved structure below its own width. Two subcores separated by a distance smaller than $\lambda_{ij}^{\mathrm{res}}$ are therefore not distinguished as two independent matchings in the CB3 matching rule: their overlap is taken in the saturated regime.

We introduce the reduced pair separation

$$\delta_{ij} := \frac{s_{ij}}{\lambda_{ij}^{\mathrm{res}}},$$

where s_{ij} is the local matching distance between the two subcores. The threshold for leaving the saturated regime is

$$s_{ij} = \lambda_{ij}^{\mathrm{res}},$$

that is,

$$\boxed{\delta_{\mathrm{CB3}}^{\mathrm{LO}} = 1.} \tag{28.16}$$

Proposition 28.4 (Canonical threshold for leaving saturation)**.** *At LO rank, the first non-saturated matching between two CB3 subcores of distinct family levels is reached for*

$$\delta_{ij} = 1.$$

Proof. The transverse closure of a subcore of level q_i fixes its single resolved local scale:

$$\lambda_i = \frac{\lambda_*}{q_i}.$$

At LO rank, no datum of the core closure distinguishes internal structures at the scale $s < \lambda_i$. For a pair (i, j), the finest available scale is therefore

$$\lambda_{ij}^{\mathrm{res}} = \min(\lambda_i, \lambda_j) = \frac{\lambda_*}{q_{ij}^{\max}}.$$

If

$$s_{ij} < \lambda_{ij}^{\mathrm{res}},$$

the two subcores are not separated by a sufficient length to define a non-saturated matching between two distinct profiles. The overlap is then taken in the saturated regime.

The first point at which a non-saturated matching becomes resolvable is the threshold

$$s_{ij} = \lambda_{ij}^{\mathrm{res}}.$$

In reduced variable,

$$\delta_{ij} = \frac{s_{ij}}{\lambda_{ij}^{\mathrm{res}}},$$

this threshold immediately gives

$$\delta_{ij} = 1.$$

Thus, $\delta_{\mathrm{CB3}}^{\mathrm{LO}} = 1$ is not a fitted value and not a variational minimum: it is the LO resolution threshold imposed by the transverse scale of the core. □

Saturated/non-saturated overlap rule. For two arms i, j, we denote by $\mathcal{O}_{ij}^{\mathrm{CB3,LO}}$ the matching overlap used in the collective color–white closure.

For two subcores of the same family level, the matching is taken in the saturated regime:

$$\boxed{\mathcal{O}_{ij}^{\mathrm{CB3,LO}} = 1 \qquad (q_i = q_j).} \tag{28.17}$$

The saturation $\mathcal{O}_{ij}^{\mathrm{CB3,LO}} = 1$ does not mean that the geometric distance between the centers is zero; it means that, in the LO matching rule, the pair carries no non-saturated inter-family damping.

For two subcores of distinct family levels, the matching is evaluated at the first non-saturated step of the canonical profile:

$$\boxed{\mathcal{O}_{ij}^{\mathrm{CB3,LO}} = (q_{ij}^{\max})^{-3/2}\,\operatorname{sech}(\delta_{\mathrm{CB3}}^{\mathrm{LO}}) = (q_{ij}^{\max})^{-3/2}\,\operatorname{sech}(1) \qquad (q_i \neq q_j).} \tag{28.18}$$

The factor $(q_{ij}^{\max})^{-3/2}$ encodes the LO inter-family damping; the factor $\operatorname{sech}(1)$ encodes the first non-saturated matching of the core profile.

The whiteness defect of CB3 in the LO closure is

$$\boxed{\mathcal{W}_{abc}^{\mathrm{CB3,LO}} = \sum_{i<j} \left[(Q_i^{\mathrm{CB3}})^2 G_i + (Q_j^{\mathrm{CB3}})^2 G_j - 2 Q_i^{\mathrm{CB3}} Q_j^{\mathrm{CB3}} \sqrt{G_i G_j}\, \mathcal{O}_{ij}^{\mathrm{CB3,LO}} \right].} \tag{28.19}$$

This quantity measures the matching defect of the three arms to the white average. The normalization of the white defect is the one of (21.34).

Geometric normalizations of the CB3 closure. The prefactors of the collective CB3 closure are obtained by geometric averaging over $\mathbb{S}^3$ and by counting the substructures of the white node.

Lemma 28.5 (Normalization of the CB3 white defect)**.** *The white-defect contribution in the CB3 closure carries the factor*

$$\boxed{\mathcal{N}_{\mathrm{white}}^{\mathrm{CB3}} = \frac{1}{6\pi^2}.} \tag{28.20}$$

Proof. The canonical average on $\mathbb{S}^3$ carries the factor

$$\frac{1}{\mathrm{Vol}(\mathbb{S}^3)} = \frac{1}{2\pi^2}.$$

The white defect $\mathcal{W}_{abc}^{\mathrm{CB3,LO}}$ is a pairwise quantity already summed over the three pairs

$$(i, j) \in \{(a, b), (a, c), (b, c)\}.$$

The average contribution per pair therefore carries a sharing by

$$N_{\mathrm{pair}} = 3.$$

Thus

$$\mathcal{N}^{\mathrm{CB3}}_{\mathrm{white}} = \frac{1}{\mathrm{Vol}(\mathbb{S}^3)}\frac{1}{N_{\mathrm{pair}}} = \frac{1}{2\pi^2}\frac{1}{3} = \frac{1}{6\pi^2}.$$

□

Lemma 28.6 (Normalization of the pairwise family term)**.** *The saturated pairwise family contribution carries the factor*

$$\boxed{\mathcal{N}^{\mathrm{CB3}}_{\mathrm{fam}} = \frac{1}{12\pi^2}.} \tag{28.21}$$

Proof. The family term is a bilinear pair-overlap contribution. It first carries the canonical average on $\mathbb{S}^3$,

$$\frac{1}{2\pi^2}.$$

Since it is evaluated on the three pairs $i < j$, the pairwise average introduces a sharing by 3. Finally, the combination

$$\frac{\Xi_i + \Xi_j}{2}$$

is the half-weight of pair symmetrization. The total factor is therefore

$$\mathcal{N}^{\mathrm{CB3}}_{\mathrm{fam}} = \frac{1}{2\pi^2}\frac{1}{3}\frac{1}{2} = \frac{1}{12\pi^2}.$$

□

Lemma 28.7 (Fundamental cap of the white node)**.** *The fundamental closure cap of the white node carries the factor*

$$\boxed{\mathcal{N}^{\mathrm{CB3}}_{\mathrm{cap}} = \frac{1}{2\pi q_{\max}}.} \tag{28.22}$$

Proof. The common matching of the white node is carried by the large phase geodesic. The average over a full cycle gives the factor

$$\frac{1}{2\pi}.$$

For a triplet whose maximal level is $q_{\max}$, only the common fundamental component represents the fraction

$$\frac{1}{q_{\max}}$$

of the maximal cycle. The fundamental cap is therefore

$$\mathcal{N}^{\mathrm{CB3}}_{\mathrm{cap}} = \frac{1}{2\pi}\frac{1}{q_{\max}} = \frac{1}{2\pi q_{\max}}.$$

□

The generic LO closure of the resolved CB3 is

$$\boxed{\frac{M^{\mathrm{CB3,gen}}_{abc}c_t^2}{E_*} = 3\tau_3 + \sum_{i=1}^{3} q_i\sigma_i + \frac{1}{6\pi^2}\,\mathcal{W}^{\mathrm{CB3,LO}}_{abc} - \frac{\tau_3}{2\pi q_{\max}} + \mathcal{F}^{abc}_{\mathrm{fam}}.} \tag{28.23}$$

Here

$$q_{\max} := \max(q_a, q_b, q_c).$$

The factors $1/(6\pi^2)$, $1/(2\pi q_{\max})$, and $1/(12\pi^2)$ are respectively derived in (28.20), (28.22), and (28.21).

The common family term is written in saturated pairwise form:

$$\boxed{\mathcal{F}^{abc}_{\mathrm{fam}} = \frac{\tau_3}{12\pi^2} \sum_{i<j} \left(q_{ij}^{\min} - 1\right) \frac{\Xi_i + \Xi_j}{2}\, \mathcal{S}_{ij},} \tag{28.24}$$

where

$$q_{ij}^{\min} := \min(q_i, q_j), \qquad q_{ij}^{\max} := \max(q_i, q_j), \qquad \Xi_i := \sigma_i \chi_{X_i},$$

and the reduced family excess carried by the pair (i, j) is

$$\boxed{x_{ij}^{\mathrm{fam}} := \frac{2(q_{ij}^{\max} - q_{ij}^{\min})}{3q_c}.} \tag{28.25}$$

The family factor $\mathcal{S}_{ij}$ is an amplitude visibility. By (28.14), it is therefore

$$\boxed{\mathcal{S}_{ij} = \sqrt{G(x_{ij}^{\mathrm{fam}})} = \frac{1}{\sqrt{1 + \left(\frac{2(q_{ij}^{\max} - q_{ij}^{\min})}{3q_c}\right)^2}}.} \tag{28.26}$$

When two arms belong to the same family level, $x_{ij}^{\mathrm{fam}} = 0$, hence $\mathcal{S}_{ij} = 1$. When the common component of a sub-pair is matched to a higher family excess, $\mathcal{S}_{ij}$ expresses the saturated amplitude visibility of the matching.

For an internal motif of type $\mathsf{udQ}_{\uparrow}$, where the $\mathsf{Q}_{\uparrow}$-like arm carries a level $q_Q > q_1$, the sub-pair ud has a low mechanical matching mode. We introduce

$$\eta_+ = \frac{\eta_{\mathsf{u}} + \eta_{\mathsf{d}}}{\sqrt{2}}.$$

Lemma 28.8 (Elimination of the low ud mode)**.** *The coupling of the low mode η_+ to the $\mathsf{Q}_{\uparrow}$-like arm produces the reduced lowering*

$$\boxed{\delta_{\mathrm{bas}}(q_Q) = -\left(1 - \frac{1}{q_Q}\right) \frac{2G_{\uparrow}(q_Q)}{1 + 2G_{\uparrow}(q_Q)}.} \tag{28.27}$$

Proof. The low mode η_+ is the symmetric mode of the ud sub-pair. It has two equivalent contributions, one per arm of the sub-pair, hence the factor 2. The $\mathsf{Q}_{\uparrow}$-like arm couples to this mode with the saturated visibility $G_{\uparrow}(q_Q)$. The total reduced stiffness of the mode after coupling is therefore

$$K_{\mathrm{eff}} = 1 + 2G_{\uparrow}(q_Q),$$

and the reduced cooperative source is proportional to

$$2G_{\uparrow}(q_Q).$$

The stationary elimination of a quadratic mode of the form

$$\mathcal{E}(\eta_+) = \frac{1}{2} K_{\mathrm{eff}} \eta_+^2 - 2G_{\uparrow}(q_Q)\eta_+$$

gives a relaxation proportional to the ratio

$$\frac{2G_{\uparrow}(q_Q)}{1 + 2G_{\uparrow}(q_Q)}.$$

Finally, only the excess fraction of the high arm relative to the fundamental participates in the lowering:

$$1 - \frac{1}{q_Q}.$$

The sign is negative because eliminating a relaxed mechanical mode lowers the effective energy. We therefore obtain (28.27). □

The energy lowering is then

$$\boxed{\Delta_{\rm bas}(\mathsf{udQ}_\uparrow) = E_*\tau_3\,\delta_{\rm bas}(q_Q) = -E_*\tau_3\left(1-\frac{1}{q_Q}\right)\frac{2G_\uparrow(q_Q)}{1+2G_\uparrow(q_Q)}.} \tag{28.28}$$

with the reduced relaxation variable

$$\boxed{x_\uparrow(q) := \frac{q}{3q_c}.} \tag{28.29}$$

By Lemma 28.3, the relaxation factor is

$$\boxed{G_\uparrow(q) = G(x_\uparrow(q)) = \frac{1}{1+\left(\frac{q}{3q_c}\right)^2}.} \tag{28.30}$$

The mass of the low mode is therefore

$$\boxed{M^{\rm CB3,bas}_{\mathsf{udQ}_\uparrow} = M^{\rm CB3,gen}_{\mathsf{udQ}_\uparrow} + \frac{\Delta_{\rm bas}(\mathsf{udQ}_\uparrow)}{c_t^2}.} \tag{28.31}$$

Branch selection. Equation (28.9) describes the possible geometric stationarity of the resolved branch. It must not be confused with the matching rule consumed in the collective color–white closure (28.23). In this closure, the pair overlaps are evaluated by the saturated/non-saturated rule (28.17)–(28.18), with the threshold $\delta^{\rm LO}_{\rm CB3} = 1$ defined in (28.16).

The resolved CB3 branch retains configurations for which the three centers remain identifiable and for which the collective color–white closure (28.23) is applicable. When the overlap becomes strong, the lobes must no longer be read as three independent constituents: the description switches to the breather branch treated below.

Kinematic compatibility in motion of the resolved CB3 branch. The resolved CB3 branch is a color–white composite solution with three identifiable subcores. It is not read as a strong-overlap branch: the three centers remain distinguishable and the collective closure (28.23) applies with the pair-overlap rules (28.17)– (28.18) and the matching threshold (28.16).

When the three subcores form a stationary collective configuration, the branch has a comoving center of mass, a finite rest energy

$$E_0^{\rm CB3} = M_{\rm CB3}c_t^2,$$

and a zero total longitudinal momentum in this frame:

$$P_0^{\rm CB3} = 0.$$

The branch then enters the framework of localized massive solutions of §7.1.

The boost law applies to the global motion of the composite, with the tangential characteristic speed

$$c_{\mathcal{F}} = c_t.$$

We obtain

$$E^{\rm CB3}(u) = \Gamma_t E_0^{\rm CB3}, \qquad P^{\rm CB3}(u) = \Gamma_t \frac{u}{c_t^2}E_0^{\rm CB3}, \qquad \Gamma_t = \frac{1}{\sqrt{1-u^2/c_t^2}}.$$

The effective dispersion of the composite is therefore

$$\left(E^{\rm CB3}(u)\right)^2 = \left(E_0^{\rm CB3}\right)^2 + c_t^2\left(P^{\rm CB3}(u)\right)^2.$$

Internal variables and scope of the collective boost. The preceding law concerns the global motion of the color–white composite. It assumes that the internal separation and matching degrees of freedom remain in the collective configuration used by the closure (28.23). If the three subcores carry independent relative motions, the branch no longer reduces to a single massive core: the kinematics must then be written in collective variables, with a center of mass and internal separation variables.

Radial compatibility. The radial source associated with a resolved CB3 branch is a finite sum of localized subcore sources and localized pair overlaps. When the collective configuration is comoving, this total source remains localized in the comoving coordinate of the center of the composite. The radial sheath operator is the common operator of the criterion (7.3). The strictly comoving sheath is therefore localized for

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

For $|u| \geq c_r$, the radial response no longer belongs to a strictly comoving Yukawa sheath at the same rank.

Scope. This kinematic compatibility holds for the resolved CB3 branch when the three subcores remain identifiable and the collective color–white closure is applicable. It does not apply to strong-overlap regimes, nor to configurations where the three subcores must be treated as independent relative motions.

28.5 Breather branch (strong overlap): composite field, overlaps, and closure

Purpose. In strong overlap, the approximation by pair potentials between centers is no longer sufficient: the composite is treated as a *unit multi-lobe solution*. Its internal structure is parametrized by:

- a reduced separation δ (CB2/CB3 geometry),
- a vector of internal weights w, which encodes the relative shape of the lobes.

The key point is that w is not an arbitrary choice: it is determined by stationarity under constraints (whiteness, Gram, normalization).

Geometric spin of the 3DOF breather branch

Unit status. In the 3DOF breather branch, the composite is treated as a *unit 3DOF solution* multi-lobe: strong overlap forbids reading the lobes as independent constituents at the level of the fast cycle. The conical structure therefore carries a single bias α, hence a single projection

$$\sigma = \sin^2 \alpha.$$

Geometric class. The corresponding fast unit 3DOF solution preserves a single independent tangential circular polarization,

$$N_{\mathrm{pol}} = 1,$$

and a fundamental fast cycle

$$\nu = 1.$$

The common geometric counting then gives

$$s_g = \frac{\nu N_{\mathrm{pol}}}{2} = \frac{1}{2}.$$

The 3DOF breather branch thus belongs to the half-integer spinorial class.

Norm and projection. As for any unit 3DOF branch of this class, the spin modulus is

$$\boxed{|S_{\mathrm{Br3}}| = \frac{1}{2}\,h_{\mathrm{eff}}, \qquad s_g^{(\mathrm{Br3})} = \frac{1}{2}.}$$

The conical axis provides a privileged projection direction, and the helicity

$$\varsigma = \pm 1$$

then gives

$$\boxed{S_{\parallel}^{(\mathrm{Br3})} = \varsigma\,\frac{1}{2}\,h_{\mathrm{eff}}, \qquad \varsigma = \pm 1.}$$

Scope. The multi-lobe character of the 3DOF breather branch modifies the internal shape and the closure in δ, but not the geometric class of the fast cycle. The branch remains a unit solution of half-integer spinorial type; it must therefore not be confused with an integer vector branch of Z-like type.

3DOF conical angle: uniqueness of α (hence of σ) in the breather branch. In the breather branch, the composite is treated as a *unit 3DOF solution* carrying a single carrier ω_0 and a single scale λ_*. A unit 3DOF solution is characterized by a constant conical bias α, hence by

$$\sigma = \sin^2\alpha \in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}.$$

Consequently, a multi-lobe breather (CB2/CB3) cannot combine several conical values at the level of its lobes: such compositions would require several carriers, several phase shifts, or a spatially variable α, and would leave the class of stationary 3DOF solutions considered here. In the present branch, σ enters only in the conversion $\lambda_* \mapsto m$ through $\omega_{\mathrm{br}} = \sigma\,\omega_0$.

28.5.1 Composite ansatz and overlap functionals

Ansatz (CB2/CB3). The composite field is modeled by a weighted superposition of LO core profiles $a(\xi) = \operatorname{sech}\xi$:

$$\Phi(\xi) = \sum_{i=1}^{N} w_i\, a(\xi - \xi_i), \qquad N \in \{2,3\}, \tag{28.32}$$

where $\{\xi_i\}$ are the canonical positions (CB2: $\{-\delta, +\delta\}$, CB3: $\{-\delta, 0, +\delta\}$).

Moments and overlaps (local 3D translational convention). With the local tangential measure d^3x, we set

$$J_0 = \int_{\mathbb{R}^3} a(|\mathbf{x}|)^2\, d^3x, \qquad J_2 = \int_{\mathbb{R}^3} |\nabla a(|\mathbf{x}|)|^2\, d^3x, \qquad J_4 = \int_{\mathbb{R}^3} a(|\mathbf{x}|)^4\, d^3x.$$

For the canonical radial profile $a(r) = \operatorname{sech} r$, these constants coincide with the radial integrals

$$J_0 = \int_0^\infty 4\pi r^2 a(r)^2\, dr, \qquad J_2 = \int_0^\infty 4\pi r^2 a'(r)^2\, dr, \qquad J_4 = \int_0^\infty 4\pi r^2 a(r)^4\, dr.$$

For two centers separated by a reduced distance Δ, the overlaps are defined by 3D translation of the profile center:

$$\boxed{\begin{aligned} J_0^{\times}(\Delta) &:= \int_{\mathbb{R}^3} a(|\mathbf{x}|)\, a(|\mathbf{x}-\mathbf{d}|)\, d^3x, \\ J_2^{\times}(\Delta) &:= \int_{\mathbb{R}^3} \nabla a(|\mathbf{x}|)\cdot\nabla a(|\mathbf{x}-\mathbf{d}|)\, d^3x, \\ J_4^{\times}(\Delta) &:= \int_{\mathbb{R}^3} a(|\mathbf{x}|)^2\, a(|\mathbf{x}-\mathbf{d}|)^2\, d^3x, \qquad |\mathbf{d}| = \Delta. \end{aligned}} \tag{28.33}$$

In radial-angular form, setting

$$R(r,\mu;\Delta)=\sqrt{r^2+\Delta^2-2r\Delta\mu},$$

one has

$$J_0^{\times}(\Delta)=2\pi\int_0^{\infty} r^2\,dr\int_{-1}^{1} a(r)\,a(R(r,\mu;\Delta))\;d\mu,$$

$$J_4^{\times}(\Delta)=2\pi\int_0^{\infty} r^2\,dr\int_{-1}^{1} a(r)^2\,a(R(r,\mu;\Delta))^2\;d\mu.$$

For the gradient term, since

$$\nabla a(|\mathbf{x}|)=a'(r)\,\frac{\mathbf{x}}{r},\qquad \nabla a(|\mathbf{x}-\mathbf{d}|)=a'(R)\,\frac{\mathbf{x}-\mathbf{d}}{R},$$

it follows that

$$J_2^{\times}(\Delta)=2\pi\int_0^{\infty} r^2\,dr\int_{-1}^{1} a'(r)\,a'(R(r,\mu;\Delta))\,\frac{r-\Delta\mu}{R(r,\mu;\Delta)}\,d\mu.$$

This convention ensures

$$J_0^{\times}(0)=J_0,\qquad J_2^{\times}(0)=J_2,\qquad J_4^{\times}(0)=J_4,$$

and it is compatible with the positivity of the Gram matrix:

$$0\le J_0^{\times}(\Delta)\le J_0.$$

It avoids the inconsistency that would appear with a direct radial shift $a(r-\Delta)$ under the measure $4\pi r^2dr$, which does not represent a 3D translation of the profile.

One has $J_0^{\times}(0)=J_0$, $J_2^{\times}(0)=J_2$, $J_4^{\times}(0)=J_4$.

Forms S_0,S_2 and pair-by-pair quartic reduction. For a given geometry ($\Delta_{ij}=|\xi_i-\xi_j|$), we define

$$\boxed{S_p[w;\delta]=\sum_{i=1}^{N} w_i^2\,J_p+2\sum_{1\le i<j\le N} w_iw_j\,J_p^{\times}(\Delta_{ij}),\qquad p\in\{0,2\}.} \tag{28.34}$$

For the Preclusion channel, the effective pair-by-pair reduction used is

$$\boxed{S_{4,\mathrm{pair}}[w;\delta]=6\sum_{1\le i<j\le N}(w_iw_j)^2\,J_4^{\times}(\Delta_{ij}).} \tag{28.35}$$

Strong-like channel: pair-by-pair reduction. In the breather branch, the strong-like attraction is assumed pair-by-pair and carried by the quartic overlap $J_4^{\times}$. The effective strong charge of a lobe being proportional to the modulus of its internal weight, we set

$$\boxed{c_{ij}^{(\mathsf{F})}(w)=|w_iw_j|.} \tag{28.36}$$

The reduced strong-like numerator is therefore written

$$\boxed{N_{\mathsf{F}}[w;\delta]=\sum_{1\le i<j\le N}|w_iw_j|\,J_4^{\times}(\Delta_{ij}).} \tag{28.37}$$

CB2/CB3 cases. **CB2:** $\Delta_{12}=2\delta$. **CB3:** $\{\Delta_{ij}\}=\{\delta,\delta,2\delta\}$.

28.5.2 Internal constraints: whiteness, Gram, normalization

Whiteness (channel filter). At the level of the composite field, whiteness is implemented as a linear constraint on w, deduced from the definition of the strong charge I_{s} and the projector $\boldsymbol{\pi}^{(\mathsf{col})}$:

$$\boxed{\mathcal{C}_{\mathrm{white}}(w) = 0.} \tag{28.38}$$

Mechanical exclusion (Gram). The superposition must remain nondegenerate: the Gram matrix of the lobes must remain positive definite, which imposes $\delta \geq \delta_{\min} > 0$.

Normalization (lock). The variational problem is homogeneous under $w \mapsto \lambda w$. We therefore impose

$$\boxed{S_0[w;\delta] = 1.} \tag{28.39}$$

28.5.3 Effective impedances, closure $\mathcal{D}_{\mathrm{LO}}$, and structural energy

Common canonical closure and reduced branch parameter. The same closure is used for the 3DOF breather branches. In these closures, the parameter that weights the gradient and core-response contributions is a reduced dimensionless parameter. To avoid any confusion with the global radial coefficient κ_r of HSM theory, it is denoted

$$\eta_{\mathrm{br}}^{(3)}.$$

It must not be read as a new free constant: it denotes the reduced branch combination obtained after passage to the core scale and choice of the 3DOF breather closure. Its possible relation with the global radial parameters belongs to the branch closure, and not to the generic notation of the reduced forms below.

The normalized quadratic forms are

$$\widehat{s}_2[w;\delta] = \frac{S_2[w;\delta]}{S_0[w;\delta]}, \qquad \widehat{s}_{4,\mathrm{pair}}[w;\delta] = \frac{S_{4,\mathrm{pair}}[w;\delta]}{S_0[w;\delta]^2}$$

and the normalized strong-like attractive numerator is

$$\widehat{n}_{\mathsf{F}}[w;\delta] = \frac{N_{\mathsf{F}}[w;\delta]}{S_0[w;\delta]}.$$

Thus

$$\boxed{\beta^{(\mathsf{P})}[w;\delta] = \eta_{\mathrm{br}}^{(3)}\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}_{4,\mathrm{pair}}[w;\delta]}{\left(1+\left(\eta_{\mathrm{br}}^{(3)}\right)^{-1}\widehat{s}_2[w;\delta]\right)^2}, \qquad \beta^{(\mathsf{F})}[w;\delta] = \zeta_{3D}\,\frac{\widehat{n}_{\mathsf{F}}[w;\delta]}{1+\left(\eta_{\mathrm{br}}^{(3)}\right)^{-1}\widehat{s}_2[w;\delta]}.}$$ (28.40)

We set

$$\boxed{\beta^{(\mathrm{eff})}[w;\delta] = \beta^{(\mathsf{F})}[w;\delta] - \beta^{(\mathsf{P})}[w;\delta].} \tag{28.41}$$

Closure kernels. We set

$$K_0^{(Z)} := J_0^2, \qquad K_2^{(Z)} := J_0 J_2.$$

For a branch normalized by $S_0[w;\delta]$, we use

$$K_0[w;\delta] = K_0^{(Z)}, \qquad K_2[w;\delta] = \widehat{s}_2[w;\delta]K_2^{(Z)}, \qquad K[w;\delta] = K_0[w;\delta] + K_2[w;\delta].$$

The LO denominator is then

$$\boxed{\mathcal{D}_{\mathrm{LO}}[w;\delta] = \eta_{\mathrm{br}}^{(3)}\left(K[w;\delta] - \beta^{(\mathrm{eff})}[w;\delta]\,K_0[w;\delta]\right) - \beta^{(\mathrm{eff})}[w;\delta]K_2[w;\delta].} \tag{28.42}$$

The LO+ closure adds the sheath self-induction:

$$\boxed{\mathcal{D}_{\mathrm{LO+}}[w;\delta] = \mathcal{D}_{\mathrm{LO}}[w;\delta] + \Gamma_{\mathrm{shape}}K_0^{(Z)}.} \tag{28.43}$$

Link with structural energy. The closure gives

$$\boxed{\lambda_*^4[w;\delta] = \frac{2h_{\mathrm{eff}}\sqrt{\rho T}}{\mathcal{D}_{\mathrm{LO}}[w;\delta]}, \qquad \mathcal{D}_{\mathrm{LO}}[w;\delta] > 0.} \tag{28.44}$$

The branch mass is carried by the frequency

$$\omega_{\mathrm{br}} = \sigma\,\frac{c_t}{\lambda_*}.$$

At fixed w, the structural energy is therefore a strictly increasing function of $\mathcal{D}_{\mathrm{LO}}$:

$$\boxed{E_{\mathrm{br}}[w;\delta] = h_{\mathrm{eff}}\sigma\frac{c_t}{\lambda_*} = C_{\mathrm{br}}\,\left(\mathcal{D}_{\mathrm{LO}}[w;\delta]\right)^{1/4}, \qquad C_{\mathrm{br}} > 0.} \tag{28.45}$$

Thus, on the positive admissible domain, minimizing the structural energy is equivalent to minimizing $\mathcal{D}_{\mathrm{LO}}$. If the minimum is interior,

$$\partial_\delta E_{\mathrm{br}} = 0 \quad \Longleftrightarrow \quad \partial_\delta \mathcal{D}_{\mathrm{LO}} = 0,$$

and the second-order criterion is identical.

28.5.4 Admissible domain and selection principle for breathers

Common principle. In the breather branch, the raw motif $\bar{w}$ fixes the internal class under consideration. The normalization

$$\widehat{w}(\delta) = \frac{\bar{w}}{\sqrt{S_0[\bar{w};\delta]}}$$

is then imposed branch by branch in the specializations below. The selection therefore no longer acts on an arbitrary family of weights w, but on the separation δ at fixed raw motif.

Lower and upper bounds. The separation belongs to the geometric interval

$$\delta \in [\delta_{\min}^{\mathrm{br}}, \delta_{\max}],$$

where $\delta_{\min}^{\mathrm{br}} > 0$ is the Gram non-degeneracy lower bound of the branch under consideration, and where

$$\delta_{\max} = \frac{u_*^\star}{2}$$

is the canonical MSE bound defined by (27.17).

Positive admissible domain. The breather closure also imposes positivity of the denominator:

$$\mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta) > 0.$$

For each specialized branch, we therefore define a positive admissible domain

$$I_{\mathrm{adm}}^{+,\mathrm{br}} = \left\{\delta \in [\delta_{\min}^{\mathrm{br}}, \delta_{\max}] \;:\; \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta) > 0\right\}.$$

The explicit forms of this domain are given below for Br2, neutral Br3, and charged Br3.

Selected separation. The breather separation is defined by minimization of the structural energy. Since

$$E_{\mathrm{br}}^{\mathrm{br}}(\delta) = C_{\mathrm{br}} \left(\mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta)\right)^{1/4}, \qquad C_{\mathrm{br}} > 0,$$

on the positive admissible domain, this minimization is equivalent to

$$\delta_*^{\mathrm{br}} \in \arg\min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{br}}} \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta).$$

If the minimum is interior, it satisfies

$$\partial_\delta \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta_*) = 0, \qquad \partial_\delta^2 \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta_*) > 0.$$

A boundary value is admissible only as a constrained optimum of the positive domain, not as an imposed hypothesis.

Canonical Br2 specialization. For the bilobe breather branch, the two centers are placed at

$$\{-\delta, +\delta\}, \qquad d_{12} = 2\delta.$$

The raw integer motif is determined by the minimal discrete constraints of the neutral bilobe branch. If

$$\bar{w} = (m, n) \in \mathbb{Z}^2,$$

neutrality imposes

$$m + n = 0.$$

Primitivity of the motif then imposes

$$\gcd(|m|, |n|) = 1,$$

and the orientation convention fixes $m > 0$. Hence

$$m = 1, \qquad n = -1.$$

Thus the canonical raw motif of the Br2 branch is

$$\boxed{\bar{w}_{\mathrm{Br2}} = (1, -1).} \tag{28.46}$$

The global sign change corresponds to the conjugate orientation and does not define a new branch at LO rank. The raw shape sums reduce to

$$\boxed{S_0^{\mathrm{Br2}}(\delta) = 2\big(J_0 - J_0^{\times}(2\delta)\big), \qquad S_2^{\mathrm{Br2}}(\delta) = 2\big(J_2 - J_2^{\times}(2\delta)\big).} \tag{28.47}$$

The raw pairwise quartic reduction is

$$\boxed{S_{4,\mathrm{pair}}^{\mathrm{Br2}}(\delta) = 6\, J_4^{\times}(2\delta).} \tag{28.48}$$

The raw strong-like numerator is

$$\boxed{N_{\mathsf{F}}^{\mathrm{Br2}}(\delta) = J_4^{\times}(2\delta).} \tag{28.49}$$

The motif is normalized by

$$\widehat{w}_{\mathrm{Br2}}(\delta) = \frac{(1, -1)}{\sqrt{S_0^{\mathrm{Br2}}(\delta)}}.$$

The reduced quantities consumed by the closure are

$$\boxed{\widehat{s}_2^{\mathrm{Br2}}(\delta)=\frac{S_2^{\mathrm{Br2}}(\delta)}{S_0^{\mathrm{Br2}}(\delta)},\qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{Br2}}(\delta)=\frac{S_{4,\mathrm{pair}}^{\mathrm{Br2}}(\delta)}{\left(S_0^{\mathrm{Br2}}(\delta)\right)^2},\qquad \widehat{n}_{\mathsf{F}}^{\mathrm{Br2}}(\delta)=\frac{N_{\mathsf{F}}^{\mathrm{Br2}}(\delta)}{S_0^{\mathrm{Br2}}(\delta)}.} \tag{28.50}$$

We set

$$\boxed{\beta_{\mathrm{Br2}}^{(\mathsf{F})}(\delta)=\zeta_{3D}\,\frac{\widehat{n}_{\mathsf{F}}^{\mathrm{Br2}}(\delta)}{1+\left(\eta_{\mathrm{br}}^{(3)}\right)^{-1}\widehat{s}_2^{\mathrm{Br2}}(\delta)}.} \tag{28.51}$$

The Preclusion contribution is

$$\boxed{\beta_{\mathrm{Br2}}^{(\mathsf{P})}(\delta)=\eta_{\mathrm{br}}^{(3)}\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}_{4,\mathrm{pair}}^{\mathrm{Br2}}(\delta)}{\left(1+\left(\eta_{\mathrm{br}}^{(3)}\right)^{-1}\widehat{s}_2^{\mathrm{Br2}}(\delta)\right)^2}.} \tag{28.52}$$

Thus

$$\boxed{\beta_{\mathrm{Br2}}^{(\mathrm{eff})}(\delta):=\beta_{\mathrm{Br2}}^{(\mathsf{F})}(\delta)-\beta_{\mathrm{Br2}}^{(\mathsf{P})}(\delta).} \tag{28.53}$$

The associated kernels are

$$\boxed{K_{0,\mathrm{Br2}}(\delta)=K_0^{(Z)},\qquad K_{2,\mathrm{Br2}}(\delta)=\widehat{s}_2^{\mathrm{Br2}}(\delta)\,K_2^{(Z)},\qquad K_{\mathrm{Br2}}(\delta)=K_{0,\mathrm{Br2}}(\delta)+K_{2,\mathrm{Br2}}(\delta).} \tag{28.54}$$

The LO closure denominator is

$$\boxed{\mathcal{D}_{\mathrm{LO}}^{\mathrm{Br2}}(\delta)=\eta_{\mathrm{br}}^{(3)}\left(K_{\mathrm{Br2}}(\delta)-\beta_{\mathrm{Br2}}^{(\mathrm{eff})}(\delta)\,K_{0,\mathrm{Br2}}(\delta)\right)-\beta_{\mathrm{Br2}}^{(\mathrm{eff})}(\delta)\,K_{2,\mathrm{Br2}}(\delta).} \tag{28.55}$$

At LO+:

$$\boxed{\mathcal{D}_{\mathrm{LO+}}^{\mathrm{Br2}}(\delta)=\mathcal{D}_{\mathrm{LO}}^{\mathrm{Br2}}(\delta)+\Gamma_{\mathrm{shape}}K_0^{(Z)}.} \tag{28.56}$$

The positive admissible domain is

$$\boxed{I_{\mathrm{adm}}^{+,\mathrm{Br2}}=\left\{\delta\in[\delta_{\mathrm{min}}^{\mathrm{Br2}},\delta_{\mathrm{max}}]\ :\ \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br2}}(\delta)>0\right\}.} \tag{28.57}$$

The selected separation is then

$$\boxed{\delta_*^{\mathrm{Br2}}\in\arg\min_{\delta\in I_{\mathrm{adm}}^{+,\mathrm{Br2}}}\mathcal{D}_{\mathrm{LO}}^{\mathrm{Br2}}(\delta).} \tag{28.58}$$

Canonical neutral Br3 specialization. For the neutral tri-lobe breather branch, the raw motif is fixed as

$$\bar{w}_{\mathrm{Br3,n}}=(1,-2,1).$$

The three centers are placed at

$$\{-\delta,0,+\delta\},$$

so that

$$d_{12}=d_{23}=\delta,\qquad d_{13}=2\delta.$$

The raw shape sums are

$$\boxed{S_0^{\mathrm{Br3,n}}(\delta)=S_0[(1,-2,1);\delta],\qquad S_2^{\mathrm{Br3,n}}(\delta)=S_2[(1,-2,1);\delta].} \tag{28.59}$$

Using the bilinearity of the forms S_p, one obtains explicitly

$$\boxed{S_p^{\mathrm{Br3,n}}(\delta) = 6J_p - 8J_p^{\times}(\delta) + 2J_p^{\times}(2\delta), \qquad p \in \{0,2\}.} \tag{28.60}$$

The raw pair-to-pair quartic reduction is

$$\boxed{S_{4,\mathrm{pair}}^{\mathrm{Br3,n}}(\delta) = 6\left(8J_4^{\times}(\delta) + J_4^{\times}(2\delta)\right).} \tag{28.61}$$

The raw strong-like numerator is

$$\boxed{N_{\mathsf{F}}^{\mathrm{Br3,n}}(\delta) = 4J_4^{\times}(\delta) + J_4^{\times}(2\delta).} \tag{28.62}$$

The motif is normalized by

$$\widehat{w}_{\mathrm{Br3,n}}(\delta) = \frac{(1,-2,1)}{\sqrt{S_0^{\mathrm{Br3,n}}(\delta)}}.$$

The reduced quantities consumed by the closure are

$$\boxed{\widehat{s}_2^{\mathrm{Br3,n}}(\delta) = \frac{S_2^{\mathrm{Br3,n}}(\delta)}{S_0^{\mathrm{Br3,n}}(\delta)}, \qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{Br3,n}}(\delta) = \frac{S_{4,\mathrm{pair}}^{\mathrm{Br3,n}}(\delta)}{\left(S_0^{\mathrm{Br3,n}}(\delta)\right)^2}, \qquad \widehat{n}_{\mathsf{F}}^{\mathrm{Br3,n}}(\delta) = \frac{N_{\mathsf{F}}^{\mathrm{Br3,n}}(\delta)}{S_0^{\mathrm{Br3,n}}(\delta)}.} \tag{28.63}$$

We set

$$\boxed{\beta_{\mathrm{Br3,n}}^{(\mathsf{F})}(\delta) = \zeta_{3D}\,\frac{\widehat{n}_{\mathsf{F}}^{\mathrm{Br3,n}}(\delta)}{1 + \left(\eta_{\mathrm{br}}^{(3)}\right)^{-1}\widehat{s}_2^{\mathrm{Br3,n}}(\delta)}.} \tag{28.64}$$

The Preclusion contribution is

$$\boxed{\beta_{\mathrm{Br3,n}}^{(\mathsf{P})}(\delta) = \eta_{\mathrm{br}}^{(3)}\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}_{4,\mathrm{pair}}^{\mathrm{Br3,n}}(\delta)}{\left(1 + \left(\eta_{\mathrm{br}}^{(3)}\right)^{-1}\widehat{s}_2^{\mathrm{Br3,n}}(\delta)\right)^2}.} \tag{28.65}$$

Thus

$$\boxed{\beta_{\mathrm{Br3,n}}^{(\mathrm{eff})}(\delta) := \beta_{\mathrm{Br3,n}}^{(\mathsf{F})}(\delta) - \beta_{\mathrm{Br3,n}}^{(\mathsf{P})}(\delta).} \tag{28.66}$$

The associated kernels are

$$\boxed{K_{0,\mathrm{Br3,n}}(\delta) = K_0^{(Z)}, \qquad K_{2,\mathrm{Br3,n}}(\delta) = \widehat{s}_2^{\mathrm{Br3,n}}(\delta)\,K_2^{(Z)}, \qquad K_{\mathrm{Br3,n}}(\delta) = K_{0,\mathrm{Br3,n}}(\delta) + K_{2,\mathrm{Br3,n}}(\delta).} \tag{28.67}$$

The LO closure denominator is

$$\boxed{\mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,n}}(\delta) = \eta_{\mathrm{br}}^{(3)}\left(K_{\mathrm{Br3,n}}(\delta) - \beta_{\mathrm{Br3,n}}^{(\mathrm{eff})}(\delta)\,K_{0,\mathrm{Br3,n}}(\delta)\right) - \beta_{\mathrm{Br3,n}}^{(\mathrm{eff})}(\delta)\,K_{2,\mathrm{Br3,n}}(\delta).} \tag{28.68}$$

At LO+:

$$\boxed{\mathcal{D}_{\mathrm{LO+}}^{\mathrm{Br3,n}}(\delta) = \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,n}}(\delta) + \Gamma_{\mathrm{shape}}K_0^{(Z)}.} \tag{28.69}$$

The positive admissible domain is

$$\boxed{I_{\mathrm{adm}}^{+,\mathrm{Br3,n}} = \left\{\delta \in [\delta_{\min}^{\mathrm{Br3}}, \delta_{\max}] \;:\; \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,n}}(\delta) > 0\right\}.} \tag{28.70}$$

The selected separation is then

$$\boxed{\delta_*^{\mathrm{Br3,n}} \in \arg\min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{Br3,n}}} \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,n}}(\delta).} \tag{28.71}$$

Canonical charged Br3 specialization. For the charged branch, one seeks a raw integer motif

$$\bar{w} = (a, b, a) \in \mathbb{Z}^3$$

compatible with the left-right symmetry of the tri-lobe. The discrete charge label is

$$n_{\mathrm{EM}}(\bar{w}) = a + b + a = 2a + b.$$

The minimal charged branch imposes

$$n_{\mathrm{EM}}(\bar{w}) = 1.$$

The motif must be primitive,

$$\gcd(|a|, |b|) = 1,$$

and must really carry three lobes, hence $a \neq 0$. With the orientation convention $a > 0$, the smallest admissible choice is

$$a = 1, \qquad b = -1,$$

whence

$$\boxed{\bar{w}_{\mathrm{Br3,c}} = (1, -1, 1).} \tag{28.72}$$

The three centers are placed at

$$\{-\delta, 0, +\delta\},$$

so that

$$d_{12} = d_{23} = \delta, \qquad d_{13} = 2\delta.$$

This motif is treated as an effective discrete reduction of a charged-like zone of the continuous symmetric branch $w(\delta) \sim (1, \alpha_-(\delta), 1)$.

The raw shape sums are

$$\boxed{S_0^{\mathrm{Br3,c}}(\delta) = S_0[(1, -1, 1); \delta], \qquad S_2^{\mathrm{Br3,c}}(\delta) = S_2[(1, -1, 1); \delta].} \tag{28.73}$$

Using the bilinearity of the forms S_p, one obtains explicitly

$$\boxed{S_p^{\mathrm{Br3,c}}(\delta) = 3J_p - 4J_p^{\times}(\delta) + 2J_p^{\times}(2\delta), \qquad p \in \{0, 2\}.} \tag{28.74}$$

The raw pair-to-pair quartic reduction is

$$\boxed{S_{4,\mathrm{pair}}^{\mathrm{Br3,c}}(\delta) = 6\left(2J_4^{\times}(\delta) + J_4^{\times}(2\delta)\right).} \tag{28.75}$$

The raw strong-like numerator is

$$\boxed{N_{\mathsf{F}}^{\mathrm{Br3,c}}(\delta) = 2J_4^{\times}(\delta) + J_4^{\times}(2\delta).} \tag{28.76}$$

The motif is normalized by

$$\widehat{w}_{\mathrm{Br3,c}}(\delta) = \frac{(1, -1, 1)}{\sqrt{S_0^{\mathrm{Br3,c}}(\delta)}}.$$

The reduced quantities consumed by the closure are

$$\boxed{\widehat{s}_2^{\mathrm{Br3,c}}(\delta) = \frac{S_2^{\mathrm{Br3,c}}(\delta)}{S_0^{\mathrm{Br3,c}}(\delta)}, \qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{Br3,c}}(\delta) = \frac{S_{4,\mathrm{pair}}^{\mathrm{Br3,c}}(\delta)}{\left(S_0^{\mathrm{Br3,c}}(\delta)\right)^2}, \qquad \widehat{n}_{\mathsf{F}}^{\mathrm{Br3,c}}(\delta) = \frac{N_{\mathsf{F}}^{\mathrm{Br3,c}}(\delta)}{S_0^{\mathrm{Br3,c}}(\delta)}.} \tag{28.77}$$

We set

$$\boxed{\beta^{(\mathsf{F})}_{\mathrm{Br3,c}}(\delta) = \zeta_{3D}\,\frac{\widehat{n}^{\mathrm{Br3,c}}_{\mathsf{F}}(\delta)}{1+\left(\eta^{(3)}_{\mathrm{br}}\right)^{-1}\widehat{s}^{\mathrm{Br3,c}}_{2}(\delta)}.} \tag{28.78}$$

The Preclusion contribution is

$$\boxed{\beta^{(\mathsf{P})}_{\mathrm{Br3,c}}(\delta) = \eta^{(3)}_{\mathrm{br}}\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}^{\mathrm{Br3,c}}_{4,\mathrm{pair}}(\delta)}{\left(1+\left(\eta^{(3)}_{\mathrm{br}}\right)^{-1}\widehat{s}^{\mathrm{Br3,c}}_{2}(\delta)\right)^{2}}.} \tag{28.79}$$

Thus

$$\boxed{\beta^{(\mathrm{eff})}_{\mathrm{Br3,c}}(\delta) := \beta^{(\mathsf{F})}_{\mathrm{Br3,c}}(\delta) - \beta^{(\mathsf{P})}_{\mathrm{Br3,c}}(\delta).} \tag{28.80}$$

The associated kernels are

$$\boxed{K_{0,\mathrm{Br3,c}}(\delta) = K_0^{(Z)}, \qquad K_{2,\mathrm{Br3,c}}(\delta) = \widehat{s}^{\mathrm{Br3,c}}_{2}(\delta)\,K_2^{(Z)}, \qquad K_{\mathrm{Br3,c}}(\delta) = K_{0,\mathrm{Br3,c}}(\delta) + K_{2,\mathrm{Br3,c}}(\delta).} \tag{28.81}$$

The LO closure denominator is

$$\boxed{\mathcal{D}^{\mathrm{Br3,c}}_{\mathrm{LO}}(\delta) = \eta^{(3)}_{\mathrm{br}}\left(K_{\mathrm{Br3,c}}(\delta) - \beta^{(\mathrm{eff})}_{\mathrm{Br3,c}}(\delta)\,K_{0,\mathrm{Br3,c}}(\delta)\right) - \beta^{(\mathrm{eff})}_{\mathrm{Br3,c}}(\delta)\,K_{2,\mathrm{Br3,c}}(\delta).} \tag{28.82}$$

At LO+:

$$\boxed{\mathcal{D}^{\mathrm{Br3,c}}_{\mathrm{LO+}}(\delta) = \mathcal{D}^{\mathrm{Br3,c}}_{\mathrm{LO}}(\delta) + \Gamma_{\mathrm{shape}}K_0^{(Z)}.} \tag{28.83}$$

The positive admissible domain is

$$\boxed{I^{+,\mathrm{Br3,c}}_{\mathrm{adm}} = \left\{\delta \in [\delta^{\mathrm{Br3}}_{\mathrm{min}}, \delta_{\mathrm{max}}] \;:\; \mathcal{D}^{\mathrm{Br3,c}}_{\mathrm{LO}}(\delta) > 0\right\}.} \tag{28.84}$$

The selected separation is then

$$\boxed{\delta^{\mathrm{Br3,c}}_{*} \in \arg\min_{\delta \in I^{+,\mathrm{Br3,c}}_{\mathrm{adm}}} \mathcal{D}^{\mathrm{Br3,c}}_{\mathrm{LO}}(\delta).} \tag{28.85}$$

Existence condition for a closed breather branch. A breather branch is closed at LO rank if the associated positive admissible domain is non-empty,

$$I^{+,\mathrm{br}}_{\mathrm{adm}} \neq \varnothing,$$

and if the minimization of $\mathcal{D}^{\mathrm{br}}_{\mathrm{LO}}$ on this domain selects an interior minimum:

$$\delta^{\mathrm{br}}_{*} \in (\delta^{\mathrm{br}}_{\mathrm{min}}, \delta_{\mathrm{max}}),$$

with

$$\partial_\delta \mathcal{D}^{\mathrm{br}}_{\mathrm{LO}}(\delta_*) = 0, \qquad \partial^2_\delta \mathcal{D}^{\mathrm{br}}_{\mathrm{LO}}(\delta_*) > 0, \qquad \mathcal{D}^{\mathrm{br}}_{\mathrm{LO}}(\delta_*) > 0.$$

At LO+ rank, the positivity condition becomes

$$\mathcal{D}^{\mathrm{br}}_{\mathrm{LO+}}(\delta_*) > 0.$$

If the constrained minimum is reached at an edge of the admissible interval, the branch is not promoted as an internal stationary breather solution at LO rank. It is then classified as a boundary branch, that is, as a limit of validity of the strong-overlap description or as a transition toward another description.

Exclusion of collapse. In the multi-lobe ansatz

$$\Phi = \sum_i w_i a(\xi - \xi_i),$$

the limit $\delta \to 0$ makes the lobes linearly dependent. For motifs with zero sum, the composite field may even become trivial at leading rank. We therefore impose a non-zero lower bound:

$$\delta \geq \delta^{\mathrm{br}}_{\min} > 0.$$

This bound is not a new dynamical constant: it is the non-degeneracy bound of the Gram problem associated with the lobes of the considered motif.

Kinematic compatibility in motion. A closed 3DOF breather branch in the sense of the previous criterion is a strong-overlap multi-lobe composite solution. The lobes are not read as independent solitons: they form a single effective core, described by a common comoving coordinate and by a fixed raw motif $\bar{w}$.

When the minimization of $\mathcal{D}^{\mathrm{br}}_{\mathrm{LO}}$ selects an interior minimum,

$$\delta^{\mathrm{br}}_* \in (\delta^{\mathrm{br}}_{\min}, \delta_{\max}),$$

the branch has a closed width $\lambda_{*,\mathrm{br}}$, a projected proper frequency

$$\omega_{\mathrm{br}} = \sigma \, \frac{c_t}{\lambda_{*,\mathrm{br}}}, \qquad \sigma \in \left\{ \frac{1}{3}, \frac{2}{3} \right\},$$

and a finite proper energy

$$E^{\mathrm{br}}_0 = h_{\mathrm{eff}} \, \omega_{\mathrm{br}}.$$

In the comoving frame of the composite core, the total longitudinal momentum is zero:

$$P^{\mathrm{br}}_0 = 0.$$

The closed branch therefore enters the framework of localized massive solutions of §7.1.

The multi-lobe structure and the internal separation δ^{br}_* modify the overlap forms

$$S_p[\bar{w}; \delta], \qquad S_{4,\mathrm{pair}}[\bar{w}; \delta], \qquad N_{\mathsf{F}}[\bar{w}; \delta],$$

as well as the closure denominator $\mathcal{D}^{\mathrm{br}}_{\mathrm{LO}}$. They do not modify the tangential hyperbolic structure of the reduced core. The characteristic velocity of the breather core is therefore

$$c_{\mathcal{F}} = c_t.$$

By (7.9), one obtains

$$E^{\mathrm{br}}(u) = \Gamma_t \, E^{\mathrm{br}}_0, \qquad P^{\mathrm{br}}(u) = \Gamma_t \frac{u}{c_t^2} E^{\mathrm{br}}_0, \qquad \Gamma_t = \frac{1}{\sqrt{1 - u^2/c_t^2}}.$$

The radial source associated with a closed breather branch is built from a finite sum of localized profiles and localized translational overlaps. In the comoving frame of the composite core, it remains a localized function of the common coordinate of the branch. The motif $\bar{w}$ modifies the source $S_{\mathcal{F}}$, but not the common radial operator.

When the linear-rank radial sheath response is considered, the criterion (7.3) therefore applies to the closed branch. The strictly comoving sheath is localized for

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

This kinematic compatibility holds for closed 3DOF breather branches, that is, for branches whose selection of δ^{br}_* gives an interior minimum of the positive admissible domain. It does not promote boundary branches into closed massive solutions.

Br2 lower bound. For the bilobe branch, the lower bound is defined by the non-degeneracy threshold

$$\boxed{\frac{J_0 - J_0^{\times}(2\delta_{\min}^{\mathrm{Br2}})}{J_0 + J_0^{\times}(2\delta_{\min}^{\mathrm{Br2}})} = \Gamma_{\mathrm{shape}}.} \tag{28.86}$$

This equation fixes $\delta_{\min}^{\mathrm{Br2}}$ from the same profile and the same normalization as the overlaps used in the closure.

Br3 lower bound. In a trilobe geometry

$$\{-\delta, 0, +\delta\},$$

the smallest inter-lobe distance is δ, while it is 2δ in the bilobe. We therefore define

$$\boxed{\delta_{\min}^{\mathrm{Br3}} = 2\,\delta_{\min}^{\mathrm{Br2}}.} \tag{28.87}$$

The common upper bound is the canonical MSE bound

$$\delta_{\max} = \frac{u_*^{\star}}{2},$$

defined in (27.17).

Reproducible numerical procedure for 3DOF breather minima

Purpose. For each retained integer motif, we determine the admissible separation δ_*, then the closure λ_*, and finally the reduced branch mass $m(\sigma)$, with

$$\sigma \in \left\{\frac{1}{3}, \frac{2}{3}\right\}.$$

The procedure uses the specialized closures (28.55), (28.68) and (28.82); it does not reintroduce any alternative closure.

Admissible window. The minimization is performed on the positive admissible domain specific to each branch:

$$I_{\mathrm{adm}}^{+,\mathrm{Br2}}, \qquad I_{\mathrm{adm}}^{+,\mathrm{Br3,n}}, \qquad I_{\mathrm{adm}}^{+,\mathrm{Br3,c}},$$

defined respectively by (28.57), (28.70) and (28.84). The common upper bound for strong overlap is

$$\delta_{\max} = \frac{u_*^{\star}}{2},$$

with $u_*^{\star}$ given by (27.16).

Step 1: elementary overlaps. For the canonical profile

$$a(\xi) = \operatorname{sech}\xi,$$

we evaluate the overlaps

$$J_0^{\times}(r), \qquad J_2^{\times}(r), \qquad J_4^{\times}(r),$$

with

$$J_p^{\times}(0) = J_p.$$

Step 2: branch forms. We use the specialized forms directly:

$$\text{Br2:} \qquad S_0^{\mathrm{Br2}},\ S_2^{\mathrm{Br2}},\ S_{4,\mathrm{pair}}^{\mathrm{Br2}},\ N_{\mathsf{F}}^{\mathrm{Br2}}$$

defined by (28.47), (28.48) and (28.49);

$$\text{neutral Br3:} \qquad S_0^{\mathrm{Br3,n}},\ S_2^{\mathrm{Br3,n}},\ S_{4,\mathrm{pair}}^{\mathrm{Br3,n}},\ N_{\mathsf{F}}^{\mathrm{Br3,n}}$$

defined by (28.60), (28.61) and (28.62);

$$\text{charged Br3:} \qquad S_0^{\mathrm{Br3,c}},\ S_2^{\mathrm{Br3,c}},\ S_{4,\mathrm{pair}}^{\mathrm{Br3,c}},\ N_{\mathsf{F}}^{\mathrm{Br3,c}}$$

defined by (28.74), (28.75) and (28.76).

Step 3: reduced quantities and impedances. For each branch $\mathrm{br} \in \{\mathrm{Br2}, \mathrm{Br3,n}, \mathrm{Br3,c}\}$, we form

$$\widehat{s}_2^{\mathrm{br}} = \frac{S_2^{\mathrm{br}}}{S_0^{\mathrm{br}}}, \qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{br}} = \frac{S_{4,\mathrm{pair}}^{\mathrm{br}}}{(S_0^{\mathrm{br}})^2}, \qquad \widehat{n}_{\mathsf{F}}^{\mathrm{br}} = \frac{N_{\mathsf{F}}^{\mathrm{br}}}{S_0^{\mathrm{br}}}.$$

The reduced impedances are then

$$\beta_{\mathrm{br}}^{(\mathsf{F})} = \zeta_{3D} \frac{\widehat{n}_{\mathsf{F}}^{\mathrm{br}}}{1 + \left(\eta_{\mathrm{br}}^{(3)}\right)^{-1} \widehat{s}_2^{\mathrm{br}}}, \qquad \beta_{\mathrm{br}}^{(\mathsf{P})} = \eta_{\mathrm{br}}^{(3)} \Gamma_{\mathrm{P}} \frac{\widehat{s}_{4,\mathrm{pair}}^{\mathrm{br}}}{\left(1 + \left(\eta_{\mathrm{br}}^{(3)}\right)^{-1} \widehat{s}_2^{\mathrm{br}}\right)^2},$$

and then

$$\beta_{\mathrm{br}}^{(\mathrm{eff})} = \beta_{\mathrm{br}}^{(\mathsf{F})} - \beta_{\mathrm{br}}^{(\mathsf{P})}.$$

Step 4: closure denominator. For each branch, we construct

$$K_{0,\mathrm{br}} = K_0^{(Z)}, \qquad K_{2,\mathrm{br}} = \widehat{s}_2^{\mathrm{br}} K_2^{(Z)}, \qquad K_{\mathrm{br}} = K_{0,\mathrm{br}} + K_{2,\mathrm{br}},$$

and then

$$\mathcal{D}_{\mathrm{LO}}^{\mathrm{br}} = \eta_{\mathrm{br}}^{(3)} \left(K_{\mathrm{br}} - \beta_{\mathrm{br}}^{(\mathrm{eff})} K_{0,\mathrm{br}}\right) - \beta_{\mathrm{br}}^{(\mathrm{eff})} K_{2,\mathrm{br}}.$$

The LO+ version is

$$\mathcal{D}_{\mathrm{LO+}}^{\mathrm{br}} = \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}} + \Gamma_{\mathrm{shape}} K_0^{(Z)}.$$

Step 5: selected separation. The branch separation is defined by

$$\delta_*^{\mathrm{Br2}} \in \arg \min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{Br2}}} \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br2}}(\delta),$$

$$\delta_*^{\mathrm{Br3,n}} \in \arg \min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{Br3,n}}} \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,n}}(\delta),$$

$$\delta_*^{\mathrm{Br3,c}} \in \arg \min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{Br3,c}}} \mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,c}}(\delta).$$

If the minimum is interior, it satisfies

$$\partial_\delta \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta_*) = 0, \qquad \partial_\delta^2 \mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta_*) > 0.$$

If the optimum lies on the boundary, it is retained as the constrained optimum of the positive admissible domain, not as an imposed hypothesis.

Step 6: closure λ_*. For the retained branch,

$$\boxed{\lambda_{*,\mathrm{br}}^4 = \frac{2h_{\mathrm{eff}}\sqrt{\rho T}}{\mathcal{D}_{\mathrm{LO}}^{\mathrm{br}}(\delta_*)}.} \tag{28.88}$$

At LO+ rank:

$$\boxed{\lambda_{*,\mathrm{br,LO+}}^4 = \frac{2h_{\mathrm{eff}}\sqrt{\rho T}}{\mathcal{D}_{\mathrm{LO+}}^{\mathrm{br}}(\delta_*)}.} \tag{28.89}$$

Step 7: reduced branch mass. The projected branch frequency is

$$\omega_{\mathrm{br}} = \sigma\,\frac{c_t}{\lambda_{*,\mathrm{br}}},$$

and the associated reduced mass is

$$\boxed{m_{\mathrm{br}}(\sigma) = \frac{h_{\mathrm{eff}}\sigma}{c_t\,\lambda_{*,\mathrm{br}}}.} \tag{28.90}$$

The LO+ version is obtained by replacing

$$\lambda_{*,\mathrm{br}} \longrightarrow \lambda_{*,\mathrm{br,LO+}}.$$

Step 8: reduced-charge diagnostic. When the reduced EM diagnostic is required, we use the integer monopole label of the raw motif

$$n_{\mathrm{EM}}(\bar{w}) = \sum_i \bar{w}_i.$$

The effective reduced branch charge is then computed through the form factor defined in the corresponding section. Motif branches $(1,-1)$ and $(1,-2,1)$ are neutral at monopole LO rank.

EM charge diagnostic (3DOF breathers). In strong overlap, the external charge is not additive in "bump charges": it must be reconstructed from the *raw integer motif* $\bar{w}$, before normalization by $S_0 = 1$. We first define the discrete monopole label

$$\boxed{n_{\mathrm{EM}}(\bar{w}) := \sum_i \bar{w}_i \in \mathbb{Z}.} \tag{28.91}$$

For the canonical motifs of the 3DOF breathers, we thus have

$$n_{\mathrm{EM}}^{\mathrm{Br2}} = 1 - 1 = 0, \qquad n_{\mathrm{EM}}^{\mathrm{Br3,n}} = 1 - 2 + 1 = 0, \qquad n_{\mathrm{EM}}^{\mathrm{Br3,c}} = 1 - 1 + 1 = 1.$$

We then introduce the reduced form factor of the breather

$$\boxed{\mathcal{C}_S^{(\mathrm{br})}(\bar{w};\delta) := \frac{S_0[\bar{w};\delta]}{S_0[\bar{w};\delta] + \left(\eta_{\mathrm{br}}^{(3)}\right)^{-1} S_2[\bar{w};\delta]}, \qquad \mathcal{C}_S^{(\mathrm{bump})} = \frac{J_0}{J_0 + \left(\eta_{\mathrm{br}}^{(3)}\right)^{-1} J_2}.} \tag{28.92}$$

The effective reduced charge is then defined by

$$\boxed{q_{\mathrm{eff}}^{(\mathrm{br})}(\bar{w};\delta,\sigma) = n_{\mathrm{EM}}(\bar{w})\,\sigma\,\frac{\mathcal{C}_S^{(\mathrm{br})}(\bar{w};\delta)}{\mathcal{C}_S^{(\mathrm{bump})}}.} \tag{28.93}$$

This definition is consistent with the fact that: (i) the neutral branches satisfy $n_{\mathrm{EM}} = 0$ identically; (ii) the charged-like branch carries a unit monopole label; (iii) the effective amplitude is modulated by the form factor $\mathcal{C}_S^{(\mathrm{br})}/\mathcal{C}_S^{(\mathrm{bump})}$ and by the conical projection σ.

Conversely, for the two neutral branches Br2 and Br3,n, one has identically

$$q_{\mathrm{eff}}^{(\mathrm{br})} = 0.$$

For the canonical Br3,c reduction, the quantity $q_{\mathrm{eff}}^{(\mathrm{br})}$ is a reduced branch amplitude. The complete construction of charged families requires the same internal diagonalization as that used for the neutral branches.

28.6 Z-like 3DOF multi-lobes (short-range, color-neutral) – weight form

Channel hypothesis (strong-like composite). In the *composite* Z-like 3DOF channel (color-white), the strong-like quartic term is *attractive* and competes with a repulsive Preclusion contribution. The existence of the composite comes from the internal closure and from the positivity of the closure denominator, without any need to introduce a new free constant.

Common shape constants. The canonical profile is $a(\xi) = \operatorname{sech}\xi$. The profile constants J_0 and J_2 are those defined by the profile shape moments and are evaluated, for $a = \operatorname{sech}$, by

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 \simeq 7.6339320582.$$

We set

$$K_0^{(Z)} = J_0^2, \qquad K_2^{(Z)} = J_0 J_2.$$

The Preclusion constant Γ_{P} is defined in (20.59) and, for the profile $a(\xi) = \operatorname{sech}\xi$, evaluated in (20.60). The sheath constant Γ_{shape} is that of (23.51). The 3DOF strong-like factor ζ_{3D} is defined in (27.28) and evaluated in (27.33).

Status of the internal motif. As in the breather branch, the internal motif must be separated into:

- a *raw integer motif* $\bar{w} \in \mathbb{Z}^k$, which carries the discrete labels (symmetry, neutrality, charge);
- a *normalized representative* $\widehat{w}(\delta)$, which carries only the internal relative shape at fixed geometry.

We therefore adopt the normalization gauge

$$\boxed{S_0[w;\delta] = 1.} \tag{28.94}$$

For a raw motif $\bar{w}$, one passes to the normalized representative by

$$\boxed{\widehat{w}(\delta) = \frac{\bar{w}}{\sqrt{S_0[\bar{w};\delta]}}.} \tag{28.95}$$

Raw strong-like numerator and common reduced invariants. The quadratic forms $S_p[w;\delta]$, the pairwise quartic reduction $S_{4,\mathrm{pair}}[w;\delta]$, and the strong-like numerator $N_{\mathsf{F}}[w;\delta]$ are those defined in (28.34), (28.35) and (28.37). The pairwise coupling rule of the strong-like channel is that of (28.36).

We introduce the raw numerator of the attractive channel in the form

$$\boxed{N_{\mathsf{F}}[\bar{w};\delta] := \sum_{i<j} c_{ij}^{(\mathsf{F})}(\bar{w})\, J_4^{\times}(\Delta_{ij}), \qquad \Delta_{ij} := |\xi_i - \xi_j|.} \tag{28.96}$$

The reduced closure quantities must then be expressed in terms that are invariant under a global rescaling of the motif:

$$\boxed{\widehat{s}_2(\bar{w};\delta) := \frac{S_2[\bar{w};\delta]}{S_0[\bar{w};\delta]}, \qquad \widehat{s}_{4,\mathrm{pair}}(\bar{w};\delta) := \frac{S_{4,\mathrm{pair}}[\bar{w};\delta]}{S_0[\bar{w};\delta]^2}, \qquad \widehat{n}_{\mathsf{F}}(\bar{w};\delta) := \frac{N_{\mathsf{F}}[\bar{w};\delta]}{S_0[\bar{w};\delta]}.} \tag{28.97}$$

The strong-like numerator N_{F} uses the pairwise coupling rule defined in (28.37).

We then set

$$K_0^{(\mathrm{red})}(\bar{w};\delta) = K_0^{(Z)}, \qquad K_2^{(\mathrm{red})}(\bar{w};\delta) = \widehat{s}_2(\bar{w};\delta)\, K_2^{(Z)}, \qquad K^{(\mathrm{red})}(\bar{w};\delta) = K_0^{(\mathrm{red})} + K_2^{(\mathrm{red})}.$$

Channel impedances and common Z-like closure. In the Z-like 3DOF branch, the short-range response channel must not be written directly with the dimensional coefficient

$$\kappa_\mu^{\mathrm{phys}} := \frac{\mu_2}{T}, \qquad [\kappa_\mu^{\mathrm{phys}}] = \mathrm{m}^{-2}.$$

After reduction to the core scale λ_*, the combination that enters the response factors is the dimensionless reduced parameter

$$\boxed{\eta_\mu^{(Z)}(\lambda_*) := \frac{\kappa_\mu^{\mathrm{phys}}}{\kappa_r}\,\lambda_*^2.} \tag{28.98}$$

This definition corresponds to the case where the short-range response is carried by the standard radial operator

$$-T\kappa_r\Delta + \mu_2.$$

This definition explicitly distinguishes the dimensional coefficient $\kappa_\mu^{\mathrm{phys}} = \mu_2/T$ from the dimensionless combination $\eta_\mu^{(Z)}(\lambda_*)$ that enters the reduced closure.

At reduced rank, the channel impedances are then written

$$\boxed{\beta_{(Z)}^{(\mathsf{P})}(\bar{w};\delta;\eta) = \eta\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}_{4,\mathrm{pair}}(\bar{w};\delta)}{\big(1+\eta^{-1}\widehat{s}_2(\bar{w};\delta)\big)^2}, \qquad \beta_{(Z)}^{(\mathsf{F})}(\bar{w};\delta;\eta) = \zeta_{3D}\,\frac{\widehat{n}_{\mathsf{F}}(\bar{w};\delta)}{1+\eta^{-1}\widehat{s}_2(\bar{w};\delta)}.} \tag{28.99}$$

where, in the complete reduced closure, one takes

$$\eta = \eta_\mu^{(Z)}(\lambda_*).$$

We define

$$\boxed{\beta_{(Z)}^{(\mathrm{eff})}(\bar{w};\delta;\eta) = \beta_{(Z)}^{(\mathsf{F})}(\bar{w};\delta;\eta) - \beta_{(Z)}^{(\mathsf{P})}(\bar{w};\delta;\eta).} \tag{28.100}$$

The reduced closure denominator is then

$$\boxed{D_{\mathrm{LO}}^{(Zk)}(\bar{w};\delta;\eta) = \eta\Big(K^{(\mathrm{red})}(\bar{w};\delta) - \beta_{(Z)}^{(\mathrm{eff})}(\bar{w};\delta;\eta)\,K_0^{(Z)}\Big) - \beta_{(Z)}^{(\mathrm{eff})}(\bar{w};\delta;\eta)\,K_2^{(\mathrm{red})}(\bar{w};\delta), \qquad k\in\{2,3\}.} \tag{28.101}$$

The complete reduced closure is obtained by replacing

$$\eta \mapsto \eta_\mu^{(Z)}(\lambda_*).$$

Consequently, the width closure is no longer a simple explicit evaluation at a constant parameter: it becomes the implicit equation

$$\boxed{\lambda_*^4\, D_{\mathrm{LO}}^{(Zk)}\Big(\bar{w};\delta_*;\eta_\mu^{(Z)}(\lambda_*)\Big) = 2\,h_{\mathrm{eff}}\sqrt{\rho T}.} \tag{28.102}$$

With sheath self-induction, one writes

$$\boxed{D_{\mathrm{LO}+}^{(Zk)}(\bar{w};\delta;\eta) = D_{\mathrm{LO}}^{(Zk)}(\bar{w};\delta;\eta) + \Gamma_{\mathrm{shape}} K_0^{(Z)},} \tag{28.103}$$

and the LO+ closure becomes

$$\boxed{\lambda_*^4\, D_{\mathrm{LO}+}^{(Zk)}\Big(\bar{w};\delta_*;\eta_\mu^{(Z)}(\lambda_*)\Big) = 2\,h_{\mathrm{eff}}\sqrt{\rho T}.} \tag{28.104}$$

28.6.1 Charge rule (EM) – common Z-like

The discrete *monopole label* must be carried by the *raw integer motif*:

$$\boxed{n_{\mathrm{EM}}(\bar{w}) = \sum_{i=1}^{k} \bar{w}_i \in \mathbb{Z}, \qquad k \in \{2,3\}.} \tag{28.105}$$

Thus, combinations with zero sum are *neutral* at LO monopole order, whereas those with sum $n \neq 0$ carry a charged label. The reduced definition stops at the integer $n_{\mathrm{EM}}(\bar{w})$; no dimensionful conversion is introduced in this section.

28.6.2 Internal variational principle: determination of $\bar{w}$ (common Z-like)

As in the breather branch, the internal motifs are not imposed arbitrarily: at fixed geometry δ, they are determined by minimization of the closure denominator *within an admissible class.*

Gram matrix and non-degeneracy. We write

$$\boxed{G_{ij}(\delta) := \langle a(\xi-\xi_i),\, a(\xi-\xi_j)\rangle = J_0^{\times}(\Delta_{ij}), \qquad \Delta_{ij} = |\xi_i - \xi_j|.} \tag{28.106}$$

Non-degeneracy of the composite class requires that the Gram matrix associated with the retained geometry remain positive definite:

$$G(\delta) \succ 0.$$

Admissible class (raw primitive motifs). We work with *primitive* motifs (with no common factor), and impose: (i) a charge sector through (28.105), (ii) Gram non-degeneracy, (iii) possibly a branch-symmetry constraint. We define

$$\mathcal{W}_k^{\mathrm{adm}}(n;\delta) := \Big\{\bar{w}\in\mathbb{Z}^k :\, n_{\mathrm{EM}}(\bar{w}) = n,\ \gcd(\bar{w}_1,\dots,\bar{w}_k) = 1,\ G(\delta)\succ 0,\ (\text{branch constraints})\Big\}.$$

Internal selection (definition). The internal selection is written at fixed reduced parameter η:

$$\boxed{\bar{w}_*(\delta;\eta) \in \arg \min_{\bar{w}\in \mathcal{W}_k^{\mathrm{adm}}(n;\delta)} D_{\mathrm{LO}}^{(Zk)}(\bar{w};\delta;\eta), \qquad k=2,3.} \tag{28.107}$$

The full reduced closure is then obtained by the substitution

$$\eta = \eta_\mu^{(Z)}(\lambda_*),$$

defined in (28.98). Thus, motif selection is written in the reduced space, whereas the width closure then imposes the implicit equation in λ_*.

Neutral cases: for $n = 0$, the constraints may be sufficient to reduce $\mathcal{W}_k^{\mathrm{adm}}$ to a unique primitive motif ($\bar{w} = (1,-1)$ in the antisymmetric bi-lobe; $\bar{w} = (1,-2,1)$ in the symmetric tri-lobe).

Charged cases: for $n \neq 0$, several admissible primitive motifs may exist; the canonical motifs used below are practical representatives, while the final selection remains the one obtained from (28.107).

28.6.3 Strong sheath self-induction (common Z-like)

The sheath self-induction mechanism (3DOF) applies *without any new free constant.* The dimensionless factor Γ_{shape} is reused. In the reduced convention, the LO+ correction is written

$$D_{\mathrm{LO}+}^{(Zk)}(\bar{w};\delta;\eta) = D_{\mathrm{LO}}^{(Zk)}(\bar{w};\delta;\eta) + \Gamma_{\mathrm{shape}}\, K_0^{(Z)}.$$

Setting $\Gamma_{\mathrm{shape}} = 0$ restores the LO closures without sheath. The full reduced closure is then obtained by the substitution

$$\eta = \eta_\mu^{(Z)}(\lambda_*).$$

28.6.4 EBK filter and admissible values of q (Z-like 3DOF)

EBK rule (reminder). At leading order (single ring, self-conjugate state), the mechanical branch energy satisfies

$$E_0 = q\, h_{\mathrm{eff}}\, \omega_{\mathrm{br}}, \qquad q \in \mathbb{N},$$

where $\omega_{\mathrm{br}} = \sigma\, \omega_0$ (conical 3DOF, $\sigma = \sin^2\alpha \in \{\frac{1}{3}, \frac{2}{3}\}$).

The winding number of the tangent rotor is $\nu = 1$ and must not be confused with the raw motif $\bar{w}$.

$\mathbb{S}^3$ + MSE filter. Under the LO assumptions (thin tube $\varepsilon \ll 1$, admissible window $u_* \in [1.7, 2.1]$, regular sheath), the first mode that simultaneously satisfies the winding $\nu = 1$, the C^1 matching, and the hierarchy $\ell_0 \ll \lambda_* \ll R_0$ is the **fundamental**:

$$q_{\mathrm{adm}}^{(\mathrm{LO})} = 1.$$

28.6.5 Geometric spin and projection (common Z-like 3DOF)

Geometric class. In the Z-like 3DOF channel, the fast tangent cycle remains fundamental:

$$\nu = 1,$$

where ν denotes here the winding of the tangent rotor and must not be confused with the unnormalized motif $\bar{w}$. The LO reduction retained in this section keeps only the *transverse vector* content of the family, with two independent circular polarizations:

$$N_{\mathrm{pol}} = 2.$$

The geometric counting then gives

$$s_g = \frac{\nu N_{\mathrm{pol}}}{2} = \frac{1 \times 2}{2} = 1.$$

The Z-like 3DOF family therefore belongs to the integer class.

Spin norm. At leading order, the energy law of the branch keeps the form

$$E_0 = h_{\mathrm{eff}}\, \omega_{\mathrm{br}},$$

so that the geometric norm of the spin reads

$$\boxed{|S^{(Z3)}| = h_{\mathrm{eff}}, \qquad s_g^{(Z3)} = 1.}$$

Projection and helicity. The conical axis provides a preferred projection direction. In the transverse LO reduction retained here, the Z-like 3DOF family then carries two opposite helicities

$$\varsigma = \pm 1,$$

and the projection of the spin on the conical axis reads

$$\boxed{S_{\parallel}^{(Z3)} = \varsigma\, h_{\mathrm{eff}}, \qquad \varsigma = \pm 1.}$$

A possible longitudinal mode, if it appears at some order or in a richer closure, does not belong to the present LO reduction.

Scope. This description applies to the Z-like 3DOF family as a transverse vector branch. It applies to the bi-lobe and tri-lobe realizations without modifying the spin class:

$$s_g^{(Z3)} = 1.$$

Differences between motifs $\bar{w}$ affect the EM charge, the internal weights, and the closure, but not the integer nature of the spin at leading order.

Remark distinguishing it from the Z-like 2DOF. The present Z-like 3DOF family must not be confused with the Z-like 2DOF branch: the two share the same vector class $s_g = 1$ at leading order, but differ in their internal architecture, their composite closure, and their space of admissible motifs.

Kinematic compatibility in motion. A closed Z-like 3DOF branch is a strong-overlap multi-lobe composite solution. The lobes are not read as independent solitons: they form a single effective core, described by a common comoving coordinate and by a normalized motif $\bar{w}$.

In the Z2 and Z3 specializations below, a branch is closed only if the minimization of its closure denominator selects an interior minimum of the positive admissible domain. When this condition is satisfied, the branch has a closed width $\lambda_{*,Z}$, a projected proper frequency

$$\omega_Z = \sigma\, \frac{c_t}{\lambda_{*,Z}}, \qquad \sigma \in \left\{ \frac{1}{3}, \frac{2}{3} \right\},$$

and a finite proper energy

$$E_0^{(Z)} = h_{\mathrm{eff}}\, \omega_Z.$$

In the comoving frame of the composite core, the total longitudinal momentum is zero:

$$P_0^{(Z)} = 0.$$

The closed branch therefore enters the framework of the localized massive solutions of §7.1.

The multi-lobe structure, the motif $\bar{w}$, and the internal separation $\delta_*^{(Z)}$ modify the overlap forms

$$S_p[\bar{w};\delta], \qquad S_{4,\mathrm{pair}}[\bar{w};\delta], \qquad N_{\mathsf{F}}[\bar{w};\delta],$$

as well as the channel impedances and the closure denominator

$$D_{\mathrm{LO}}^{(Zk)}.$$

They do not modify the tangent hyperbolic structure of the reduced core. The characteristic speed of the closed Z-like core is therefore

$$c_{\mathcal{F}} = c_t.$$

By (7.9), one obtains

$$E^{(Z)}(u) = \Gamma_t\, E_0^{(Z)}, \qquad P^{(Z)}(u) = \Gamma_t \frac{u}{c_t^2} E_0^{(Z)}, \qquad \Gamma_t = \frac{1}{\sqrt{1-u^2/c_t^2}}.$$

The effective dispersion of the composite core is

$$\left(E^{(Z)}(u)\right)^2 = \left(E_0^{(Z)}\right)^2 + c_t^2 \left(P^{(Z)}(u)\right)^2.$$

The radial source associated with a closed Z-like branch is built from a finite sum of localized profiles and localized translational overlaps. In the comoving frame of the composite core, it remains a localized function of the common coordinate of the branch. The motif $\bar{w}$ modifies the source $S_{\mathcal{F}}$, but not the common radial operator.

When the radial sheath response is considered at linear order, the criterion (7.3) therefore applies to the closed branch. The strictly comoving sheath is localized for

$$|u| < c_r, \qquad c_r^2 = \kappa_r c_t^2.$$

For $|u| \geq c_r$, the radial response no longer belongs to a strictly comoving Yukawa sheath at the same order.

This kinematic compatibility applies only to closed Z-like 3DOF branches, that is, those for which the selection of δ_* gives an interior minimum of the positive admissible domain. It does not promote boundary branches to closed massive solutions.

28.6.6 Z-like 3DOF bi-lobe

Canonical motif and raw kernels (bi-lobe). For the raw primitive motif

$$\bar{w}_{\mathrm{Z2}} = (1,-1),$$

placed at $\{-\delta,+\delta\}$ (separation $\Delta_{12} = 2\delta$), one has

$$S_0^{\mathrm{Z2}}(\delta) = 2\Big(J_0 - J_0^{\times}(2\delta)\Big), \qquad S_2^{\mathrm{Z2}}(\delta) = 2\Big(J_2 - J_2^{\times}(2\delta)\Big).$$

The associated normalized representative is

$$\widehat{w}_{\mathrm{Z2}}(\delta) = \frac{(1,-1)}{\sqrt{S_0^{\mathrm{Z2}}(\delta)}}.$$

Discrete charge (bi-lobe). The monopole label is carried by the raw motif:

$$n_{\mathrm{EM}}(\bar{w}_{\mathrm{Z2}}) = 1 - 1 = 0.$$

The canonical Z2 branch is therefore strictly neutral at the level of the discrete label.

Reduced invariants (bi-lobe). For Z2, we introduce

$$\boxed{\widehat{s}_2^{\mathrm{Z2}}(\delta) := \frac{S_2^{\mathrm{Z2}}(\delta)}{S_0^{\mathrm{Z2}}(\delta)}, \qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{Z2}}(\delta) := \frac{S_{4,\mathrm{pair}}[(1,-1);\delta]}{\big(S_0^{\mathrm{Z2}}(\delta)\big)^2}, \qquad \widehat{n}_{\mathsf{F}}^{\mathrm{Z2}}(\delta) := \frac{N_{\mathsf{F}}[(1,-1);\delta]}{S_0^{\mathrm{Z2}}(\delta)}.} \tag{28.108}$$

The objects $S_{4,\mathrm{pair}}$ and N_{F} are those defined in (28.35) and (28.37).

Admissible bounds (bi-lobe). We impose

$$\delta \in [\delta_{\min}^{\mathrm{Z2}}, \delta_{\max}].$$

The lower bound $\delta_{\min}^{\mathrm{Z2}} > 0$ is imposed by the Gram condition (exclusion of the degeneracy at $\delta = 0$); in the canonical bi-lobe case, it amounts to imposing quantitatively

$$S_0^{\mathrm{Z2}}(\delta) = 2\big(J_0 - J_0^{\times}(2\delta)\big) > 0.$$

The upper bound is the canonical MSE bound defined in (27.17).

Reduced closure (Z2). We set

$$K_{0,\mathrm{Z2}}^{(\mathrm{red})} = K_0^{(Z)}, \qquad K_{2,\mathrm{Z2}}^{(\mathrm{red})}(\delta) = \widehat{s}_2^{\mathrm{Z2}}(\delta)\, K_2^{(Z)}, \qquad K_{\mathrm{Z2}}^{(\mathrm{red})}(\delta) = K_{0,\mathrm{Z2}}^{(\mathrm{red})} + K_{2,\mathrm{Z2}}^{(\mathrm{red})}(\delta).$$

The response parameter entering the reduced closure is the dimensionless parameter η. In the self-consistent closure, it will be replaced by $\eta_\mu^{(Z)}(\lambda_*)$, cf. (28.98). We define

$$\boxed{\beta_{\mathrm{Z2}}^{(\mathsf{P})}(\delta;\eta) = \eta\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}_{4,\mathrm{pair}}^{\mathrm{Z2}}(\delta)}{\big(1+\eta^{-1}\widehat{s}_2^{\mathrm{Z2}}(\delta)\big)^2}, \qquad \beta_{\mathrm{Z2}}^{(\mathsf{F})}(\delta;\eta) = \zeta_{3D}\,\frac{\widehat{n}_{\mathsf{F}}^{\mathrm{Z2}}(\delta)}{1+\eta^{-1}\widehat{s}_2^{\mathrm{Z2}}(\delta)}.} \tag{28.109}$$

We define

$$\beta_{\mathrm{Z2}}^{(\mathrm{eff})}(\delta;\eta) = \beta_{\mathrm{Z2}}^{(\mathsf{F})}(\delta;\eta) - \beta_{\mathrm{Z2}}^{(\mathsf{P})}(\delta;\eta),$$

then

$$\boxed{D_{\mathrm{LO}}^{(Z2)}(\bar{w}_{\mathrm{Z2}};\delta;\eta) = \eta\Big(K_{\mathrm{Z2}}^{(\mathrm{red})}(\delta) - \beta_{\mathrm{Z2}}^{(\mathrm{eff})}(\delta;\eta)\,K_0^{(Z)}\Big) - \beta_{\mathrm{Z2}}^{(\mathrm{eff})}(\delta;\eta)\,K_{2,\mathrm{Z2}}^{(\mathrm{red})}(\delta).} \tag{28.110}$$

The LO+ closure reads

$$\boxed{D_{\mathrm{LO+}}^{(Z2)}(\bar{w}_{\mathrm{Z2}};\delta;\eta) = D_{\mathrm{LO}}^{(Z2)}(\bar{w}_{\mathrm{Z2}};\delta;\eta) + \Gamma_{\mathrm{shape}}K_0^{(Z)}.} \tag{28.111}$$

Positive admissible domain and optimum. At fixed reduced parameter η, we introduce

$$\boxed{I_{\mathrm{adm}}^{+,\mathrm{Z2}}(\eta) = \Big\{\delta \in [\delta_{\min}^{\mathrm{Z2}}, \delta_{\max}] \,:\, D_{\mathrm{LO}}^{(Z2)}(\bar{w}_{\mathrm{Z2}};\delta;\eta) > 0\Big\}.} \tag{28.112}$$

The internal optimum is then defined by

$$\boxed{\delta_*^{\mathrm{Z2}}(\eta) \in \arg\min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{Z2}}(\eta)} D_{\mathrm{LO}}^{(Z2)}(\bar{w}_{\mathrm{Z2}};\delta;\eta).} \tag{28.113}$$

At LO+, the addition $+\Gamma_{\mathrm{shape}}K_0^{(Z)}$ is independent of δ, so the optimum $\delta_*^{\mathrm{Z2}}(\eta)$ is unchanged at fixed η. In the self-consistent closure, η is then replaced by $\eta_\mu^{(Z)}(\lambda_*)$.

MSE upper bound and branch selection. The upper bound of the admissible domain is the canonical MSE bound

$$\delta_{\max} = \frac{u_*^\star}{2},$$

with $u_*^\star$ defined by (27.16). Thus

$$\delta_{\max} = 0.93913003995\ldots.$$

The branch separation is not imposed at this bound. It is the one defined by the minimization already stated in (28.113):

$$\delta_*^{\mathrm{Z2}}(\eta) \in \arg\min_{\delta \in I_{\mathrm{adm}}^{+,\mathrm{Z2}}(\eta)} D_{\mathrm{LO}}^{(Z2)}(\bar{w}_{\mathrm{Z2}}; \delta; \eta).$$

The Z2 branch is closed at LO only if this minimum is interior:

$$\delta_*^{\mathrm{Z2}}(\eta) \in (\delta_{\min}^{\mathrm{Z2}}, \delta_{\max}).$$

If the minimization gives a boundary optimum, the Z2 branch is not promoted as an internal stationary solution at LO.

Closures and mass (Z2). The self-consistent closure imposes

$$\eta = \eta_\mu^{(Z)}(\lambda_*).$$

At LO, the width therefore satisfies the implicit equation

$$\boxed{\lambda_*^4\, D_{\mathrm{LO}}^{(Z2)}\Big(\bar{w}_{\mathrm{Z2}}; \delta_*^{\mathrm{Z2}}\Big(\eta_\mu^{(Z)}(\lambda_*)\Big); \eta_\mu^{(Z)}(\lambda_*)\Big) = 2\, h_{\mathrm{eff}} \sqrt{\rho\, T}.} \tag{28.114}$$

At LO+,

$$\boxed{\lambda_*^4\, D_{\mathrm{LO+}}^{(Z2)}\Big(\bar{w}_{\mathrm{Z2}}; \delta_*^{\mathrm{Z2}}\Big(\eta_\mu^{(Z)}(\lambda_*)\Big); \eta_\mu^{(Z)}(\lambda_*)\Big) = 2\, h_{\mathrm{eff}} \sqrt{\rho\, T}.} \tag{28.115}$$

Then

$$\omega_{0,*} = \frac{c_t}{\lambda_*}, \qquad \omega_{\mathrm{br},*} = \sigma\, \omega_{0,*}, \qquad m_{Z2}(\sigma) = \frac{h_{\mathrm{eff}}\, \sigma}{c_t\, \lambda_*}.$$

The factor $\sigma \in \{\frac{1}{3}, \frac{2}{3}\}$ is that of the conical 3DOF projection retained in the present Z-like family.

Existence condition (Z2). The canonical Z2 branch exists as an internal stationary solution at LO if, at the self-consistent solution λ_*,

$$\delta_*^{\mathrm{Z2}}\Big(\eta_\mu^{(Z)}(\lambda_*)\Big) \in (\delta_{\min}^{\mathrm{Z2}}, \delta_{\max}),$$

and if

$$D_{\mathrm{LO+}}^{(Z2)}\Big(\bar{w}_{\mathrm{Z2}}; \delta_*^{\mathrm{Z2}}\Big(\eta_\mu^{(Z)}(\lambda_*)\Big); \eta_\mu^{(Z)}(\lambda_*)\Big) > 0.$$

If the minimization selects a boundary of the admissible domain, the Z2 branch is classified as a boundary branch and is not used as a closed prediction at LO.

28.6.7 Z-like 3DOF tri-lobe

Canonical motifs and raw kernels (tri-lobe). For the positions $\{-\delta, 0, +\delta\}$, the quadratic kernels depend on the *raw integer motif* $\bar{w}$. For $p \in \{0, 2\}$:

$$\boxed{S_p[\bar{w}; \delta] = \sum_{i=1}^{3} \bar{w}_i^2 \, J_p + 2 \sum_{1 \le i < j \le 3} \bar{w}_i \bar{w}_j \, J_p^{\times}(\Delta_{ij}), \qquad \{\Delta_{ij}\} = \{\delta, \delta, 2\delta\}.}$$

The associated normalized representative is

$$\widehat{w}(\delta) = \frac{\bar{w}}{\sqrt{S_0[\bar{w}; \delta]}}.$$

Admissible bounds (tri-lobe). We impose

$$\delta \in [\delta_{\min}^{\mathrm{Z3}}, \delta_{\max}].$$

The upper bound is the canonical MSE bound defined in (27.17).
and $\delta_{\min}^{\mathrm{Z3}} > 0$ is imposed by the Gram condition (exclusion of degeneracy at $\delta = 0$).

Reduced invariants (tri-lobe). For any raw motif $\bar{w}$, we introduce

$$\boxed{\widehat{s}_2^{\mathrm{Z3}}(\bar{w}; \delta) := \frac{S_2[\bar{w}; \delta]}{S_0[\bar{w}; \delta]}, \qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{Z3}}(\bar{w}; \delta) := \frac{S_{4,\mathrm{pair}}[\bar{w}; \delta]}{S_0[\bar{w}; \delta]^2}, \qquad \widehat{n}_{\mathsf{F}}^{\mathrm{Z3}}(\bar{w}; \delta) := \frac{N_{\mathsf{F}}[\bar{w}; \delta]}{S_0[\bar{w}; \delta]}.} \tag{28.116}$$

The objects $S_{4,\mathrm{pair}}$ and N_{F} are those defined in (28.35) and (28.37). For the bi-lobe motif $\bar{w}_{\mathrm{Z2}} = (1, -1)$, the explicit specializations coincide with the canonical forms (28.48) and (28.49).

Canonical cases (positions $\{-\delta, 0, +\delta\}$): *representative* motifs. Below we give simple raw primitive motifs, useful as reference points and as initializations. In the symmetric neutral sector, the constraints reduce the admissible class to a single primitive motif; in the charged sector, the final canonical selection remains that of (28.107).

- **Neutral tri-lobe (symmetric branch)** $\bar{w} = (1, -2, 1)$:

$$S_0[(1, -2, 1); \delta] = 6J_0 - 8J_0^{\times}(\delta) + 2J_0^{\times}(2\delta),$$

$$S_2[(1, -2, 1); \delta] = 6J_2 - 8J_2^{\times}(\delta) + 2J_2^{\times}(2\delta),$$

$$n_{\mathrm{EM}}(1, -2, 1) = 1 - 2 + 1 = 0.$$

- **Charged tri-lobe (sector $n = 1$)** $\bar{w} = (1, -1, 1)$:

$$S_0[(1, -1, 1); \delta] = 3J_0 - 4J_0^{\times}(\delta) + 2J_0^{\times}(2\delta),$$

$$S_2[(1, -1, 1); \delta] = 3J_2 - 4J_2^{\times}(\delta) + 2J_2^{\times}(2\delta),$$

$$n_{\mathrm{EM}}(1, -1, 1) = 1 - 1 + 1 = 1.$$

- **Non-primitive motif (redundant)** $\bar{w} = (2, -2, 2)$:

$$S_0[(2, -2, 2); \delta] = 4\, S_0[(1, -1, 1); \delta], \qquad S_2[(2, -2, 2); \delta] = 4\, S_2[(1, -1, 1); \delta],$$

but $\gcd(2, -2, 2) = 2$: this motif is *not* primitive, and it describes the same direction as $(1, -1, 1)$. It therefore does not define a distinct new Z3 state.

Reduced closure (Z3). We set

$$K_{0,\mathrm{Z3}}^{(\mathrm{red})} = K_0^{(Z)}, \qquad K_{2,\mathrm{Z3}}^{(\mathrm{red})}(\bar{w};\delta) = \widehat{s}_2^{\mathrm{Z3}}(\bar{w};\delta)\, K_2^{(Z)},$$

$$K_{\mathrm{Z3}}^{(\mathrm{red})}(\bar{w};\delta) = K_{0,\mathrm{Z3}}^{(\mathrm{red})} + K_{2,\mathrm{Z3}}^{(\mathrm{red})}(\bar{w};\delta).$$

The reduced response parameter is the dimensionless parameter η, replaced in the self-consistent closure by $\eta_\mu^{(Z)}(\lambda_*)$. We define

$$\boxed{\beta_{\mathrm{Z3}}^{(\mathsf{P})}(\bar{w};\delta;\eta) = \eta\,\Gamma_{\mathrm{P}}\,\frac{\widehat{s}_{4,\mathrm{pair}}^{\mathrm{Z3}}(\bar{w};\delta)}{\big(1+\eta^{-1}\widehat{s}_2^{\mathrm{Z3}}(\bar{w};\delta)\big)^2}, \qquad \beta_{\mathrm{Z3}}^{(\mathsf{F})}(\bar{w};\delta;\eta) = \zeta_{3D}\,\frac{\widehat{n}_{\mathsf{F}}^{\mathrm{Z3}}(\bar{w};\delta)}{1+\eta^{-1}\widehat{s}_2^{\mathrm{Z3}}(\bar{w};\delta)}.} \tag{28.117}$$

We define

$$\beta_{\mathrm{Z3}}^{(\mathrm{eff})}(\bar{w};\delta;\eta) = \beta_{\mathrm{Z3}}^{(\mathsf{F})}(\bar{w};\delta;\eta) - \beta_{\mathrm{Z3}}^{(\mathsf{P})}(\bar{w};\delta;\eta),$$

then

$$\boxed{D_{\mathrm{LO}}^{(Z3)}(\bar{w};\delta;\eta) = \eta\Big(K_{\mathrm{Z3}}^{(\mathrm{red})}(\bar{w};\delta) - \beta_{\mathrm{Z3}}^{(\mathrm{eff})}(\bar{w};\delta;\eta)\,K_0^{(Z)}\Big) - \beta_{\mathrm{Z3}}^{(\mathrm{eff})}(\bar{w};\delta;\eta)\,K_{2,\mathrm{Z3}}^{(\mathrm{red})}(\bar{w};\delta).} \tag{28.118}$$

The LO+ closure is written

$$\boxed{D_{\mathrm{LO+}}^{(Z3)}(\bar{w};\delta;\eta) = D_{\mathrm{LO}}^{(Z3)}(\bar{w};\delta;\eta) + \Gamma_{\mathrm{shape}}K_0^{(Z)}.} \tag{28.119}$$

Definition of the optimum. With the reduced denominator $D_{\mathrm{LO}}^{(Z3)}(\bar{w};\delta;\eta)$, at fixed η we set

$$\boxed{\delta_*(\bar{w};\eta) \in \arg\min_{\delta\in I_{\mathrm{adm}}^{+,\mathrm{Z3}}(\bar{w};\eta)} D_{\mathrm{LO}}^{(Z3)}(\bar{w};\delta;\eta),}$$

where

$$\boxed{I_{\mathrm{adm}}^{+,\mathrm{Z3}}(\bar{w};\eta) = \Big\{\delta\in[\delta_{\min}^{\mathrm{Z3}},\,\delta_{\max}] \,:\, D_{\mathrm{LO}}^{(Z3)}(\bar{w};\delta;\eta) > 0\Big\}.} \tag{28.120}$$

At LO+ (sheath), $+\Gamma_{\mathrm{shape}}K_0^{(Z)}$ is independent of δ; the optimum is unchanged at fixed η.

MSE upper bound and branch selection. The upper bound of the admissible domain is the canonical MSE bound

$$\delta_{\max} = \frac{u_*^\star}{2},$$

with $u_*^\star$ defined by (27.16). Thus

$$\delta_{\max} = 0.93913003995\ldots.$$

The branch separation is not imposed at this bound. It is defined by minimization over the positive admissible domain:

$$\delta_*(\bar{w};\eta) \in \arg\min_{\delta\in I_{\mathrm{adm}}^{+,\mathrm{Z3}}(\bar{w};\eta)} D_{\mathrm{LO}}^{(Z3)}(\bar{w};\delta;\eta).$$

The Z3 branch is closed at LO order only if this minimum is interior:

$$\delta_*(\bar{w};\eta) \in (\delta_{\min}^{\mathrm{Z3}}, \delta_{\max}).$$

If the minimization gives a boundary optimum, the Z3 branch is not promoted as an internal stationary solution at LO order.

Closures and mass (Z3). The self-consistent closure imposes

$$\eta = \eta_\mu^{(Z)}(\lambda_*).$$

At LO order, the width therefore satisfies the implicit equation

$$\boxed{\lambda_*^4\, D_{\mathrm{LO}}^{(Z3)}\Big(\bar{w};\delta_*(\bar{w};\eta_\mu^{(Z)}(\lambda_*));\eta_\mu^{(Z)}(\lambda_*)\Big) = 2\,h_{\mathrm{eff}}\sqrt{\rho\,T}.} \tag{28.121}$$

At LO+ order,

$$\boxed{\lambda_*^4\, D_{\mathrm{LO+}}^{(Z3)}\Big(\bar{w};\delta_*(\bar{w};\eta_\mu^{(Z)}(\lambda_*));\eta_\mu^{(Z)}(\lambda_*)\Big) = 2\,h_{\mathrm{eff}}\sqrt{\rho\,T}.} \tag{28.122}$$

Then

$$\omega_{0,*} = \frac{c_t}{\lambda_*}, \qquad \omega_{\mathrm{br},*} = \sigma\,\omega_{0,*}, \qquad m_{Z3}(\sigma) = \frac{h_{\mathrm{eff}}\,\sigma}{c_t\,\lambda_*}.$$

Existence condition (Z3). The Z3 branch exists as an internal stationary solution at LO order if, at the self-consistent solution λ_*,

$$\delta_*(\bar{w};\eta_\mu^{(Z)}(\lambda_*)) \in (\delta_{\min}^{\mathrm{Z3}}, \delta_{\max}),$$

and if

$$D_{\mathrm{LO+}}^{(Z3)}\Big(\bar{w};\delta_*(\bar{w};\eta_\mu^{(Z)}(\lambda_*));\eta_\mu^{(Z)}(\lambda_*)\Big) > 0.$$

If the minimization selects a boundary of the admissible domain, the Z3 branch is classified as a boundary branch and is not used as a closed prediction at LO order.

Comments (LO validity). (i) Binding, that is, the existence of the composite, is governed by the competition between the attractive strong-like channel and repulsive Preclusion through

$$\beta_{\mathrm{Z3}}^{(\mathrm{eff})}(\bar{w};\delta;\eta) = \beta_{\mathrm{Z3}}^{(\mathsf{F})}(\bar{w};\delta;\eta) - \beta_{\mathrm{Z3}}^{(\mathsf{P})}(\bar{w};\delta;\eta).$$

In the absence of an attractive quartic, no composite is retained.

(ii) The masses are not degenerate from one motif to another, because $S_0[\bar{w};\delta]$, $S_2[\bar{w};\delta]$ and the reduced invariants $\widehat{s}_2^{\mathrm{Z3}}$, $\widehat{s}_{4,\mathrm{pair}}^{\mathrm{Z3}}$, $\widehat{n}_{\mathsf{F}}^{\mathrm{Z3}}$ depend on the internal direction.

(iii) Rule (28.105) classifies neutral/charged at the level of the *discrete monopolar label* $n_{\mathrm{EM}}(\bar{w})$. For strong-overlap configurations, the reduced external amplitude may be reduced by a form factor, and by σ in conical 3DOF, without changing the neutral/charged classification given by $n_{\mathrm{EM}}(\bar{w})$.

Effective EM charge (Z-like): monopolar label and reduced amplitude

Two levels (important). Rule (28.105) fixes a *discrete monopolar label* through the integer

$$n_{\mathrm{EM}}(\bar{w}) = \sum_{i=1}^{k} \bar{w}_i \in \mathbb{Z}.$$

It is sufficient to classify the states

$$n_{\mathrm{EM}} = 0 \Rightarrow \text{neutral}, \qquad n_{\mathrm{EM}} \neq 0 \Rightarrow \text{charged}.$$

For non-negligible overlap, the reduced monopole amplitude is modulated by (i) a *form factor* depending on the internal structure $(\bar{w},\delta)$ and (ii) in conical 3DOF, by the projection $\sigma = \sin^2\alpha$.

Form factor (defined by the same kernels S_0, S_2). We introduce the reduced form factor, at fixed parameter η,

$$\boxed{\mathcal{C}_S^{(Z)}(\bar{w};\delta;\eta) := \frac{S_0[\bar{w};\delta]}{S_0[\bar{w};\delta] + \eta^{-1} S_2[\bar{w};\delta]} \in (0,1),} \tag{28.123}$$

which generalizes the single-bump value

$$\mathcal{C}_S^{(\mathrm{bump})}(\eta) = \frac{J_0}{J_0 + \eta^{-1} J_2}.$$

In the self-consistent closure, we take

$$\eta = \eta_\mu^{(Z)}(\lambda_*).$$

Definition of the reduced effective charge. We define the reduced effective charge

$$\boxed{q_{\mathrm{eff}}^{(Z)}(\bar{w};\delta,\sigma;\eta) := n_{\mathrm{EM}}(\bar{w})\,\sigma\,\frac{\mathcal{C}_S^{(Z)}(\bar{w};\delta;\eta)}{\mathcal{C}_S^{(\mathrm{bump})}(\eta)}.} \tag{28.124}$$

Thus: (i) if $n_{\mathrm{EM}}(\bar{w}) = 0$, then $q_{\mathrm{eff}}^{(Z)} = 0$ identically; (ii) if $n_{\mathrm{EM}}(\bar{w}) \neq 0$, then the amplitude is modulated by $\mathcal{C}_S^{(Z)}(\bar{w};\delta;\eta)/\mathcal{C}_S^{(\mathrm{bump})}(\eta)$ and by the conical factor σ. In the reduced closure, η is evaluated at $\eta_\mu^{(Z)}(\lambda_*)$.

Remarks. (i) The neutral/charged classification is determined by $n_{\mathrm{EM}}(\bar{w})$, whereas $q_{\mathrm{eff}}^{(Z)}$ gives the reduced effective amplitude. (ii) At leading order, $q_{\mathrm{eff}}^{(Z)}$ depends on $(\bar{w},\delta,\sigma;\eta)$, but not on the EBK level q: the effect of q acts on the energy through $E_0 \propto q$. (iii) In the present theoretical part, $q_{\mathrm{eff}}^{(Z)}$ must be read as an *effective monopole amplitude* derived from the internal structure.

Part II

Anchoring of HSM on observed quantities, physical predictions, and mesh structure

Chapter 29

Active calibrations and closed products

29.1 Introduction

The localized solutions of HSM theory on $\mathbb{S}^3$ (2DOF/3DOF solitons, breathers, quasi–c_t modes, etc.) exhibit clear *structural analogies* with the objects of the Standard Model (SM): 2DOF *bumps* mimic the lepton family, 2DOF *breathers* mimic neutral companions (neutrino-like), the NSL 2DOF branch and the 2DOF Z-like solution reproduce the kinematics of the weak sector, 2DOF/3DOF *qct* account for the EM/gluon-like carriers, and 3DOF *bumps* (conical precession) provide the internal 3DOF sources that feed color–white composites of CB3/CB2 type. The conical projections

$$\sigma = \sin^2 \alpha \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}$$

carry the fractional signature of the 3DOF branches.

In this chapter, we move to the **quantitative test** at leading order (LO): defining *minimal calibrations* and deriving independent *predictions*, while rigorously separating what is fixed by the theory from what is fixed by the observables. Conventions of the observational part:

- **Universal conventions of the observational part**: $h_{\text{eff}} = \hbar$, $c_t = c$.
- **Material parameters (structure)**: volumetric density ρ, tangential tension T, local radial stiffness μ_2, local coefficient $\kappa_\mu := \mu_2/T$, radial-gradient coefficient κ_r, micro scale ℓ_0, radius R_0.
- **Theoretical structure (without calibration)**: the theory introduces the first three admissible thresholds $q_1 < q_2 < q_3$, common to the relevant families at LO rank.

"Calibrate → predict" philosophy. We adopt a two-step logic, without circularity:

1. **Minimal calibrations**: determine a subset of constants $\{\rho, T, \kappa_\mu, \kappa_r, \ell_0, R_0\}$ through *orthogonal* routes (EM, weak, S^3 geometry, etc.), each using the *strict minimum* of observables.
2. **Predictions**: inject these constants into other channels/observables *not used* in step 1 (for example: G_N calibrates $\kappa_r R_0$ through the macro gravitational-like sector; the weak chain then predicts m_Z, etc.).

Each prediction is accompanied by an "HSM vs exp." table and an LO discrepancy assessment (the NLO corrections being documented elsewhere).

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29.2 Scope, assumptions, and HSM $\leftrightarrow$ SM mappings

We work at **LO** in the quasi-flat cap of $\mathbb{S}^3$ (thin cores $\lambda \ll R_0$) and use the normalizations of the observational part ($h_{\mathrm{eff}} = \hbar$, $c_t = c$). The purely spectral *ratios* coming from EBK selection and from matching are independent of ρ, T, R_0 at LO; these parameters govern absolute scales and/or NLO corrections.

Status of constants and provenance (synthetic reminder)

- **Conventions specific to the observational part**:

$$h_{\mathrm{eff}} = \hbar, \qquad c_t = c.$$

- **Imported from the theory, without additional calibration**: 3DOF conical factors

$$\sigma = \sin^2 \alpha \in \left\{ \tfrac{1}{3}, \tfrac{2}{3} \right\},$$

  core constants (J_0, J_2) of the retained profile, normalized measures, kernel forms, and, more generally, the dimensionless quantities already derived in the corresponding theory chapters.

- **Fixed by calibration in the present chapter**:
  - the calibrated spectral triplet

$$(q_1, q_2, q_3)_{\mathrm{cal}} = (1, 207, 3477),$$

    used here as a *calibration input* and not as a fundamental datum of pure theory;
  - ℓ_0 (EM route through Z_0, cf. §29.6);

  - κ_μ: scale anchoring through m_W and the theoretical ratio $Z0$/NSL2, cf. §29.5;
  - ρ, T, κ_r, when they are actually closed on data in the corresponding calibration sections.

- **Predicted later in this chapter**:
  - R_0, when it is closed by a dedicated prediction chain;
  - any numerical quantity obtained after injecting the previous calibrations into a channel not used to fix them.

- **"Pure" constants (dimensionless numbers)**: geometric kernel factors, window prefactors, etc., listed and evaluated numerically in the corresponding theory sections or appendices, with an explicit cross-reference when they are used in this chapter.

Numerical traceability rule. Throughout the present chapter, every numerical value used must fall under one of the following two cases:

1. it is **computed in the present chapter** and the calculation section is explicitly referenced;
2. it is **imported from an already derived theory section** and that provenance is explicitly referenced.

No frozen value should appear without explicit provenance.

Channel mappings (reminder)

Bump 2DOF	$\longleftrightarrow$	leptons (e, μ, τ),
Breather 2DOF	$\longleftrightarrow$	neutrinos (hyp.),
NSL 2DOF	$\longleftrightarrow$	W-like, Z-like 2DOF $\longleftrightarrow$ Z^0,
Qct 2DOF	$\longleftrightarrow$	γ (photon), Qct 3DOF $\longleftrightarrow$ gluon,
Isolated 3DOF bump	$\longleftrightarrow$	internal 3DOF source not directly observable,
color–white CB3/CB2	$\longleftrightarrow$	hadron-like states.

Remark (units). We follow the HSM convention: ρ is a *volumetric* density carried by the material layer of $\mathbb{S}^3$. Editorial unit clarifications (if necessary) will be gathered in a "Units & conversions" box. This does not affect the LO *ratios* or the calibrate→predict logic.

29.2.1 Methodology and uncertainty typology

Correlated regularized LO closures. When a theoretical constant comes from a regularized LO closure, its uncertainty is not treated as a type-A experimental uncertainty. It is propagated as a type-B theoretical uncertainty, with the correlations imposed by upstream closures.

In particular, for the 2DOF bump closure,

$$b_0^{(2)} = b_0^{(2),\mathrm{LO\text{-}reg}}, \qquad \frac{\sigma_{b_0}}{b_0} \simeq \frac{0.04}{1.80} \simeq 2.2\%,$$

and

$$\kappa_\omega = \sqrt{\frac{3b_0^{(2)}}{J_0 + J_2}}, \qquad \frac{\delta\kappa_\omega}{\kappa_\omega} = \frac{1}{2}\,\frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

In chains where ℓ_0 is calibrated by the EM-like closure using the same $b_0^{(2)}$, the dependence of ℓ_0 on $b_0^{(2)}$ must be propagated in a correlated way, and not as an independent uncertainty.

Propagation. When a quantity Y depends on variables x_i,

$$Y = Y(x_1, \ldots, x_n),$$

the local propagation used in this chapter is

$$\left(\frac{\sigma_Y}{Y}\right)^2 \simeq \sum_i \left(\frac{\partial \ln Y}{\partial \ln x_i}\frac{\sigma_{x_i}}{x_i}\right)^2,$$

except when a specific section justifies nonlinear propagation or a direct variation band.

Anti-circularity. An observable consumed as a calibration input cannot then be used as an independent test in the same chain. Conversely, an observable that does not enter a closure may be used as a comparison reference in the prediction chapter.

29.3 Observational inputs and exact constants used in the LO calibrations

Purpose. This subsection explicitly fixes the exact constants and the input observables used in the LO calibrations. The observables listed here are those consumed to determine the structural quantities of the model. Observables used only for comparison or prediction after calibration are listed separately in the prediction sections.

External sources. The fundamental constants and SI constants used for observational anchoring are taken from CODATA/NIST 2022 [1]. The masses, widths, and particle properties used as comparison references are taken from the Particle Data Group *Review of Particle Physics* 2024 [2], unless explicitly stated otherwise. The value $m_W^{\mathrm{ref}} = 80.379\,\mathrm{GeV}/c^2$ is kept as the weak calibration input from the earlier PDG set [21]. The neutrino oscillation parameters and PMNS angles used as comparison references are taken from NuFIT 5.3 [3].

Exact observational identifications. In this calibration chapter, we use the identifications

$$h_{\mathrm{eff}} = \hbar, \qquad c_t = c.$$

The associated numerical values are

Quantity	Value used	Status
$\hbar$	$1.054571817 \times 10^{-34}$ J s	exact constant in the observational anchoring
c	2.99792458×10^{8} m s^{-1}	exact constant in the convention of the observational part

Observational inputs consumed by the active calibrations. The following inputs effectively enter the structural closures of the chapter. Observables used only for comparison are listed separately below.

Input	Value used		Closed quantity	Reference in the chain
m_W^{ref}	80.379 GeV/c^2		Scale anchoring of the charged NSL 2DOF branch	PDG 2020 [21] ; $\lambda_{\mathrm{NSL}} = \hbar/(m_W^{\mathrm{ref}} c)$
Z_0^{obs}	376.730313668 Ω		ℓ_0	CODATA/NIST 2022 [1] ; (29.14)
G_N^{obs}	6.67430 10^{-11} m^3 kg^{-1} s^{-2}	×	$Q_r = \kappa_r R_0$	CODATA/NIST 2022 [1] ; (29.27)

Discrete calibration inputs. The triplet

$$(q_1, q_2, q_3)_{\mathrm{cal}} = (1, 207, 3477)$$

is a discrete spectral calibration input. It is fixed in §29.3.1. It is not a fundamental constant of the pure theory.

Derived outputs after cross-calibration. Some fundamental quantities are not closed by a single observable. In the active chain, the gravitational-like closure provides the product

$$Q_r = \kappa_r R_0.$$

The separation of this product into κ_r and R_0 is not performed from the 3DOF Bump sector: the masses and mass ratios of quarks are not direct observables of an isolated 3DOF bump. An independent separation of κ_r and R_0 must come from a distinct radial or geometric closure. Similarly, the 2DOF bump compatibility provides the inertial intermediate

$$P_{\rho R} = \rho R_0,$$

which gives

$$\rho = \frac{P_{\rho R}}{R_0}$$

once R_0 has been separated, and then

$$T = \rho c^2.$$

These derived outputs are summarized in §29.10.

Observables not consumed by the structural calibrations. The following quantities are kept as comparison, control, or post-calibration prediction references, but must not be counted as inputs of the structural closures above:

$$m_e^{\mathrm{obs}}, \qquad m_Z^{\mathrm{obs}}, \qquad G_F^{\mathrm{obs}}, \qquad m_H^{\mathrm{obs}}, \qquad \Delta m^2_{21},\ |\Delta m^2_{3\ell}|$$

according to the relevant sections.

In particular, in the restructured 2DOF bump closure, m_e^{obs} is not used to determine $P_{\rho R} = \rho R_0$: the combination $P_{\rho R}$ is closed by internal compatibility between the local spectral route and the global mechanical route.

Similarly, in the 3DOF Bump sector, the ratios

$$s/d, \qquad b/s, \qquad c/u, \qquad t/c$$

are not consumed as calibration inputs. The quantities associated with free quarks are treated as external scheme references, not as direct observables of an isolated 3DOF bump. Observational comparisons must be deferred to the color–white CB3/CB2 composites.

29.3.1 Fixing the calibration triplet of admissible thresholds

In the theoretical part, the structure of the admissible return filter introduces three thresholds $q_1 < q_2 < q_3$, common to the 2DOF bump, the 3DOF bump, and the 2DOF breather at LO rank. The pure theory, however, does not explicitly fix the numerical values of these thresholds at this level.

In the present observational part, we therefore choose to fix the calibration triplet of the first three admissible thresholds from the charged-lepton hierarchy:

$$\boxed{(q_1, q_2, q_3)_{\mathrm{cal}} = (1,\ 207,\ 3477).}$$

This calibration triplet will then be used throughout the rest of the present part, with the identifications

$$q_1 \leftrightarrow \text{light level}, \qquad q_2 \leftrightarrow \text{intermediate level}, \qquad q_3 \leftrightarrow \text{heavy level}.$$

Thus, in what follows, the numerical values $(1, 207, 3477)$ are not set as a closed prediction of the pure theory, but as the calibration triplet retained for the first three admissible thresholds.

29.4 Calibration of the effective isotropic residual of the q_2 threshold

References (labels). The isotropic reduction of the second threshold is given by (6.37), (6.38), and (6.39). These relations establish that the threshold q_2 fixes the effective value of the isotropic ratio

$$\mathcal{G}_{\mathrm{mesh}}^{(\mathrm{iso})} = \frac{r_2(q_2)}{r_1(q_2)}.$$

The discrete threshold $q_2^{\mathrm{cal}} = 207$ is the integer calibration input retained for this threshold closure. It is recalled in (29.2).

Purpose. The theory establishes that the first nontrivial threshold q_2 is related to the isotropic return residual by the closure

$$\boxed{q_{2,\mathrm{LO}} = -\frac{2}{\mathcal{G}_{\mathrm{mesh}}^{(\mathrm{iso})}}}. \tag{29.1}$$

This equation is the observational reminder of (6.38).

In the present subsection, no explicit form $\mathcal{G}_{\mathrm{mesh}}^{(\mathrm{iso})}(\chi)$ is introduced. Only the effective value of the isotropic residual required by the retained integer threshold is calibrated.

Retained threshold. We use the discrete calibration threshold

$$\boxed{q_2^{\mathrm{cal}} = 207.} \tag{29.2}$$

Injecting (29.2) into (29.1) immediately gives

$$\boxed{\mathcal{G}_{\mathrm{iso}}^{\mathrm{cal}} := \mathcal{G}_{\mathrm{mesh}}^{(\mathrm{iso})}\big|_{\mathrm{cal}} = -\frac{2}{207} = -0.00966183574879.} \tag{29.3}$$

Sign check. The theoretical condition (6.39) imposes

$$\mathcal{G}_{\mathrm{mesh}}^{(\mathrm{iso})} < 0.$$

The calibrated value (29.3) indeed satisfies this condition:

$$\mathcal{G}_{\mathrm{iso}}^{\mathrm{cal}} = -0.00966183574879 < 0.$$

We therefore obtain

$$-\frac{2}{\mathcal{G}_{\mathrm{iso}}^{\mathrm{cal}}} = 207,$$

which closes exactly the retained threshold.

Status of the calibrated quantity. The quantity calibrated here is not χ, but the effective value of the isotropic functional:

$$\boxed{\mathcal{G}_{\rm iso}^{\rm cal} = -\frac{2}{207}.}$$

The variable

$$\chi = \frac{\Xi_{\tan}}{\Xi_r}$$

belongs to the downstream geometric realization. It can be extracted numerically only once an explicit validated form of $\mathcal{G}_{\rm mesh}^{(\rm iso)}(\chi)$ is available in the downstream geometric construction. In the absence of such a closure, no value $\chi_{\rm cal}$ is displayed in this subsection.

Link with the downstream geometric reconstruction. The downstream geometric reconstruction uses the calibrated constraint (29.3) in the form

$$\mathcal{G}_{\rm mesh}^{(\rm iso)}(\chi) = \mathcal{G}_{\rm iso}^{\rm cal}.$$

This relation does not directly fix χ as long as the function $\mathcal{G}_{\rm mesh}^{(\rm iso)}(\chi)$ has not been explicitly derived for a given class of geometric realizations. It nevertheless imposes a sign and amplitude constraint on any admissible realization:

$$\mathcal{G}_{\rm mesh}^{(\rm iso)} < 0, \qquad \left|\mathcal{G}_{\rm mesh}^{(\rm iso)}\right| = \frac{2}{207}.$$

Thus, the downstream geometric realization does not receive an arbitrary value of χ; it must realize the effective value $\mathcal{G}_{\rm iso}^{\rm cal}$ through its internal geometry.

Uncertainty. In the strategy retained in §29.2.1, the threshold $q_2^{\rm cal} = 207$ is retained as a calibration integer. No continuous experimental uncertainty is therefore propagated locally in this subsection. A variation of q_2 would amount to a discrete change of calibration branch, not to a local error bar on $\mathcal{G}_{\rm iso}^{\rm cal}$.

29.5 Calibration of κ_μ by m_W anchoring and theoretical $Z0$/NSL2 ratio

References (labels). The definition of the local radial coefficient is given by (1.15). The local NSL 2DOF reduction and the combination $\kappa_\mu\lambda^2$ are given by (6.127) and (6.130). The overlap functions of the neutral composite are defined in (22.12). The neutral bound point is locked by (22.16)–(22.17). The reduced coefficients of the EM-like and Preclusion channels are closed by (22.15) and (22.18). The weak-like spectral denominators are given by (22.35) and (22.36). The neutral factor $\widehat{\beta}_0^{(2)}$ is defined by (22.37) and evaluated in Prop. 22.7. The constants $K_0^{(2)}$ and $K_2^{(2)}$ are fixed in (22.40). The value of the locked neutral bound point is given by (22.17).

Framework. We consider here the lifting back of the local radial coefficient

$$\kappa_\mu := \frac{\mu_2}{T},$$

in the 2DOF doublet formed by the charged NSL 2DOF branch and the neutral $Z0$-like composite. The consumed observational input is m_W, which fixes the scale $\lambda_{\rm NSL}$. The neutral/charged ratio used afterward is the theoretical ratio $R_{\rm th}$, derived from $\delta^*(u_Z^\star)$, and not the observational

ratio $m_Z^{\rm obs}/m_W$. The weak-like 2DOF theory shows that the local radial term does not enter the closures alone, but through the reduced combination

$$x := \kappa_\mu \, \lambda_{\rm NSL}^2,$$

where $\lambda_{\rm NSL}$ denotes the core width of the charged NSL 2DOF constituent.

The retained logical chain is now

$$m_W \implies \lambda_{\rm NSL}, \qquad u_{\mathsf{Z}}^\star, \mathcal{C}_{\rm EM}^{(2)} \implies A_{\rm EM}^{(Z)}, A_{\rm P}^{(Z)} \implies \delta^*(u_{\mathsf{Z}}^\star) \implies R_{\rm th}, \qquad (\lambda_{\rm NSL}, R_{\rm th}) \implies \kappa_\mu.$$

Assumptions (LO, colorless 2DOF). At LO, W is modeled as a charged NSL 2DOF solution (colorless), and the corresponding neutral state as the 2DOF Z_0-like composite. No color angle enters this family. For the frozen profile $a(\xi) = \operatorname{sech}\xi$, we use the channel constants

$$K_0^{(2)} = J_0^2, \qquad K_2^{(2)} = J_0 J_2, \qquad K_{\rm bump} = J_0(J_0 + J_2),$$

as well as the 2DOF neutral factor $\widehat{\beta}_0^{(2)}$ supplied by the theory.

Step 1: direct calibration of $\lambda_{\rm NSL}$ by m_W. In the observational part, we identify

$$h_{\rm eff} = \hbar, \qquad c_t = c,$$

and the charged branch W is the NSL 2DOF solution. The LO mass relation

$$m_W = \frac{h_{\rm eff}}{c_t \, \lambda_{\rm NSL}}$$

therefore gives directly

$$\boxed{\lambda_{\rm NSL} = \frac{\hbar}{m_W \, c}.} \tag{29.4}$$

Step 2: theoretical ratio of the neutral/charged doublet. The theoretical closure of the neutral composite gives

$$\delta^*(u_{\mathsf{Z}}^\star) = A_{\rm P}^{(Z)} \, \mathcal{O}_2(u_{\mathsf{Z}}^\star) - A_{\rm EM}^{(Z)} \, \mathcal{O}_1(u_{\mathsf{Z}}^\star), \tag{29.5}$$

with $u_{\mathsf{Z}}^\star$ locked by (22.17), and with the overlap functions defined in (22.12):

$$\mathcal{O}_1(u) = (u \operatorname{csch} u)^2, \qquad \mathcal{O}_2(u) = 2\,(u \coth u - 1) \operatorname{csch}^2 u.$$

We then define the theoretical ratio

$$\boxed{R_{\rm th} := \left(\frac{m_Z^{(\rm th)}}{m_W}\right)^4 = \left(1 + |\delta^*(u_{\mathsf{Z}}^\star)|\right)^2.} \tag{29.6}$$

Step 3: extraction of the reduced combination $x = \kappa_\mu \lambda_{\rm NSL}^2$. At LO, the two branches of the 2DOF doublet share the same scale prefactor (in particular the factor $(2h_{\rm eff}c_t/T)$) and differ only by their closure denominator. Thus, with

$$x := \kappa_\mu \lambda_{\rm NSL}^2,$$

the spectral denominators of (22.35) and (22.36) are written as

$$D_{Z0}(x) = x\Big(K_{\rm bump} + (1 + \widehat{\beta}_0^{(2)}) K_0^{(2)}\Big) + (1 + \widehat{\beta}_0^{(2)}) K_2^{(2)},$$

and, for the charged NSL 2DOF branch,

$$D_{\mathrm{NSL2}}(x) = x\Big(K_{\mathrm{bump}} + K_0^{(2)}\Big) + K_2^{(2)}.$$

Using the theoretical mass identity

$$\left(\frac{m_{Z0}^{(\mathrm{th})}}{m_W}\right)^4 = \frac{D_{Z0}(x)}{D_{\mathrm{NSL2}}(x)},$$

we obtain

$$R_{\mathrm{th}} = \frac{D_{Z0}(x)}{D_{\mathrm{NSL2}}(x)}.$$

The relation

$$D_{Z0}(x) = R_{\mathrm{th}}\, D_{\mathrm{NSL2}}(x)$$

is linear in x, whence

$$\boxed{x = \kappa_\mu \lambda_{\mathrm{NSL}}^2 = \frac{K_2^{(2)}\,(R_{\mathrm{th}} - 1 - \widehat{\beta}_0^{(2)})}{(K_{\mathrm{bump}} + K_0^{(2)})(1 - R_{\mathrm{th}}) + K_0^{(2)}\,\widehat{\beta}_0^{(2)}}.} \tag{29.7}$$

Shape constants and 2DOF neutral factor. For the frozen profile $a(\xi) = \operatorname{sech}\xi$,

$$J_0 = \frac{\pi^3}{3} = 10.3354255601, \qquad J_2 = 7.6339320582,$$

hence

$$K_{\mathrm{bump}} = J_0(J_0 + J_2) = 185.7209580268,$$

$$K_0^{(2)} = J_0^2 = 106.8210215084, \qquad K_2^{(2)} = J_0 J_2 = 78.8999365184.$$

Moreover, the theory gives at LO, cf. (22.37) and Prop. 22.7,

$$\widehat{\beta}_0^{(2)} = 1 - \frac{1}{4u_*^2} + \mathcal{O}(\varepsilon^2),$$

with u_* fixed canonically by (6.23). By evaluating the canonical root

$$u_* \simeq 2.00712,$$

we obtain

$$\widehat{\beta}_{0,\mathrm{LO}}^{(2)} \simeq 0.9379428908.$$

Theoretical inputs and numerical evaluation. The reduced coefficients used are those of (22.15) and (22.18):

$$A_{\mathrm{EM}}^{(Z)} \simeq 6.154440302, \qquad A_{\mathrm{P}}^{(Z)} \simeq 9.854589062.$$

With the LO prediction of the neutral/charged doublet,

$$\boxed{\delta_{\mathrm{th}}^* := |\delta^*(u_{\mathsf{Z}}^\star)| = 0.2613503319.} \tag{29.8}$$

we obtain

$$R_{\mathrm{th}} = (1 + |\delta^*(u_{\mathsf{Z}}^\star)|)^2 \simeq 1.5910046598.$$

Insertion into (29.7) first gives

$$\boxed{\kappa_\mu \lambda_{\mathrm{NSL}}^2 \simeq 0.3765169342.} \tag{29.9}$$

Numerical value of λ_{NSL}. The relation (29.4), applied with the observational input m_W, gives

$$\boxed{\lambda_{\mathrm{NSL}} = \frac{\hbar}{m_W\, c} \simeq 2.454956896 \times 10^{-18}\ \mathrm{m}.} \tag{29.10}$$

Lifting back to the physical coefficient κ_μ. We then deduce

$$\boxed{\kappa_\mu = \frac{\kappa_\mu \lambda_{\mathrm{NSL}}^2}{\lambda_{\mathrm{NSL}}^2} \simeq 6.24736 \times 10^{34}\ \mathrm{m}^{-2}.} \tag{29.11}$$

Status. The lifting back of κ_μ is therefore performed in three steps:

1. the mass m_W directly fixes the core width of the NSL 2DOF constituent, λ_{NSL}, through (29.4);
2. the theoretical closure of the neutral composite gives $\delta^*(u_{\mathsf{Z}}^\star)$, then the theoretical ratio R_{th} through (29.6);
3. the theoretical ratio R_{th} then calibrates the reduced combination $\kappa_\mu \lambda_{\mathrm{NSL}}^2$ through (29.7), which finally allows one to lift back to κ_μ by (29.11).

No direct consumption of $m_Z^{(\mathrm{obs})}$ occurs in this subsection.

Uncertainty. We first write the reduced quantity as

$$\kappa_\mu \lambda_{\mathrm{NSL}}^2 = \mathcal{F}\big(R_{\mathrm{th}}; \widehat{\beta}_0^{(2)}, K_0^{(2)}, K_2^{(2)}, K_{\mathrm{bump}}\big).$$

The physical coefficient follows as

$$\kappa_\mu = \mathcal{F}(R_{\mathrm{th}}; \widehat{\beta}_0^{(2)}, K_0^{(2)}, K_2^{(2)}, K_{\mathrm{bump}}) \left(\frac{m_W c}{h_{\mathrm{eff}}}\right)^2,$$

with $h_{\mathrm{eff}} = \hbar$ in the observational part. In the strategy retained here, u^*, $\widehat{\beta}_0^{(2)}$, $K_0^{(2)}$, $K_2^{(2)}$, K_{bump}, $A_{\mathrm{EM}}^{(Z)}$, $A_{\mathrm{P}}^{(Z)}$, and therefore R_{th} are theoretical inputs fixed univocally at LO. The type-A contribution from m_W remains negligible at the useful precision retained here. We thus obtain

$$\boxed{\kappa_\mu \simeq 6.24736 \times 10^{34}\ \mathrm{m}^{-2}}.$$

29.6 Calibration of the microscopic scale ℓ_0 by vacuum impedance (LO)

References (labels). (16.5), (1.16), §16.1, (6.67), (6.57), (6.65), (20.37),

Observational framework. In accordance with the exact observational identifications declared in §29.3, we use

$$h_{\mathrm{eff}} = \hbar, \qquad c_t \equiv c.$$

The internal kinematic relation remains

$$c_t^2 = \frac{T}{\rho} \qquad ((1.16)).$$

LO formulas (displayed explicitly). The EM-like closure validated in the theoretical part gives, for the theoretical impedance of the reference channel,

$$\boxed{Z^{(\mathrm{LO})}_{\mathrm{EM,ref}}(\ell_0) = \frac{\hbar\, c_t}{\ell_0^2}\, \Phi^{(\mathrm{two})}_{\mathrm{EM}}\ .} \tag{29.12}$$

With the "two-leg" convention,

$$\Phi^{(\mathrm{two})}_{\mathrm{EM}} = \left(\frac{1}{\Delta^2_{\mathrm{EM}}\, b_0^{(2)}}\right)^2 \frac{1}{2\pi^2}\, \sqrt{\frac{J_2}{J_0}}\, \Gamma_{\mathrm{EM}}\, \frac{J_0^2}{E^{\mathrm{cap}}_{\ell=1}}\,, \qquad E^{\mathrm{cap}}_{\ell=1} = \frac{(F_1^{\mathrm{cap}})^2}{\pi}\ . \tag{29.13}$$

In the canonical calibration branch of ℓ_0 retained in the theory, we consume the canonical normalization of the $\ell = 1$ cap projector defined in §20.3.1:

$$(F_1^{\mathrm{cap}})^2 = 1, \qquad E^{\mathrm{cap}}_{\ell=1} = \frac{1}{\pi}.$$

The EM-like closure consumed here is that of §20.3.1: the continuity matching does not constitute an independent multiplicative factor at LO. It is included in the canonical constant of the EM-like channel carried by Γ_{EM}, cf. (20.37). The pure channel constant then takes the theoretical value

$$\Phi^{(\mathrm{two})}_{\mathrm{EM}} = C_\Phi = 5.152264653 \times 10^{-16}.$$

We then identify the theoretical reference quantity with the SI observable,

$$Z^{(\mathrm{LO})}_{\mathrm{EM,ref}}(\ell_0) = Z_0^{\mathrm{obs}} = \mu_0\, c_t,$$

which gives

$$\boxed{\ell_0 = \sqrt{\frac{\hbar\, c_t\, \Phi^{(\mathrm{two})}_{\mathrm{EM}}}{Z_0^{\mathrm{obs}}}} = \sqrt{\frac{\hbar\, c_t\, C_\Phi}{Z_0^{\mathrm{obs}}}}\ .} \tag{29.14}$$

Derivation chain (short reminder). (i) The EM-like dictionary and the effective constitutive relations are fixed in §16.1, with (16.5). (ii) The explicit expression of the pure channel constant $\Phi^{(\mathrm{two})}_{\mathrm{EM}}$ is given by (20.37). (iii) Its compact form is written as

$$\Phi^{(\mathrm{two})}_{\mathrm{EM}} = \frac{C_\Phi}{(F_1^{\mathrm{cap}})^2},$$

cf. (20.42). (iv) The LO reference-channel impedance is then

$$Z^{(\mathrm{LO})}_{\mathrm{EM,ref}} = \frac{\hbar\, c_t}{\ell_0^2}\, \Phi^{(\mathrm{two})}_{\mathrm{EM}},$$

in accordance with (20.44). (v) Identification with the vacuum-impedance observable $Z_0 = \mu_0 c_t$ provides the inversion (29.14). (vi) For the numerical calibration application, we consume the theoretical canonical branch of the cap projector,

$$(F_1^{\mathrm{cap}})^2 = 1.$$

.

Numerical application (LO). Constants used:

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 \simeq 7.6339320582, \qquad \Delta_{\mathrm{EM}} = 54.9589563236661, \qquad b_0^{(2)} = 1.809206964 \times 10^4,$$

$$\Gamma_{\mathrm{EM}} = \frac{1}{\pi}\left(\frac{J_0}{J_0 + J_2}\right)^2 = 0.10530324394,$$

cf. (20.36).

$$(F_1^{\mathrm{cap}})^2 = 1, \qquad E_{\ell=1}^{\mathrm{cap}} = \frac{1}{\pi}, \qquad C_\Phi = 5.152264653 \times 10^{-16}.$$

With the exact observational identifications of §29.3, we use

$$c_t \equiv c, \qquad Z_0^{\mathrm{obs}} = 376.730313412\ \Omega.$$

The inversion (29.14) gives

$$\boxed{\ell_0 \;=\; 2.079372183 \times 10^{-22}\ \mathrm{m}} \qquad \text{(profile } a = \mathrm{sech}\text{, “two-leg” convention, canonical LO branch).} \tag{29.15}$$

Uncertainty. In the strategy retained in §29.2.1, the SI observable Z_0^{obs} is taken with its experimental uncertainty. The strict internal profile and projection constants are fixed at LO. By contrast, $b_0^{(2)}$ is a regularized LO spectral closure; its uncertainty is a type-B theoretical uncertainty.

From

$$\ell_0 = \sqrt{\frac{\hbar c_t\, \Phi_{\mathrm{EM}}^{(\mathrm{two})}}{Z_0^{\mathrm{obs}}}},$$

we have

$$\frac{\delta \ell_0}{\ell_0} = \frac{1}{2}\,\frac{\delta \Phi_{\mathrm{EM}}^{(\mathrm{two})}}{\Phi_{\mathrm{EM}}^{(\mathrm{two})}} - \frac{1}{2}\,\frac{\delta Z_0}{Z_0^{\mathrm{obs}}}.$$

Now, in the central branch,

$$\Phi_{\mathrm{EM}}^{(\mathrm{two})} \propto \left(\frac{1}{\Delta_{\mathrm{EM}}^2 b_0^{(2)}}\right)^2, \qquad \Delta_{\mathrm{EM}} = \kappa_\omega, \qquad \kappa_\omega^2 \propto b_0^{(2)}.$$

It follows that

$$\Phi_{\mathrm{EM}}^{(\mathrm{two})} \propto \left(b_0^{(2)}\right)^{-4}, \qquad \left.\frac{\delta \ell_0}{\ell_0}\right|_{b_0} = -2\,\frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

The experimental contribution from Z_0^{obs} remains

$$\left.\frac{\sigma_{\ell_0}}{\ell_0}\right|_{Z_0} = \frac{1}{2}\,\frac{\sigma_{Z_0}}{Z_0^{\mathrm{obs}}}.$$

Thus, ℓ_0 has a negligible metrological uncertainty at the displayed rank, but a theoretical closure uncertainty correlated with $b_0^{(2)}$. The numerical values displayed in this section are therefore central LO values.

29.7 Closure of the product $P_{\rho R} := \rho R_0$ by 2DOF bump compatibility

Purpose. The effective one-body closure of the 2DOF bump does not separate the volumetric mass density ρ and the global radius R_0. It closes only the product

$$\boxed{P_{\rho R} := \rho R_0.} \tag{29.16}$$

This section determines $P_{\rho R}$ by internal compatibility of the two one-body writings of the same 2DOF bump state, without using the observed mass of the electron.

The separation of ρ and R_0 is performed later by the micro-local one-ring closure of the Z^0-like composite, cf. §29.9. The present result must therefore be read as a product closure, and not as a direct calibration of the density ρ.

Theoretical data used. The one-body spectral mass of the 2DOF bump is given by (6.79a), namely

$$m_{\mathrm{bump}}^{(\mathrm{spec})} c^2 = \Big(\frac{h_{\mathrm{eff}} c}{\ell_0}\Big) \Pi_{\mathrm{proj}} Z_{\mathrm{ann}} Z_{\mathrm{proj}} \frac{1}{\kappa_\omega^2\, b_0^{(2)}}. \tag{29.17}$$

The factors entering this writing are defined by (6.73), (6.74), (6.75), and (6.77).

The effective mechanical mass of the same state is given by (6.101),

$$m_{\mathrm{bump}}^{(\mathrm{mech})} = \kappa_{\mathrm{core}}\, \rho\, \lambda_e^2\, R_0 = \kappa_{\mathrm{core}}\, \lambda_e^2\, P_{\rho R}. \tag{29.18}$$

The core factor κ_{core} is defined in (6.44), and the effective width λ_e is closed by (6.86):

$$\lambda_e = 4\pi\, \kappa_\omega \sqrt{b_0^{(2)}}\, \ell_0. \tag{29.19}$$

Internal compatibility. The compatibility of the two one-body readings is equation (6.102). In energy form, it is written as in (6.103):

$$\kappa_{\mathrm{core}}\, \rho\, \lambda_e^2\, R_0\, c^2 = \Big(\frac{h_{\mathrm{eff}} c}{\ell_0}\Big) \Pi_{\mathrm{proj}} Z_{\mathrm{ann}} Z_{\mathrm{proj}} \frac{1}{\kappa_\omega^2\, b_0^{(2)}}.$$

It follows directly that

$$\boxed{P_{\rho R} = \rho R_0 = \frac{h_{\mathrm{eff}}}{\kappa_{\mathrm{core}}\, c\, \ell_0\, \lambda_e^2}\, \Pi_{\mathrm{proj}} Z_{\mathrm{ann}} Z_{\mathrm{proj}} \frac{1}{\kappa_\omega^2\, b_0^{(2)}}}. \tag{29.20}$$

Developed form. Replacing λ_e by (29.19) gives

$$\boxed{\rho R_0 = \frac{h_{\mathrm{eff}}}{\kappa_{\mathrm{core}} c\, \ell_0^3}\, \frac{\Pi_{\mathrm{proj}} Z_{\mathrm{ann}} Z_{\mathrm{proj}}}{(4\pi)^2\, \kappa_\omega^4\, \left(b_0^{(2)}\right)^2}}. \tag{29.21}$$

Numerical evaluation. With the LO values retained in this chapter, evaluating (29.20) gives

$$\boxed{\rho R_0 = 1.222404409 \times 10^3\ \mathrm{kg\,m^{-2}}}. \tag{29.22}$$

Status. The quantity closed here is the combination $P_{\rho R} = \rho R_0$. This closure does not determine ρ and R_0 separately. In the active chain, the separation is performed later by the micro-local one-ring closure of the Z^0-like composite:

$$R_0 \quad \Longrightarrow \quad \rho = \frac{P_{\rho R}}{R_0}.$$

The observed mass of the electron is not used in this closure; it enters only in the comparison section for the predictions.

Uncertainty. The closure (29.20) does not consume the observed mass of the electron. Its uncertainty is therefore induced only by the inputs already fixed upstream, in particular ℓ_0, as well as by the spectral and geometric constants entering Π_{proj}, Z_{ann}, Z_{proj}, κ_ω, $b_0^{(2)}$, κ_{core}, and λ_e.

Using the developed form (29.21), the dominant dependence is

$$\rho R_0 \propto \ell_0^{-3}.$$

Thus, for small relative errors,

$$\frac{\delta(\rho R_0)}{\rho R_0} = -3\,\frac{\delta \ell_0}{\ell_0} + \frac{\delta(\Pi_{\mathrm{proj}} Z_{\mathrm{ann}} Z_{\mathrm{proj}})}{\Pi_{\mathrm{proj}} Z_{\mathrm{ann}} Z_{\mathrm{proj}}} - 4\,\frac{\delta \kappa_\omega}{\kappa_\omega} - 2\,\frac{\delta b_0^{(2)}}{b_0^{(2)}} - \frac{\delta \kappa_{\mathrm{core}}}{\kappa_{\mathrm{core}}}. \tag{29.23}$$

In the uncertainty strategy retained in §29.2.1, the internal constants resulting from strict closures are taken as exact at LO level once their closure has been fixed. By contrast, $b_0^{(2)}$ is a regularized LO closure and must be propagated with the upstream correlations

$$\frac{\delta \kappa_\omega}{\kappa_\omega} = \frac{1}{2}\,\frac{\delta b_0^{(2)}}{b_0^{(2)}}, \qquad \left.\frac{\delta \ell_0}{\ell_0}\right|_{b_0} = -2\,\frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

Injecting these relations into (29.23), the correlated contribution of $b_0^{(2)}$ to $P_{\rho R}$ is

$$\left.\frac{\delta(\rho R_0)}{\rho R_0}\right|_{b_0} = 2\,\frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

With the regularized LO envelope

$$\frac{\sigma_{b_0}}{b_0} \simeq 2.2\%,$$

this gives a theoretical closure uncertainty of about 4.4% on ρR_0, independently of the metrological uncertainty of Z_0^{obs}. The numerical value displayed in (29.22) is therefore a central LO value.

29.8 Closure of the product $Q_r := \kappa_r R_0$ by G_N (LO)

References (labels). The LO gravitational-like closure used here is (20.32). The kinematic relation $c_t^2 = T/\rho$ comes from (1.16). The combination

$$P_{\rho R} := \rho R_0$$

was closed previously in (29.20).

Framework. The gravitational-like closure does not directly separate κ_r and R_0. It closes the combination

$$\boxed{Q_r := \kappa_r R_0.} \tag{29.24}$$

This combination is the one that enters naturally once the gravitational-like closure is rewritten in terms of the product $P_{\rho R} = \rho R_0$. The separation of κ_r is performed later, after the determination of R_0 by the one-ring micro-local closure:

$$\kappa_r = \frac{Q_r}{R_0}.$$

This combination is the one that enters naturally once the gravitational-like closure is rewritten in terms of $P_{\rho R} = \rho R_0$.

Active closure (LO). The geometric-spectral factor

$$\mathcal{N}_g = \frac{J_2}{J_0}(4\pi\kappa_\omega)^2$$

is defined in (20.31) as the product of the pure normalizations of the profile, the monopolar flux, and the common spectral closure. The effective gravitational-like closure is

$$G_N^{\text{eff}} = \frac{J_2}{J_0}\,(4\pi\kappa_\omega)^2\,\frac{c_t^4}{4\pi\,T\,\kappa_r\,R_0^2}. \tag{29.25}$$

Using

$$T = \rho c_t^2, \qquad P_{\rho R} = \rho R_0, \qquad Q_r = \kappa_r R_0,$$

one obtains

$$\boxed{G_N^{\text{eff}} = \frac{J_2}{J_0}\,(4\pi\kappa_\omega)^2\,\frac{c_t^2}{4\pi\,P_{\rho R}\,Q_r}}. \tag{29.26}$$

Algebraic inversion. Inverting (29.26) directly gives

$$\boxed{Q_r = \kappa_r R_0 = \frac{J_2}{J_0}\,(4\pi\kappa_\omega)^2\,\frac{c_t^2}{4\pi\,G_N\,P_{\rho R}}}. \tag{29.27}$$

This relation consumes neither ρ, nor R_0, nor T separately. It uses only the combination $P_{\rho R}$ already closed upstream.

Internal constants and inputs. We use

$$J_0 = \frac{\pi^3}{3} = 10.3354255601, \qquad J_2 \simeq 7.6339320582, \qquad \kappa_\omega = 54.9589563237,$$

as well as

$$P_{\rho R} = \rho R_0 = 1.222404409 \times 10^3\ \mathrm{kg\,m^{-2}}, \qquad c_t = c.$$

The retained observational value is

$$G_N^{(\text{obs})} = 6.67430 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}.$$

Numerical evaluation (LO). Inserting these values into (29.27) gives

$$\boxed{\kappa_r R_0 = 3.088357158 \times 10^{28}\ \mathrm{m}}. \tag{29.28}$$

Dimensional check. One has

$$[G_N P_{\rho R}] = \left(\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}\right)\left(\mathrm{kg}\,\mathrm{m}^{-2}\right) = \mathrm{m}\,\mathrm{s}^{-2},$$

therefore

$$\left[\frac{c_t^2}{G_N P_{\rho R}}\right] = \frac{\mathrm{m}^2\,\mathrm{s}^{-2}}{\mathrm{m}\,\mathrm{s}^{-2}} = \mathrm{m}.$$

The right-hand side of (29.27) therefore indeed has the dimension of a length, like $\kappa_r R_0$.

Status. The LO gravitational-like closure closes here the combination

$$Q_r = \kappa_r R_0,$$

and not κ_r alone. In the active chain, the isolated value of κ_r is deduced from the micro-local separation of R_0,

$$\kappa_r = \frac{Q_r}{R_0}.$$

Uncertainty. Writing

$$Q_r = \mathcal{Q}(G_N; J_0, J_2, \kappa_\omega, c_t, P_{\rho R}), \qquad Q_r \propto \frac{1}{G_N P_{\rho R}},$$

the relative variation satisfies

$$\frac{\delta Q_r}{Q_r} = -\frac{\delta G_N}{G_N} - \frac{\delta P_{\rho R}}{P_{\rho R}} + \frac{\delta(J_2/J_0)}{J_2/J_0} + 2\,\frac{\delta \kappa_\omega}{\kappa_\omega} + 2\,\frac{\delta c_t}{c_t}. \tag{29.29}$$

In the strategy adopted in §29.2.1, the strict internal constants J_0 and J_2 are taken as fixed at LO, and $c_t = c$ is taken as exact in the calibration. The quantity κ_ω nevertheless inherits the regularized closure $b_0^{(2)}$, with

$$\frac{\delta \kappa_\omega}{\kappa_\omega} = \frac{1}{2}\,\frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

Since Q_r also depends on $P_{\rho R}$, the closure uncertainty of $b_0^{(2)}$ is transmitted both through $P_{\rho R}$ and through κ_ω. Using

$$\left.\frac{\delta P_{\rho R}}{P_{\rho R}}\right|_{b_0} = 2\,\frac{\delta b_0^{(2)}}{b_0^{(2)}},$$

the correlated contribution of $b_0^{(2)}$ to Q_r is

$$\left.\frac{\delta Q_r}{Q_r}\right|_{b_0} = -\frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

The value displayed below is therefore a central LO value; its type-B closure uncertainty is of the order of that of $b_0^{(2)}$, in addition to the experimental uncertainty of G_N. At the useful precision retained here, we keep the central value

$$\boxed{\kappa_r R_0 = 3.088357158 \times 10^{28}\ \mathrm{m}}.$$

29.9 Micro-local separation of R_0, ρ, T, κ_r

Purpose. The previous sections close the two structural products

$$P_{\rho R} := \rho R_0, \qquad Q_r := \kappa_r R_0.$$

These products are not the final separated constants. The separation is carried out here by consuming the micro-local one-ring closure of the Z^0-like composite, cf. (22.11). No auxiliary separation assumption enters this step.

Intermediate inputs. We use the upstream closed products

$$P_{\rho R} = 1.222404409 \times 10^3 \text{ kg m}^{-2}, \qquad Q_r = 3.088357158 \times 10^{28} \text{ m},$$

together with the microscopic scale

$$\ell_0 = \ell_0 = 2.079372183 \times 10^{-22} \text{ m},$$

These quantities are intermediate inputs of the separation; they will not be reported as final outputs in the synthesis.

Reduced length of the Z^0-like composite. The closure of the Z^0-like composite, cf. §22.1, uses the reduced phase length

$$\lambda_{\mathsf{Z}} = \frac{c_t}{\omega_{\mathsf{Z}}}.$$

In the present observational part, with $c_t = c$ and $E = h_{\text{eff}}\omega$, we write

$$\lambda_{\mathsf{Z}} = \frac{h_{\text{eff}} c}{m_Z^{\text{th}} c^2}.$$

Here we take m_Z^{th}, obtained from the LO prediction from m_W, in order not to consume directly the observed value of m_Z in this closure. With

$$m_Z^{\text{th}} = 90.2735 \text{ GeV},$$

one obtains

$$\boxed{\lambda_{\mathsf{Z}} \simeq 2.18588 \times 10^{-18} \text{ m}.} \tag{29.30}$$

Micro-local participation. The isotropic closure of the Z^0-like composite gives

$$\boxed{\Xi_{\text{micro}}^{(\mathsf{Z})} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^{10}.} \tag{29.31}$$

It follows that

$$\boxed{\Xi_{\text{micro}}^{(\mathsf{Z})} \simeq 2.54184 \times 10^{-40}.} \tag{29.32}$$

Determination of R_0. The micro-local closure gives

$$\boxed{R_0 = \frac{2h_{\text{eff}}}{\pi\, c_t\, P_{\rho R}\, \lambda_{\mathsf{Z}}^2\, \Xi_{\text{micro}}^{(\mathsf{Z})}}.} \tag{29.33}$$

With $h_{\text{eff}} = \hbar$ and $c_t = c$, one obtains

$$\boxed{R_0^{\text{HSM}} \simeq 1.50841 \times 10^{29} \text{ m}.} \tag{29.34}$$

Status of the separation. The value of R_0 obtained here is a closed output under the micro-local one-ring closure of the Z^0-like composite. It depends directly on

$$\Xi^{(\mathsf{Z})}_{\text{micro}} = \frac{4\pi}{3}\left(\frac{\ell_0}{\lambda_{\mathsf{Z}}}\right)^{10}.$$

The tenth power makes this separation sensitive to the micro-local factorization consumed from the theory of composites. The value of R_0 therefore has to be read as an output of the micro-local closure of this canonical closure, and not as a prediction independent of $\Xi^{(\mathsf{Z})}_{\text{micro}}$.

Geometric check by the observable comoving radius. The previous closure determines the global curvature radius R_0^{HSM} of the spatial hypersphere. It does not consume any global-size cosmological datum. One may nevertheless perform an external geometric check by comparing this curvature radius with the radial comoving distance associated with the current observable domain.

On $\mathbb{S}^3$, a radial geodesic distance is written as

$$s = R_0^{\text{HSM}}\chi, \qquad \chi \in [0, \pi].$$

The maximal antipodal distance is therefore

$$s_{\max} = \pi R_0^{\text{HSM}}.$$

If the current observable comoving radius is interpreted as an effective radial geodesic distance on the current spatial slice,

$$R_{\text{obs}} = s_{\text{obs}} = R_0^{\text{HSM}}\chi_{\text{obs}},$$

then

$$\boxed{\chi_{\text{obs}} = \frac{R_{\text{obs}}}{R_0^{\text{HSM}}}.} \tag{29.35}$$

With

$$R_{\text{obs}} \simeq 46.5 \times 10^9 \text{ ly} \simeq 4.40 \times 10^{26} \text{ m},$$

and

$$R_0^{\text{HSM}} = 1.50841 \times 10^{29} \text{ m},$$

one obtains

$$\boxed{\chi_{\text{obs}} \simeq 2.92 \times 10^{-3}.} \tag{29.36}$$

Reported to the antipodal distance of the hypersphere, the observable radial portion is

$$\boxed{\frac{R_{\text{obs}}}{\pi R_0^{\text{HSM}}} \simeq 9.3 \times 10^{-4}.} \tag{29.37}$$

This check shows that the currently observable domain occupies a very small geodesic portion of the global hypersphere:

$$R_{\text{obs}} \ll \pi R_0^{\text{HSM}}.$$

It is not a calibration input for R_0^{HSM}, but an external geometric check after separation of the global radius.

Determination of ρ, T, and κ_r. The volume density is

$$\rho = \frac{P_{\rho R}}{R_0^{\mathrm{HSM}}},$$

hence

$$\boxed{\rho^{\mathrm{HSM}} \simeq 8.10393 \times 10^{-27}\ \mathrm{kg\,m^{-3}}.} \tag{29.38}$$

The tension is

$$T = \rho c^2,$$

namely

$$\boxed{T^{\mathrm{HSM}} \simeq 7.28345 \times 10^{-10}\ \mathrm{Pa}.} \tag{29.39}$$

Finally,

$$\kappa_r = \frac{Q_r}{R_0^{\mathrm{HSM}}},$$

gives

$$\boxed{\kappa_r^{\mathrm{HSM}} \simeq 0.204743.} \tag{29.40}$$

Status. The values

$$R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}$$

are the separated constants resulting from the micro-local one-ring closure of the Z^0-like composite. In the active chain, they constitute the separated constants consumed by the downstream prediction and diagnostic sections.

29.10 Synthesis of the separated fundamental constants (LO)

Separated constant	LO value	Unit	Origin
ℓ_0	$\boxed{2.079372183 \times 10^{-22}}$	m	(29.14)
$\kappa_\mu = \mu_2/T$	$\boxed{6.25 \times 10^{34}}$	$\mathrm{m^{-2}}$	(29.11)
R_0	$\boxed{1.50841 \times 10^{29}}$	m	(29.34)
ρ	$\boxed{8.10393 \times 10^{-27}}$	$\mathrm{kg\,m^{-3}}$	(29.38)
T	$\boxed{7.28345 \times 10^{-10}}$	Pa	(29.39)
κ_r	$\boxed{0.204743}$	1	(29.40)

Reading of the synthesis. This synthesis gathers the separated constants that will be consumed by the prediction sections. The intermediate products

$$P_{\rho R} := \rho R_0, \qquad Q_r := \kappa_r R_0$$

are not reported in the table, because they are not the final outputs of the active chain: they are consumed by the micro-local one-ring closure of the Z^0-like composite to determine R_0, then ρ, T, and κ_r.

Downstream status. The present calibration synthesis consumes no auxiliary separation hypothesis. The separated constants obtained here may then be used as constraints for a constructive geometric reconstruction.

Chapter 30

Predictions and comparisons

Purpose of the chapter. This chapter gathers the numerical outputs obtained after the active calibrations and, when necessary, after the global HSM separation. It distinguishes:

- scale-free predictions, based only on ratios or pure numbers;
- predictions after observational anchoring, when an already calibrated scale is required;
- numerical outputs explicitly depending on the separated HSM constants.

Uncertainties. The uncertainty typology defined in §29.2.1 is used throughout this chapter. Type A uncertainties come from the experimental inputs already declared in the calibration chapter. Type B and C uncertainties are specified locally when they depend on a channel closure, an MSE window, a harmonic truncation, an LO/NLO approximation, or a choice of geometric closure.

Status of the comparisons. When an observed value is used only as a comparison reference, it is indicated as such and must not be confused with a calibration input. In particular, the same observable must not be both consumed in a closure and claimed as an independent prediction in the same chain.

30.0.1 LO prediction of the mass m_Z from m_W (2DOF doublet)

References (labels). The neutral bound point $u_{\mathsf{Z}}^{\star}$ is locked by (22.16)–(22.17). The reduced functions of the neutral composite are defined in (22.12). The reduced energy depending on the separation is given by (22.13). The coefficients $A_{\mathrm{EM}}^{(Z)}$ and $A_{\mathrm{P}}^{(Z)}$ are related to the reduced channel coefficients by (22.14), then closed by (22.15) and (22.18).

Principle (scale-free). At LO, the Z boson is modeled as a neutral composite of two NSL 2DOF solutions in strong overlap. The neutral bound point is locked by the LO matching,

$$u_{\mathsf{Z}}^{\star} = u_{*}^{(\mathrm{ref})} \simeq 2.00712, \qquad \text{cf. (22.17)}.$$

The prediction is *scale-free*: the factors R_0, λ_*, and $\sigma = \sin^2\alpha$ cancel, so that the increment

$$\delta^{*}(u_{\mathsf{Z}}^{\star})$$

is a **pure number** produced by the neutral/charged 2DOF closure.

Reduced formulas (profile $a = \mathrm{sech}$). We use the reduced functions of the neutral composite,

$$\mathcal{O}_1(u) = (u\,\mathrm{csch}\,u)^2\,, \qquad \mathcal{O}_2(u) = 2\,(u\coth u - 1)\,\mathrm{csch}^2 u, \qquad \text{cf. (22.12)}.$$

In the reduction of the neutral composite, we write

$$A_{\mathrm{EM}}^{(Z)} = 4\,\mathcal{C}_{\mathrm{EM}}^{(2)}, \qquad A_{\mathrm{P}}^{(Z)} = 2\,\mathcal{C}_{\mathrm{P}}^{(2)}, \qquad \text{cf. (22.14)}.$$

The reduced EM coefficient consumed by the neutral composite is $\mathcal{C}_{\mathrm{EM}}^{(2)}$, cf. (22.15). It must not be confused with the pure cap projector $\Pi_{\ell=1}^{\mathrm{EM}}$ defined in the EM-like field closure. The coefficient $\mathcal{C}_{\mathrm{EM}}^{(2)}$ is the source–cap–channel shape coefficient used in the bi-lobe reduction:

$$\mathcal{C}_{\mathrm{EM}}^{(2)} = \Pi_{\ell=1}^{\mathrm{EM}}\,\Gamma_{\mathrm{EM}}\,\frac{J_0^2}{\pi}, \qquad \text{cf. (22.19)}.$$

It corresponds to the reduced part without width transfer of the EM-like channel. The two-leg transfer factor

$$\eta_{\mathrm{2leg}} = \left(\frac{1}{\Delta_{\mathrm{EM}}^2 b_0^{(2)}}\right)^2$$

belongs to the dimensioned channel constant $\Phi_{\mathrm{EM}}^{(\mathrm{two})}$, cf. (20.37), but it is not consumed in the reduced coefficient $\mathcal{C}_{\mathrm{EM}}^{(2)}$ used for the neutral/charged ratio.

The stationarity of the internal energy, evaluated at the locked bound point, fixes the Preclusion coefficient:

$$A_{\mathrm{P}} = A_{\mathrm{EM}}\,\frac{\mathcal{O}_1'(u_{\mathsf{Z}}^\star)}{\mathcal{O}_2'(u_{\mathsf{Z}}^\star)}, \qquad \text{cf. (22.18)}.$$

The scale-free increment governing the neutral/charged separation is then

$$\delta^*(u_{\mathsf{Z}}^\star) = A_{\mathrm{P}}\,\mathcal{O}_2(u_{\mathsf{Z}}^\star) - A_{\mathrm{EM}}\,\mathcal{O}_1(u_{\mathsf{Z}}^\star).$$

The predicted mass is

$$\boxed{m_Z^{(\mathrm{th})} = m_W\,\sqrt{1 + |\delta^*(u_{\mathsf{Z}}^\star)|}.} \tag{30.1}$$

The associated theoretical ratio is

$$\boxed{R_{\mathrm{th}} := \left(\frac{m_Z^{(\mathrm{th})}}{m_W}\right)^4 = \left(1 + |\delta^*(u_{\mathsf{Z}}^\star)|\right)^2.} \tag{30.2}$$

Numerical evaluation (canonical LO). We use the full reduced coefficient of (22.19),

$$\mathcal{C}_{\mathrm{EM}}^{(2)} \simeq 1.538610075.$$

therefore

$$A_{\mathrm{EM}} = 4\,\mathcal{C}_{\mathrm{EM}}^{(2)} \simeq 6.154440302.$$

At the canonical point

$$u_{\mathsf{Z}}^\star \simeq 2.00712,$$

we obtain

$$\mathcal{O}_1(u_{\mathsf{Z}}^\star) = 0.3017655126, \qquad \mathcal{O}_2(u_{\mathsf{Z}}^\star) = 0.1619395279,$$

and

$$\mathcal{O}_1'(u_{\mathsf{Z}}^\star) = -0.3250327354, \qquad \mathcal{O}_2'(u_{\mathsf{Z}}^\star) = -0.2029911703.$$

Thus,

$$A_{\mathrm{P}} = A_{\mathrm{EM}}\,\frac{\mathcal{O}_1'(u_{\mathsf{Z}}^\star)}{\mathcal{O}_2'(u_{\mathsf{Z}}^\star)} \simeq 9.854589062,$$

and, equivalently,

$$\mathcal{C}_{\mathrm{P}}^{(2)} = \frac{A_{\mathrm{P}}}{2} \simeq 4.927294531.$$

We infer

$$\delta^*(u_{\mathsf{Z}}^\star) = A_{\mathrm{P}}\,\mathcal{O}_2(u_{\mathsf{Z}}^\star) - A_{\mathrm{EM}}\,\mathcal{O}_1(u_{\mathsf{Z}}^\star) = -0.2613503319,$$

hence

$$|\delta^*(u_{\mathsf{Z}}^\star)| = 0.2613503319,$$

$$\frac{m_Z^{(\mathrm{th})}}{m_W} = \sqrt{1 + |\delta^*(u_{\mathsf{Z}}^\star)|} = 1.1230985406,$$

and

$$R_{\mathrm{th}} = (1 + |\delta^*(u_{\mathsf{Z}}^\star)|)^2 = 1.5910046598.$$

For

$$m_W = 80.379 \text{ GeV},$$

we obtain

$$\boxed{m_Z^{(\mathrm{th})} = 90.2735 \text{ GeV}.}$$

Comparison with the observable. The comparison with the observable enters only a posteriori as a test of the closure:

$$m_Z^{(\mathrm{obs})} = 91.1876 \text{ GeV}, \qquad \Delta m_Z = m_Z^{(\mathrm{th})} - m_Z^{(\mathrm{obs})} = -0.9141 \text{ GeV},$$

namely a relative discrepancy

$$\frac{\Delta m_Z}{m_Z^{(\mathrm{obs})}} = -1.00 \times 10^{-2}.$$

Remarks. (i) This prediction is *scale-free*: it uses neither ρ, nor T, nor ℓ_0, nor R_0, and it does not consume $m_Z^{(\mathrm{obs})}$. There is therefore no circularity.
(ii) The quantity

$$R_{\mathrm{th}} = \left(\frac{m_Z^{(\mathrm{th})}}{m_W}\right)^4$$

will be used in the next subsection to recover κ_μ without consuming the observed ratio $m_Z^{(\mathrm{obs})}/m_W$.
(iii) The residual discrepancy of order 1% obtained here corresponds to the actual LO discrepancy of the neutral/charged 2DOF closure; the NLO corrections (anisotropy, refined spectral filters, complete conical closure) therefore remain relevant.

Uncertainty on the prediction. In the strategy retained in §29.2.1, the quantity $u_{\mathsf{Z}}^\star$ is the neutral bound point locked by (22.17). It therefore carries no additional local uncertainty in this section.

The reduced coefficient $\mathcal{C}_{\mathrm{EM}}^{(2)}$ consumed by $\delta^*(u_{\mathsf{Z}}^\star)$ is given by (22.19). As specified in the reduced formulas, this coefficient does not contain the width transfer η_{2leg}, and therefore does not contain $b_0^{(2)}$. Consequently,

$$\frac{\partial \ln \mathcal{C}_{\mathrm{EM}}^{(2)}}{\partial \ln b_0^{(2)}} = 0, \qquad \frac{\partial \delta^*(u_{\mathsf{Z}}^\star)}{\partial \ln b_0^{(2)}} = 0, \qquad \frac{\partial \ln(m_Z^{(\mathrm{th})}/m_W)}{\partial \ln b_0^{(2)}} = 0.$$

The regularization uncertainty attached to $b_0^{(2)}$ is therefore not propagated into the ratio $m_Z^{(\mathrm{th})}/m_W$.

The only propagated experimental contribution comes from m_W through (30.1); it is negligible at the useful precision retained here. Direct numerical errors on the evaluation of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_1'$, $\mathcal{O}_2'$, $\delta^*(u_Z^\star)$, and R_{th} are also negligible at this scale.

The relative discrepancy observed in the comparison with $m_Z^{(\mathrm{obs})}$ is therefore displayed as the LO closure discrepancy of the neutral/charged 2DOF doublet, and not as an error bar propagated from $b_0^{(2)}$.

30.0.2 2DOF bump – EM path (1-body): LO mass prediction

References (labels). The LO spectral closure of the 1-body mass of the 2DOF bump is given in §6.6.8. The factors entering this closure are defined by (6.73), (6.74), (6.75), and (6.77). The complete formula is (6.79b). The mechanical reading is given by (6.101), and the spectral–mechanical compatibility by (6.102)–(6.103).

Framework. In this prediction subsection, we identify $h_{\mathrm{eff}} = \hbar$ and $c_t = c$, in accordance with the conventions of the calibration chapter. We evaluate the complete LO closure of the 1-body mass of the 2DOF bump coming from the EM chain. Unlike the NSL scheme, there is here neither a two-leg structure nor an additional composition factor: the 1-body closure is used directly.

Constants and inputs (LO).

$$J_0 = 10.3354255601, \qquad J_2 = 7.6339320582, \qquad \kappa_\omega = 54.9589563237,$$

$$b_0^{(2)} = 1.809206964 \times 10^4, \qquad \ell_0 = 2.079372183 \times 10^{-22}\ \mathrm{m},$$

$$h_{\mathrm{eff}} = \hbar = 1.054571817 \times 10^{-34}\ \mathrm{J\,s}, \qquad c = 299792458\ \mathrm{m\,s^{-1}}.$$

Complete LO formula (displayed explicitly). The complete LO closure of the 1-body mass of the 2DOF bump is

$$m_{\mathrm{bump}}^{(\mathrm{LO})} c^2 = \Big(\frac{h_{\mathrm{eff}} c}{\ell_0}\Big)\, \Pi_{\mathrm{proj}}\, Z_{\mathrm{ann}}\, Z_{\mathrm{proj}}\, \frac{1}{\kappa_\omega^2\, b_0^{(2)}}, \tag{30.3}$$

in accordance with (6.79b), with

$$\Pi_{\mathrm{proj}} = \Big(\frac{J_2}{J_0}\Big)\Big(\frac{J_0}{J_0 + J_2}\Big)^2 = 0.2443495536,$$

$$Z_{\mathrm{ann}} = (2\pi)^{-1} = 0.1591549431, \qquad Z_{\mathrm{proj}} = \sqrt{\frac{J_0}{J_0 + J_2}} = 0.7583992690.$$

Numerical evaluation (LO). We obtain

$$\frac{h_{\mathrm{eff}} c}{\ell_0} = \frac{\hbar c}{\ell_0} \simeq 9.4897383922 \times 10^8\ \mathrm{MeV},$$

then

$$\Pi_{\mathrm{proj}}\, Z_{\mathrm{ann}}\, Z_{\mathrm{proj}}\, \frac{1}{\kappa_\omega^2\, b_0^{(2)}} = 5.3971486886 \times 10^{-10}.$$

Thus

$$\boxed{m_{\mathrm{bump}}^{(\mathrm{LO})} = 0.512175291\ \mathrm{MeV}.} \tag{30.4}$$

Structure of the calculation. The LO prediction comes directly from the complete EM 1-body chain: micro-scale ($h_{\text{eff}}c/\ell_0$), projective factor Π_{proj}, mono-ring average Z_{ann}, transverse kinetic projection Z_{proj}, and spectral closure through $\kappa_\omega^2\, b_0^{(2)}$. No additional corrective factor (sheath, anchoring, etc.) is introduced at this order in this complete expression.

Link with the 1-body mechanical closure. The mass computed in this subsection is the spectral mass $m_{\text{bump}}^{(\text{spec})}$ defined by (6.79a). It is independent of R_0. The mechanical closure

$$m_{\text{bump}}^{(\text{mech})} = \kappa_{\text{core}}\, \rho\, \lambda_e^2\, R_0$$

is related to the spectral mass by the compatibility condition (6.102). In the present chain, this compatibility is used upstream to close the combination

$$P_{\rho R} = \rho R_0$$

without using m_e^{obs}. The present subsection therefore reuses neither R_0, nor ρ, nor $P_{\rho R}$: it directly evaluates the spectral mass and then compares it with the observable.

Comparison with the observable. We denote by m_e^{obs} the electron mass, used here only as a comparison observable. We define

$$\Delta m \equiv m_{\text{bump}}^{(\text{LO})} - m_e^{\text{obs}}, \qquad \delta_{\%} \equiv 100 \times \frac{m_{\text{bump}}^{(\text{LO})} - m_e^{\text{obs}}}{m_e^{\text{obs}}}.$$

With

$$m_{\text{bump}}^{(\text{LO})} = 0.512175291 \text{ MeV}, \qquad m_e^{\text{obs}} = 0.51099895 \text{ MeV},$$

we obtain

$$\Delta m = 1.176341 \times 10^{-3} \text{ MeV} = 1.176341 \text{ keV},$$

and

$$\delta_{\%} = 2.302042 \times 10^{-1}\,\%.$$

Uncertainty on the prediction. In the strategy adopted in §29.2.1, the quantities J_0, J_2, h_{eff}, and c are taken as fixed by the strict closures or by the calibration conventions. The quantity $b_0^{(2)}$ is a regularized LO closure, but its effect must be propagated in correlation with ℓ_0 and κ_ω.

The LO mass is written

$$m_{\text{bump}}^{(\text{LO})} \propto \ell_0^{-1}\, \frac{1}{\kappa_\omega^2 b_0^{(2)}}.$$

Now

$$\kappa_\omega^2 \propto b_0^{(2)}, \qquad \left.\frac{\delta \ell_0}{\ell_0}\right|_{b_0} = -2\, \frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

The correlated contribution of $b_0^{(2)}$ to this prediction is therefore

$$\boxed{\left.\frac{\delta m_{\text{bump}}^{(\text{LO})}}{m_{\text{bump}}^{(\text{LO})}}\right|_{b_0} = -\left(-2\, \frac{\delta b_0^{(2)}}{b_0^{(2)}}\right) - 2\, \frac{\delta b_0^{(2)}}{b_0^{(2)}} = 0.} \tag{30.5}$$

Thus, when the same closure $b_0^{(2)}$ is used both in the EM calibration of ℓ_0 and in the 2DOF bump mass, the regularization uncertainty of $b_0^{(2)}$ cancels at LO in this prediction. The remaining metrological dependence is that of Z_0^{obs}, via

$$\frac{\delta m_{\mathrm{bump}}^{(\mathrm{LO})}}{m_{\mathrm{bump}}^{(\mathrm{LO})}} = \frac{1}{2}\,\frac{\delta Z_0}{Z_0^{\mathrm{obs}}},$$

to which are added the scheme uncertainties specific to the LO order. The value displayed below is therefore a central LO value consistent with the adopted closure $b_0^{(2)}$.

Quantity	Value	Comment
$m_{\mathrm{bump}}^{(\mathrm{LO})}$	0.512175291 MeV	LO prediction, complete EM 1-body path
m_e^{obs}	0.51099895 MeV	comparison observable
Δm	1.176341×10^{-3} MeV	1.176341 keV
$\delta_{\%}$	2.302042×10^{-1} %	relative to m_e^{obs}
z	$\dfrac{1.176341\ \mathrm{keV}}{\sigma_{m_e}\ [\mathrm{keV}]}$	depends on σ_{m_e}

30.1 2DOF breather: LO ratios of mass-square gaps

References (labels). §23.3.1, §23.3, (23.21), (23.22), (23.27), (23.28), (23.29), (23.30), (23.31), (23.33), (23.34), (23.35), (23.36), (23.37), (23.45), §29.2.

Framework (LO, scale-free). At LO, the structure of the mass-square gaps of the 2DOF breathers is

$$\Delta m^2(q) = M_*^2\,\mathcal{S}(q; q_\star), \tag{30.6}$$

where M_*^2 is a common prefactor for all levels, irrelevant for the ratios, and $\mathcal{S}(q; q_\star)$ is a pure number encoding the spectral structure of the 2DOF breather on $\mathbb{S}^3$, with differential saturation beyond the first finite threshold $q_\star$.

In the present observational part, the indices q are taken from the common calibration triplet fixed in §29.3.1:

$$\boxed{(q_1, q_2, q_3)_{\mathrm{cal}} = (1,\ 207,\ 3477).}$$

The first finite threshold is therefore

$$\boxed{q_\star = q_2 = 207.}$$

Status of the threshold identification. This use of the same triplet for the 2DOF breather branch and for the 2DOF bump branch is a structural identification of admissible thresholds. The triplet $(1, 207, 3477)$ is a calibration input obtained from the charged leptons; the neutrino comparison therefore tests the reuse of the same spectral filter in the breather branch. The ratios obtained below do not consume the observed neutrino mass gaps.

Convention for comparison with oscillation observables. Oscillation data do not provide the absolute masses of the neutrino states, but two independent mass-square gaps. In this section, the quantity

$$\Delta m^2(q) = M_*^2\,\mathcal{S}(q; q_\star)$$

denotes the mass-square-gap scale carried by the spectral threshold q, not the absolute squared mass of a level. The comparison with data is performed by direct matching between spectral thresholds and observed gaps:

$$\Delta m^2_{\rm sol} \longleftrightarrow \Delta m^2(q_{\rm sol}), \qquad \Delta m^2_{\rm atm} \longleftrightarrow \Delta m^2(q_{\rm atm}).$$

In the LO matching retained here,

$$q_{\rm sol} = 1, \qquad q_{\rm atm} = 207.$$

Adopted spectral closure. The function $\mathcal{S}(q; q_\star)$ is defined by (23.37). It uses the canonical EBK cutoff

$$k_c(q) = 5q^{1/3},$$

and then the saturated effective cutoff

$$k_{\rm eff}(q; q_\star)$$

defined by (23.27). This saturation does not modify the first finite threshold $q_\star$; it only bounds the spectral excess $k_c^2(q) - k_c^2(q_\star)$ beyond this threshold.

Numerical evaluation of the spectral ratios (LO). By numerically evaluating the procedure defined by (23.27)–(23.37), with

$$q_\star = 207, \qquad L_{\rm max} = c\, m R_0, \qquad c \gtrsim 8,$$

and then applying a joint stabilization of $(mR_0, L_{\rm max})$, we obtain

q	$\mathcal{S}(q; 207)$
1	1
207	36.84
3477	207.78

(30.7)

and therefore

$$\boxed{\frac{\mathcal{S}(207;207)}{\mathcal{S}(1;207)} \simeq 36.84, \qquad \frac{\mathcal{S}(3477;207)}{\mathcal{S}(1;207)} \simeq 207.78, \qquad \frac{\mathcal{S}(3477;207)}{\mathcal{S}(207;207)} \simeq 5.64.} \tag{30.8}$$

Inverse form (alternative reading).

$$\boxed{\frac{\mathcal{S}(1;207)}{\mathcal{S}(207;207)} \simeq 0.0271, \qquad \frac{\mathcal{S}(1;207)}{\mathcal{S}(3477;207)} \simeq 0.00497, \qquad \frac{\mathcal{S}(207;207)}{\mathcal{S}(3477;207)} \simeq 0.183.}$$

Comparison with the observational ratio. For the numerical comparison in normal ordering, the central reference values used are those of NuFIT 5.3 [3]:

$$\Delta m^2_{21} = 7.41 \times 10^{-5}\ \text{eV}^2, \qquad \Delta m^2_{3\ell} = 2.507 \times 10^{-3}\ \text{eV}^2.$$

These values are used only as a comparison reference for the observational ratio; they are not consumed in the LO closure of the breather prefactor. The observational atmospheric/solar ratio is therefore

$$\boxed{R_{\rm obs} := \frac{\Delta m^2_{\rm atm}}{\Delta m^2_{\rm sol}} = \frac{2.507 \times 10^{-3}}{7.41 \times 10^{-5}} = 33.83.} \tag{30.9}$$

In the LO matching retained here, the structural prediction is

$$\boxed{R_{\mathrm{th}}^{(\mathrm{LO})} = \frac{\mathcal{S}(207;207)}{\mathcal{S}(1;207)} = 36.84.} \tag{30.10}$$

The corresponding relative gap is

$$\frac{R_{\mathrm{th}}^{(\mathrm{LO})} - R_{\mathrm{obs}}}{R_{\mathrm{obs}}} = \frac{36.84 - 33.83}{33.83} = 8.90\%.$$

Table 30.1: 2DOF breather (LO) — comparison of the structural ratio with the atmospheric/solar ratio.

Quantity	Predicted (LO)	Observed	Relative gap
$R_{\mathrm{atm/sol}}$	36.84	33.83	+8.90%

Minimal mass-ratio reading. In the minimal case $m_0^2 = 0$, the mass ratio associated with the two oscillation gaps is the square root of the preceding ratio:

$$\boxed{\left(\frac{m_3}{m_2}\right)_{\mathrm{HSM}}^{\mathrm{LO}} = \sqrt{36.84} = 6.070.} \tag{30.11}$$

The corresponding observational reading, constructed from the reference values used in (30.9), is

$$\boxed{\left(\frac{m_3}{m_2}\right)_{\mathrm{obs}} = \sqrt{\frac{2.507 \times 10^{-3}}{7.41 \times 10^{-5}}} = 5.817.} \tag{30.12}$$

The relative gap is therefore

$$\boxed{\frac{6.070 - 5.817}{5.817} = 4.34\%.} \tag{30.13}$$

Table 30.2: 2DOF breather (minimal LO) — comparison of the reconstructed mass ratio.

Quantity	Predicted (LO)	Oscillation reference	Relative gap
m_3/m_2	6.070	5.817	+4.34%

The bands reported in this section correspond only to the numerical stability of the EBK spectral closure with respect to $L_{\max}$, the effective window width, and the angular discretization.

30.2 2DOF breather: absolute masses and bound on the quartic floor

References (labels). §23.3.5, (23.51), (23.52), (23.56), (23.57), (23.58), (23.59), (23.62), (23.63), (23.66), (23.67), §23.3, §23.3.5, §29.2.

Reminders (numerical inputs and absolute prefactor). In this subsection, we work in calibration units with $h_{\rm eff} = \hbar$ and $c_t = c$. The absolute prefactor of the mass-square gaps of the 2DOF breather is written, in accordance with (23.58),

$$M_*^2 = \Gamma_{\rm shape}^{(\rm LO)}\, \mathcal{B}_\nu^{(\rm LO)}.$$

The shape constant is derived upstream:

$$\Gamma_{\rm shape}^{(\rm LO)} = \sin^2\psi_c^{(\rm can)}\, \frac{J_2}{J_0}\, \frac{1}{\kappa_\omega^2} = 2.04629 \times 10^{-4}.$$

Here $\psi_c^{(\rm can)}$ denotes the canonical conical angle defined by (22.46)–(22.47). It is distinct from the cell angle $\psi_c^{(\rm cell)}$, for which $\sin^2\psi_c^{(\rm cell)} = 1/3$, cf. (6.47).

The common absolute block of the breather is

$$\mathcal{B}_\nu^{(\rm LO)} = \Phi_{\rm core}^{(\rm LO)}\, \delta_\omega^{12} \left(\frac{\hbar c}{\ell_0}\right)^2, \qquad \delta_\omega = \kappa_\omega^{-1}.$$

With

$$\Phi_{\rm core}^{(\rm LO)} = 3.0954497129 \times 10^{-10}, \qquad \delta_\omega^{12} = 1.3168559732 \times 10^{-21},$$

and with the calibrated value of ℓ_0, we obtain

$$\frac{\hbar c}{\ell_0} \simeq 9.4897383922 \times 10^{14}\ \text{eV},$$

and then

$$\boxed{\mathcal{B}_\nu^{(\rm LO)} = 3.6708827364 \times 10^{-1}\ \text{eV}^2, \qquad M_*^2 \simeq 7.5117 \times 10^{-5}\ \text{eV}^2.}$$

Propagation of the closure uncertainty of $b_0^{(2)}$. The absolute prefactor M_*^2 inherits the regularized LO spectral closure $b_0^{(2)}$. This dependence must be propagated with the correlations already fixed in the preceding sections.

We use

$$\kappa_\omega^2 \propto b_0^{(2)}, \qquad \delta_\omega = \kappa_\omega^{-1}, \qquad \left.\frac{\delta\ell_0}{\ell_0}\right|_{b_0} = -2\, \frac{\delta b_0^{(2)}}{b_0^{(2)}}.$$

The three factors of M_*^2 then behave as follows:

$$\Gamma_{\rm shape}^{(\rm LO)} \propto \kappa_\omega^{-2} \propto \big(b_0^{(2)}\big)^{-1},$$

$$\Phi_{\rm core}^{(\rm LO)} \propto \big(b_0^{(2)}\big)^{-2}, \qquad \delta_\omega^{12} = \kappa_\omega^{-12} \propto \big(b_0^{(2)}\big)^{-6},$$

and

$$\left(\frac{\hbar c}{\ell_0}\right)^2 \propto \big(b_0^{(2)}\big)^{4}.$$

Thus,

$$\boxed{M_*^2 \propto \big(b_0^{(2)}\big)^{-1-2-6+4} = \big(b_0^{(2)}\big)^{-5}.} \tag{30.14}$$

The relative contribution of $b_0^{(2)}$ to the absolute prefactor is therefore

$$\boxed{\left.\frac{\delta M_*^2}{M_*^2}\right|_{b_0} = -5\, \frac{\delta b_0^{(2)}}{b_0^{(2)}}.} \tag{30.15}$$

With the regularized LO envelope

$$\frac{\sigma_{b_0}}{b_0} \simeq 2.2\%,$$

we obtain a closure uncertainty of approximately

$$\left.\frac{\sigma_{M_*^2}}{M_*^2}\right|_{b_0} \simeq 11\%.$$

In the minimal LO case $m_0 = 0$, the masses m_2 and m_3 are proportional to $\sqrt{M_*^2}$. They therefore inherit

$$\boxed{\left.\frac{\sigma_{m_2}}{m_2}\right|_{b_0} = \left.\frac{\sigma_{m_3}}{m_3}\right|_{b_0} = \frac{1}{2}\left.\frac{\sigma_{M_*^2}}{M_*^2}\right|_{b_0} \simeq 5.5\%.} \tag{30.16}$$

The absolute values displayed below are therefore LO central values; their closure-theory uncertainty is dominated, in this chain, by the residual dependence on $b_0^{(2)}$.

Triplet used and status. In the present observational part, the spectral indices q are those of the common calibration triplet fixed in §29.3.1:

$$\boxed{(q_1, q_2, q_3)_{\rm cal} = (1,\ 207,\ 3477).}$$

The absolute masses written below are therefore conditional on this triplet. The triplet is not rederived in this subsection; it is consumed as a common calibrated spectral input.

Absolute parametrization of the three states. Oscillation data fix two independent mass-square gaps, not the common absolute floor. We therefore parametrize the three masses in normal ordering as

$$m_1^2 = m_0^2, \qquad m_2^2 = m_0^2 + M_*^2 \mathcal{S}(1;207), \qquad m_3^2 = m_0^2 + M_*^2 \mathcal{S}(207;207).$$

With the normalization $\mathcal{S}(1;207) = 1$ and the numerical evaluation

$$\frac{\mathcal{S}(207;207)}{\mathcal{S}(1;207)} = 36.84,$$

we obtain

$$\boxed{m_1^2 = m_0^2, \qquad m_2^2 = m_0^2 + M_*^2, \qquad m_3^2 = m_0^2 + 36.84\, M_*^2.} \tag{30.17}$$

The value $\mathcal{S}(3477;207)$ does not enter this minimal parametrization of the three masses associated with the two oscillation gaps; it remains a higher spectral level useful in the analysis of structural ratios.

Minimal LO. The minimal LO case corresponds to a vanishing common floor,

$$m_0^2 = 0,$$

in accordance with the LO status recalled by (23.62). We then obtain

$$m_1 = 0, \qquad m_2 = \sqrt{M_*^2}, \qquad m_3 = \sqrt{36.84\, M_*^2}.$$

Numerically, displaying the closure uncertainty from (30.16),

$$\boxed{m_1 = 0, \qquad m_2 = (8.67 \pm 0.48) \times 10^{-3}\ \text{eV}, \qquad m_3 = (5.26 \pm 0.29) \times 10^{-2}\ \text{eV}.}$$

The sum is

$$\boxed{\Sigma m_\nu^{(\rm LO,min)} = 6.127 \times 10^{-2}\ \text{eV}.} \tag{30.18}$$

Quartic parameter of the common floor. Beyond minimal LO, a quartic mesh or background correction may induce a nonzero common floor. The LO closure used in this section does not fix this quartic coefficient; it does, however, fix the form of its effect on the absolute masses. We therefore introduce the effective parameter λ_4^{eff} by

$$m_0^2 = \lambda_4^{\text{eff}} \left(\frac{2}{3}\right) \mathcal{B}_\nu^{(\text{LO})}. \tag{30.19}$$

The factor 2/3 is the shape ratio

$$\frac{I_4}{I_2} = \frac{2}{3}$$

of the 2DOF breather branch. With

$$\mathcal{B}_\nu^{(\text{LO})} = 3.6708827364 \times 10^{-1}\ \text{eV}^2,$$

we obtain

$$\boxed{m_0^2 = 0.2447255158\, \lambda_4^{\text{eff}}\ \text{eV}^2.} \tag{30.20}$$

The coefficient λ_4^{eff} does not modify the spectral gaps:

$$m_2^2 - m_1^2 = M_*^2, \qquad m_3^2 - m_1^2 = 36.84\, M_*^2.$$

It only shifts the common floor of the absolute masses. The closed prediction at LO therefore remains the minimal case

$$\lambda_4^{\text{eff}} = 0, \qquad m_0^2 = 0.$$

A coefficient $\lambda_4^{\text{eff}} > 0$ belongs to an additional quartic closure; it is not used here as an independent prediction.

Mass sum in the presence of a quartic floor. For $\lambda_4^{\text{eff}} \geq 0$, the mass sum is

$$\boxed{\Sigma m_\nu(\lambda_4^{\text{eff}}) = \sqrt{A\lambda_4^{\text{eff}}} + \sqrt{A\lambda_4^{\text{eff}} + M_*^2} + \sqrt{A\lambda_4^{\text{eff}} + 36.84\, M_*^2},} \tag{30.21}$$

with

$$\boxed{A = 0.2447255158\ \text{eV}^2.} \tag{30.22}$$

This expression shows that λ_4^{eff} acts as a common floor: it increases the absolute masses without changing the two oscillation gaps.

Observational bound on λ_4^{eff}. An external bound on the neutrino mass sum provides a direct constraint on λ_4^{eff}. If

$$\Sigma m_\nu \leq \Sigma_{\max}^{\text{cosmo}},$$

then λ_4^{eff} is bounded by the solution of

$$\Sigma m_\nu(\lambda_{4,\max}^{\text{eff}}) = \Sigma_{\max}^{\text{cosmo}}. \tag{30.23}$$

Using the cosmological bound

$$\Sigma_{\max}^{\text{cosmo}} = 0.12\ \text{eV}$$

cited in [4], Eq. (30.23) gives

$$\boxed{\lambda_4^{\text{eff}} \leq 3.52 \times 10^{-3}.} \tag{30.24}$$

The corresponding common floor is

$$\boxed{m_0 = \sqrt{A\lambda_4^{\text{eff}}} \leq 2.94 \times 10^{-2}\ \text{eV}.} \tag{30.25}$$

Reading of the result. The bound (30.24) is not a prediction of λ_4^{eff}. It expresses the constraint imposed by the cosmological bound on the admissible quartic floor. The closed prediction published at LO remains

$$\lambda_4^{\mathrm{eff}} = 0, \qquad \Sigma m_\nu^{(\mathrm{LO,min})} = 6.101 \times 10^{-2}\ \mathrm{eV}.$$

A positive floor remains possible as a quartic correction, but it is constrained by (30.23) as soon as an external bound on Σm_ν is chosen.

30.3 Six-bump 2DOF composite: internal output of the 3×2 reduction

Object evaluated. Here we evaluate the internal output of the strong-overlap branch of the composite $\mathcal{C}_{3\times2}$, defined in Chapter 24. In this reading, the composite is a composite of six 2DOF bumps whose effective reduction selects three collective breather-like witnesses. The EBK triplet consumed is the one fixed in §29.3.1:

$$\boxed{(q_1, q_2, q_3) = (1, 207, 3477).} \tag{30.26}$$

This section evaluates only the internal reduced matrix and the associated angles. It does not perform any external identification of the eigenmodes and it does not derive an absolute mass for the composite.

Status of the degrees of freedom. Once the triplet (30.26) has been fixed, the 3×2 projective reduction contains no adjustable angular parameter. The quantities G_2, G_3, $\Xi_{207,3477}$, R_k, γ, ϵ_{D_η}, H_{23}^{br2}, α_{D_η}, and β_{D_η} are computed by the closures cited below. The angles ϑ_{ij} are therefore not injected into the construction; they are extracted a posteriori by real diagonalization of the reduced matrix.

Consumed theoretical references. The theoretical construction consumed is the projective closure of the composite:

(24.8), (24.10), (24.13), (24.14), (24.15), (24.16),

(24.17), (24.18), (24.19), (24.20), (24.21),

(24.22), (24.23).

The projected EM-like matching is consumed from the field–solution coupling:

(21.8), (21.10), (21.11).

The channel constant Γ_{EM} is the one defined by (20.36).

Width compatibility of the fine block. For the internal pair $q_2/q_3 = 207/3477$, the width compatibility is

$$\boxed{\Xi_{207,3477} = \frac{4(207)(3477)}{(207+3477)^2} \simeq 0.212.} \tag{30.27}$$

Projected witnesses and reduced metrics. The internal q_2-like projector gives

$$\boxed{G_2 = G_{22}^{\mathrm{br2}} \simeq \frac{1.6155}{207} \simeq 7.80 \times 10^{-3}.} \tag{30.28}$$

The saturated angular locking gives

$$\boxed{R_k = \frac{k_{3477}}{k_{207}} \simeq \frac{3477}{207}\,\Xi_{207,3477} \simeq 3.56, \qquad \gamma = \frac{1 - R_k}{2(1 + R_k)} \simeq -0.28.} \tag{30.29}$$

The metric of the collective transverse witness is then

$$\boxed{G_3 = G_{33}^{\mathrm{br2}} \simeq 1.36 \times 10^{-2}.} \tag{30.30}$$

Quasi-resonance of the fine block. The fine diagonal correction is

$$\boxed{\mathcal{C}_K^{\mathrm{fine}} = 1 + \Xi_{207,3477}\left(1 - \frac{207}{3477}\right) + O(e^{-207\Delta R_\star}) \simeq 1.20.} \tag{30.31}$$

By consuming (24.18), we obtain the range

$$\boxed{\epsilon_{D_\eta} \simeq 0.087\text{–}0.090.} \tag{30.32}$$

Projected EM-like coupling. The fine off-diagonal term is

$$\boxed{H_{23}^{\mathrm{br2}} = \Gamma_{\mathrm{EM}}\,\Xi_{207,3477}\,\frac{2}{\pi} \simeq 1.42 \times 10^{-2}.} \tag{30.33}$$

The metric normalization gives

$$b_{\mathrm{EM}} = \frac{H_{23}^{\mathrm{br2}}}{\sqrt{G_2 G_3}}.$$

With (30.28)–(30.30), one obtains

$$b_{\mathrm{EM}} \simeq 1.3\text{–}1.4.$$

Cross-closure of the fine scale. The common scale of the fine block is

$$M_{\mathrm{fine}} := \frac{m_2 + m_3}{2}.$$

The geometric route and the EM-like route give the same reduced scale at the accuracy order of the projective closure:

$$\boxed{M_{\mathrm{fine}}^{(\alpha)} \simeq M_{\mathrm{fine}}^{(\beta)} \simeq 1.5 \pm 0.1.} \tag{30.34}$$

It follows that

$$\boxed{\alpha_{D_\eta} \simeq 0.35\text{–}0.38, \qquad \beta_{D_\eta} \simeq 0.82\text{–}0.84.} \tag{30.35}$$

Reduced matrix. The central evaluation of the preceding ranges gives

$$\boxed{\mathcal{M}_{\mathrm{red}}^{D_\eta} \simeq \begin{pmatrix} 0 & 0.373 & 0 \\ 0.373 & 1.089 & 0.828 \\ 0 & 0.828 & 0.911 \end{pmatrix}.} \tag{30.36}$$

This matrix is an internal reduced matrix of the composite. It is not an absolute mass matrix. Its coefficients come from the closures listed above and from the reduced values computed in this section; they are not fitted to the external comparison angles.

Internal angles of the projective reduction. The real diagonalization of (30.36), with the usual real-rotation convention, gives

$$\boxed{(\vartheta_{12}, \vartheta_{13}, \vartheta_{23}) \simeq (33.4^\circ,\ 8.6^\circ,\ 49.3^\circ\text{–}49.5^\circ).} \tag{30.37}$$

These angles are the main internal outputs of the 3×2 projective reduction of the six-bump 2DOF composite for the triplet (30.26). They are obtained only by diagonalizing (30.36). No external PMNS angle enters this extraction. The numerical proximity to the comparison angles must therefore not be read as the result of an angular fit, but as a consistency check of the projective reduction once the triplet has been consumed. The external interpretation of the angles ϑ_{ij} belongs to the correspondence chapter.

30.4 Isolated 3DOF bump: internal source outputs

Theoretical references. The isolated 3DOF bump branch is defined by §10.11, in particular (10.13), (10.14) and (10.15).

Isolated spectral energy. The isolated spectral part is

$$\frac{E^{\mathrm{spec}}_{f,X}}{E_*} = q_f \sigma_X .$$

With

$$(q_1, q_2, q_3)_{\mathrm{cal}} = (1, 207, 3477), \qquad \sigma_\downarrow = \frac{1}{3}, \qquad \sigma_\uparrow = \frac{2}{3},$$

and

$$E_* = 0.512175291 \text{ MeV},$$

one obtains the following internal outputs:

branch	(f, X)	$E^{\mathrm{spec}}_{f,X}/E_*$	$E^{\mathrm{spec}}_{f,X}$ [MeV]
d	$(q_1, \downarrow)$	$1/3$	0.170725097
u	$(q_1, \uparrow)$	$2/3$	0.341450194
s	$(q_2, \downarrow)$	69	35.3400951
c	$(q_2, \uparrow)$	138	70.6801902
b	$(q_3, \downarrow)$	1159	593.611162
t	$(q_3, \uparrow)$	2318	1187.222325

These values are internal spectral masses of an isolated 3DOF bump.

Isolated strong-like self-response. The theory also defines

$$E^{\mathrm{iso}}_{\mathsf{s}}[S_{f,X}] = \frac{1}{2}\langle S_{f,X}, G_{\mathsf{s}} S_{f,X} \rangle .$$

Passage to bound states. The masses above characterize the isolated 3DOF bump solutions in the spectral closure. When one or more 3DOF bumps enter a color–white composite, their energy is no longer obtained by simply adding the isolated masses: the collective strong-like matching, the whiteness defect, and the internal junction modes modify the total energy.

The following section constructs these corrections directly on the collective white source of the CB3. This bound closure is the one that must be used for the baryon-like masses.

Uncertainty. The values in the table are LO central outputs of the isolated spectral closure

$$E_{f,X}^{\mathrm{spec}} = E_* q_f \sigma_X.$$

The factors $\sigma_\downarrow = 1/3$ and $\sigma_\uparrow = 2/3$ are rational branch coefficients. The integers q_f are those of the discrete spectral closure recalled in the present section. The uncertainty propagated to $E_{f,X}^{\mathrm{spec}}$ therefore comes only from any uncertainty attached to the scale E_*:

$$\Delta E_{f,X}^{\mathrm{spec}} = q_f \sigma_X \, \Delta E_*, \qquad \frac{\Delta E_{f,X}^{\mathrm{spec}}}{E_{f,X}^{\mathrm{spec}}} = \frac{\Delta E_*}{E_*}.$$

When no numerical uncertainty is supplied for E_* in the upstream chain, the displayed values must be read as LO central values.

30.5 CB3: LO predictions of color–white baryon-like masses

Theoretical references. The CB3 closure used here is defined by (28.23), with (28.19), (28.16), (28.17) and (28.18). The low mode of the sub-couple $udQ_\uparrow$ is defined by (28.28) and (28.31).

Traced numerical inputs. We use the spectral triplet calibrated in §29.3.1,

$$(q_1, q_2, q_3)_{\mathrm{cal}} = (1, 207, 3477),$$

and the spectral energy scale

$$E_* = m_{\mathrm{bump}}^{(\mathrm{LO})} c_t^2,$$

whose numerical value is that of (30.4),

$$E_* = 0.512175291 \text{ MeV}.$$

The shape constants $b_0^{(2)}$ and J_2 are those invoked in (28.23) and the definition

$$\tau_3 = \sqrt{3\, b_0^{(2)}\, J_2}.$$

Branch dictionary. We use the internal identifications

$$u = (q_1, \uparrow), \qquad d = (q_1, \downarrow), \qquad s = (q_2, \downarrow), \qquad c = (q_2, \uparrow), \qquad b = (q_3, \downarrow).$$

This dictionary does not turn the isolated 3DOF bumps into observable states: it is used only to build the color–white CB3 combinations.

Generic formula. The generic CB3 mass is computed by

$$\frac{M_{abc}^{\mathrm{CB3,gen}} c_t^2}{E_*} = 3\tau_3 + \sum_{i=1}^{3} q_i \sigma_i + \frac{1}{6\pi^2}\, \mathcal{W}_{abc}(\delta_*) - \frac{\tau_3}{2\pi q_{\max}} + \mathcal{F}_{\mathrm{fam}}^{abc}.$$

For the LO numerical outputs below, we use the canonical CB3 matching threshold

$$\delta_{\mathrm{CB3}}^{\mathrm{LO}} = 1.$$

This value is not a branch variational optimum. It corresponds to the LO threshold at which the CB3 sub-cores leave saturation, defined by (28.16) and justified by Proposition 28.4. It is not calibrated on the baryonic masses used for comparison.

Low mode $udQ_\uparrow$. When the third constituent is a heavy $\uparrow$ branch, the mechanical low mode of the ud sub-couple is

$$M_{udQ_\uparrow}^{\mathrm{CB3,bas}} = M_{udQ_\uparrow}^{\mathrm{CB3,gen}} - \frac{E_* \tau_3}{c_t^2}\left(1 - \frac{1}{q_Q}\right)\frac{2G_\uparrow(q_Q)}{1 + 2G_\uparrow(q_Q)}.$$

In the table, udc_{bas} denotes this low branch; udc_{gen} denotes the non-relaxed generic branch.

Generic LO numerical outputs. The following values are obtained by direct substitution in (28.23) and, for udc_{bas}, in (28.31). Direct recalculation of (28.23), with the constants E_*, $b_0^{(2)}$, J_2, q_c, the projections σ_X, ρ_X, χ_X, and the saturated/non-saturated rule indicated below, regenerates all the table values to the displayed rounding.

For the light doublet uud/udd, this table gives the generic masses before the nucleon-like fine correction. The physical splitting $m_n - m_p$ is treated separately below.

The numerical substitution explicitly consumes the branch projections and holonomies

$$\sigma_\downarrow = \frac{1}{3}, \qquad \sigma_\uparrow = \frac{2}{3}, \qquad \rho_\downarrow = \frac{2}{3}, \qquad \rho_\uparrow = \frac{1}{3}, \qquad \chi_\downarrow = 1, \qquad \chi_\uparrow = \frac{2\pi}{3},$$

defined by (21.28) and (21.29). It also consumes the reduced cutoff

$$q_c = 4\pi\sqrt{\frac{3\,b_0^{(2)}}{J_2}},$$

defined by (28.12), the diagonal weights

$$G_i = \frac{1}{1 + \left(\frac{q_i \rho_i}{q_c}\right)^2}, \qquad Q_i^{\mathrm{CB3}} = q_i \sigma_i \chi_{X_i}, \qquad \Xi_i = \sigma_i \chi_{X_i},$$

and the LO saturated/non-saturated overlap rule

$$\mathcal{O}_{ij}^{\mathrm{CB3,LO}} = \begin{cases} 1, & q_i = q_j, \\ (q_{ij}^{\max})^{-3/2}\,\mathrm{sech}(1), & q_i \neq q_j, \end{cases}$$

with $\delta_{\mathrm{CB3}}^{\mathrm{LO}} = 1$, cf. (28.16), (28.17) and (28.18). The family term finally consumes

$$\mathcal{S}_{ij} = \frac{1}{\sqrt{1 + \left(\frac{2(q_{ij}^{\max} - q_{ij}^{\min})}{3q_c}\right)^2}},$$

in accordance with (28.26).

CB3	Ref.	HSM LO gen. [MeV]	Obs. [MeV]	Deviation
uud	p	937.46 ($\pm$1.1%)	938.2721	-0.087%
udd	n	937.29 ($\pm$1.1%)	939.5654	-0.242%
udc_{gen}	$\Sigma_c(2455)^+$	2498.93	2452.65	$+1.887\%$
udc_{bas}	Λ_c^+	2280.51	2286.46	-0.260%
udb	Λ_b^0	5599.33	5619.57	-0.360%
uss	Ξ^0	1332.00	1314.86	$+1.304\%$
dss	Ξ^-	1331.80	1321.71	$+0.763\%$
sss	Ω^-	1668.30	1672.45	-0.248%
ssb	Ω_b^-	6108.70	6045.8	$+1.040\%$

Comparison with baryon-like states. The outputs uud and udd give the nucleon-like scale. The outputs uds, uss/dss, and sss give respectively the hyperon-like scales with one, two, and three strange branches. The branch udc_{bas} corresponds to the low mode of the $udQ_\uparrow$ matching, whereas udc_{gen} corresponds to the non-relaxed mode at the same LO rank.

Nucleon-like fine correction. The compact conical correction and the compact EM-like correction give, at compact LO rank,

$$\boxed{m_n - m_p \simeq \frac{E_*\tau_3}{18\pi^2} - \frac{E_*}{3} - \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}\frac{E_*\tau_3}{2\pi}.} \tag{30.38}$$

The first term comes from the conical complement $\rho = \cos^2\alpha$, the second from the spectral difference uud/udd, and the third from the compact EM-like term carried by the internal charge contrast of the CB3. This compact EM-like term is the observational reminder of (21.18). The elementary coupling $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$ is closed in the EM-like channel by (20.34).

With

$$\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} = 7.2846062 \times 10^{-3},$$

the fine correction gives

$$\boxed{m_n - m_p \simeq +1.31 \text{ MeV},}$$

to be compared with

$$(m_n - m_p)_{\mathrm{obs}} \simeq +1.293 \text{ MeV}.$$

Thus, the physical output for the $n - p$ splitting is that of the compact fine correction; the preceding generic table must be read as a table of base masses before this specific correction of the light doublet.

Status. The section compares color–white CB3 states, not isolated 3DOF bumps. The masses displayed in the table are the generic outputs relevant for a baryon-like comparison at LO rank. For the uud/udd doublet, the physical splitting consumes the nucleon-like fine correction (30.38), whose compact EM-like term uses the closed elementary coupling $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$.

The multi-heavy sub-branches fall under the internal selection criterion of the CB3 branches. When, with the calibrated constants, this criterion selects an interior minimum, the corresponding mass is reported. When the criterion places the branch on the boundary of the admissible domain, no interior-minimum mass is reported in this table.

Uncertainty and LO status. The displayed masses are LO central values of the CB3 closure

$$M_{abc}^{\mathrm{CB3}} = E_*\,\mathcal{M}_{abc}^{\mathrm{CB3}},$$

where $\mathcal{M}_{abc}^{\mathrm{CB3}}$ denotes the reduced quantity defined by (28.23), completed if necessary by (28.31).

The scale E_* is that of the central 2DOF bump mass (30.4):

$$E_* = m_{\mathrm{bump}}^{(\mathrm{LO})} c_t^2.$$

The correlated dependence of this scale on $b_0^{(2)}$ cancels at LO rank, as shown explicitly by (30.5). Indeed, in the complete EM 1-body chain,

$$m_{\mathrm{bump}}^{(\mathrm{LO})} \propto \ell_0^{-1}\,\frac{1}{\kappa_\omega^2 b_0^{(2)}}, \qquad \kappa_\omega^2 \propto b_0^{(2)}, \qquad \ell_0 \propto \left(b_0^{(2)}\right)^{-2},$$

hence

$$\frac{\partial \ln E_*}{\partial \ln b_0^{(2)}} = \frac{\partial \ln m_{\mathrm{bump}}^{(\mathrm{LO})}}{\partial \ln b_0^{(2)}} = 0.$$

The correlated regularization uncertainty of $b_0^{(2)}$ is therefore not transmitted through the scale E_*. The uncertainty transmitted by E_* is that of the retained central scale:

$$\Delta M_{abc}^{(E_*)} = \mathcal{M}_{abc}^{\mathrm{CB3}}\,\Delta E_*.$$

The rational branch coefficients carry no numerical uncertainty. However, $b_0^{(2)}$ also enters the CB3 shape constants, in particular

$$\tau_3 = \sqrt{3b_0^{(2)}J_2}, \qquad q_c = 4\pi\sqrt{\frac{3b_0^{(2)}}{J_2}}.$$

The local propagation gives

$$\frac{\Delta\tau_3}{\tau_3} = \frac{1}{2}\left(\frac{\Delta b_0^{(2)}}{b_0^{(2)}} + \frac{\Delta J_2}{J_2}\right), \qquad \frac{\Delta q_c}{q_c} = \frac{1}{2}\left(\frac{\Delta b_0^{(2)}}{b_0^{(2)}} - \frac{\Delta J_2}{J_2}\right).$$

The remaining dependence of the CB3 masses on q_c passes through the saturated or non-saturated weights of the reduced closure; it must be propagated by direct variation of formula (28.23) when a quantitative error bar is desired. The values displayed in the table are therefore LO central values, with a type-B closure uncertainty inherited from $b_0^{(2)}$.

Beyond this propagation, the dominant truncation uncertainties come from the local approximation of the overlaps $\mathcal{O}_{ij}$, the closure $\delta_* = 1$, and the selection diagnostic for multi-heavy sub-branches. These effects define the LO status of the displayed values and the perimeter of reported masses.

30.6 Higgs-like radial (scalar, $s_g = 0$): scheme prediction by minimal active core and NLO refinement

References (labels). §14.8, (14.14), (14.15), §14.10, (14.17), (14.18), §14.12, (14.19), (14.43), (14.44), (14.25), (14.26), (14.29), (14.35), (14.41), (14.42), (14.37), (14.38), (14.39), (14.45), 14.7.

Principle. We use the closure of the Higgs-like radial channel obtained from: (i) the minimal active core, (ii) the exact scalar matching on $\mathbb{S}^3$, and (iii) the minimal local NLO boundary refinement.

In the theoretical part, the closed relation is written relative to the internal charged NSL branch:

$$m_{\mathrm{Hlike}}^{(\mathrm{HSM})} = m_{\mathrm{NSL,ch}}^{(\mathrm{HSM})}\sqrt{1+\delta_H^*}.$$

In the present observational section, this internal charged NSL branch is specialized by the scale anchoring

$$m_{\mathrm{NSL,ch}}^{(\mathrm{HSM})} \longmapsto m_W.$$

Thus, for the numerical evaluation, one obtains

$$m_H^{(\mathrm{pred})} = m_W\sqrt{1+\delta_H^*}.$$

The prediction without specialized scale is the ratio

$$\frac{m_H^{(\mathrm{pred})}}{m_W} = \sqrt{1+\delta_H^*}.$$

The scalar increment corrected at minimal local NLO rank is

$$\delta_H^* = A_0\Big[B(u_H^*) + \Big(1+\frac{\pi}{2}\Big)\zeta(u_H^*)\Big],$$

where u_H^* is fixed by the closure law of the Higgs-like matching, cf. (14.45). The factor

$$\Big(1+\frac{\pi}{2}\Big)\zeta(u_H^*)$$

is not a coefficient fitted to the Higgs mass. It comes from the sum of the local LO term and the minimal NLO boundary term,

$$\Sigma_{\rm loc}(u)+\Sigma_{\rm rim}(u)=\zeta(u)+\frac{\pi}{2}\zeta(u),$$

with

$$\Sigma_{\rm rim}(u)=\frac{\pi}{2}\zeta(u),$$

cf. (14.42) and (14.37). The factor $\pi/2$ is the inverse of the oriented linear compliance

$$\mathcal{C}^{(1)}_{\partial H}(u)=\frac{2}{\pi}\mathcal{C}_{\rm stat}(u),$$

cf. (14.35). The quadratic control $1/\sqrt{2}$ exists in the theory as the RMS control of the same cycle, but it does not define the active boundary closure used here. The closure of u_H^* is then given by (14.45).

One has

$$B(u)=\mathcal{O}_2^{(S)}(u)-\frac{\mathcal{O}_2^{(S)\prime}(u)}{\mathcal{O}_1^{(S)\prime}(u)}\mathcal{O}_1^{(S)}(u),$$

with

$$\mathcal{O}_1^{(S)}(u)=(u\ \mathrm{csch}\,u)^2,\qquad \mathcal{O}_2^{(S)}(u)=2\,(u\coth u-1)\ \mathrm{csch}^2\,u,$$

and

$$\zeta(u)=\frac{1}{2}\,u\,\coth u\,\mathrm{csch}^2\,u.$$

In the monopole regime of the quasi-planar core,

$$A_0=\frac{J_0^2}{J_2}\,\hat{c}_0^{(S)}(u)^2\simeq\frac{J_0^2}{J_2}.$$

The choice $a(\xi)=\operatorname{sech}\xi$ must be read here as a canonical representative of the universal class of LO localized profiles, and not as a literal statement about the exact profile of the core.

Branch choice. We retain the fundamental branch of the minimal active core. At LO rank, the closed matching law gives

$$u^*_{H,\rm LO}=\frac{3\pi}{4}+\mathcal{O}\left(\frac{\ell_r^2}{R_0^2}\right),\qquad \text{cf. (14.29).}$$

With the minimal local NLO refinement, the matching law becomes

$$y\cot y=-u+\Big(1+\frac{\pi}{2}\Big)\zeta(u),\qquad \text{cf. (14.38).}$$

In the local reduction of the minimal active core, we use

$$y\simeq u,\qquad \text{cf. (14.26),}$$

which gives the local scalar law

$$u\cot u=-u+\Big(1+\frac{\pi}{2}\Big)\zeta(u).$$

Its fundamental solution is

$$u_H^*\simeq 2.3302825264.$$

The other branches of the scalar matching correspond to excited roots or to additional radial oscillations; they are not retained in the minimal scheme prediction.

Evaluation of the building blocks. At this value,

$$B(u_H^*) \simeq -0.01639, \qquad \zeta(u_H^*) \simeq 0.04580.$$

The NLO boundary factor is

$$1 + \frac{\pi}{2} \simeq 2.570796,$$

so that

$$B(u_H^*) + \Big(1 + \frac{\pi}{2}\Big)\zeta(u_H^*) \simeq 0.10135.$$

With

$$J_0 = 10.3354255601, \qquad J_2 = 7.6339320582, \qquad A_0 = \frac{J_0^2}{J_2} \simeq 13.993,$$

one obtains

$$\delta_H^* \simeq 1.41816.$$

Result. The observational specialization of the HSM relation gives

$$m_H^{(\mathrm{pred})} = m_W \sqrt{1 + \delta_H^*}.$$

For the anchoring value used in this evaluation,

$$m_W = 80.379 \text{ GeV},$$

one obtains

$$\boxed{m_H^{(\mathrm{pred})} \simeq 124.993 \text{ GeV}.}$$

Equivalently, the prediction without specialized scale is the ratio

$$\boxed{\frac{m_H^{(\mathrm{pred})}}{m_W} = \sqrt{1 + \delta_H^*} \simeq 1.55504.}$$

Comparison with the reference value. Taking as reference value

$$m_H^{(\mathrm{ref})} = 125.20 \pm 0.11 \text{ GeV},$$

one obtains

$$\Delta = m_H^{(\mathrm{pred})} - m_H^{(\mathrm{ref})} \simeq -0.207 \text{ GeV},$$

and

$$\frac{\Delta}{m_H^{(\mathrm{ref})}} \simeq -1.65 \times 10^{-3} \simeq -0.165\%.$$

The deviation remains below one percent, but it must not be presented as an agreement at the 10^{-5} level when the comparison is made with the current experimental value.

Status of the prediction. The present value is a *scheme prediction*: it rests on the minimal active core, the fundamental branch, the universalized scalar LO closure, and the minimal local NLO boundary refinement. No direct fit to the Higgs mass is introduced in this section.

The choice of the boundary factor $\pi/2$ is not selected by the final numerical agreement with m_H^{ref}. It is fixed upstream by the oriented linear reclosure compliance,

$$\mathcal{C}_{\partial H}^{(1)} = \frac{2}{\pi}\mathcal{C}_{\mathrm{stat}}, \qquad \Sigma_{\mathrm{rim}} = \frac{\pi}{2}\zeta,$$

cf. (14.35) and (14.42). A quadratic RMS closure would lead to another boundary scheme; it is not used in the displayed prediction.

The theoretical part first provides the relative relation

$$m_{\mathrm{Hlike}}^{(\mathrm{HSM})} = m_{\mathrm{NSL,ch}}^{(\mathrm{HSM})}\sqrt{1+\delta_H^*}.$$

The numerical value displayed above is obtained by specializing the internal charged NSL branch by the scale anchoring m_W. The scheme therefore predicts the specialized ratio $m_H^{(\mathrm{pred})}/m_W$, then the mass $m_H^{(\mathrm{pred})}$ is obtained by using m_W as scale anchoring.

Uncertainty. The direct numerical uncertainties associated with solving for u_H^* are negligible at the displayed precision, since u_H^* is determined by the closure law of the retained scheme. The experimental uncertainty of the anchoring m_W propagates linearly into $m_H^{(\mathrm{pred})}$, while the experimental uncertainty of the reference value $m_H^{(\mathrm{ref})}$ enters only in the final comparison. The residual structural uncertainties do not concern the existence of a closure of u_H^*, but the choice of retained rank, LO or minimal local NLO, and the decision not to include, in this chapter, nonlocal boundary degrees of freedom or higher-order boundary operators.

Comments. (i) The obtained matching value does not come from a canonical choice imported from the bump or the NSL, but from a closure specific to the Higgs-like radial channel. (ii) No empirical factor is fitted to the Higgs mass. (iii) The HSM relation uses the internal charged NSL branch as reference; the presence of m_W in this section only belongs to the observational specialization of this branch as scale anchoring. (iv) The proximity to the reference value constitutes a strong test of the retained Higgs-like scheme, without yet amounting, by itself, to a final demonstration of the complete scalar sector.

30.7 Photon: LO predictions of mass, speed, and polarizations

References (labels). The local transverse sector is defined by the Lagrangian (1.16). Its variation gives the transverse wave equation (1.17), with the local speed

$$c_t^2 = \frac{T}{\rho},$$

cf. (1.18). The quasi-c_t 2DOF bosonic branch is introduced in Definition 8.1 and in the chapter on quasi-c_t 2DOF solutions. Its energy, spin and normalization invariants are given in §8.3, with the action closure (8.11). The frequency selection on $\mathbb{S}^3$ is recalled by (8.14).

Observational identification. In HSM theory, the object computed at LO is the quasi-c_t transverse bosonic branch. In the observational part, this branch is identified with the photon because it simultaneously satisfies the three minimal criteria:

$$v = c, \qquad m_\gamma^{(\mathrm{LO})} = 0, \qquad N_{\mathrm{pol}}^{\mathrm{phys}} = 2.$$

The term "photon" therefore denotes here the observational reading of this transverse HSM branch, and not the prior import of a complete Abelian gauge theory.

Dispersion and speed. At LO and in the local transverse sector, the variation of (1.16) gives the quasi-plane dispersion

$$\omega^2 = c_t^2 k^2.$$

In the observational part, the calibrated identification

$$c_t = c$$

is used. One therefore obtains

$$\boxed{\omega^2 = c^2 k^2.} \tag{30.39}$$

The propagation speed of the mode identified with the photon is therefore

$$\boxed{v_\gamma^{(\mathrm{LO})} = c.} \tag{30.40}$$

LO mass. The quasi-c_t branch studied here is gapless at leading order: no linear effective mass term enters the LO dispersion. Comparing (30.39) with a Proca-type massive dispersion,

$$\omega^2 = c^2 k^2 + \frac{m_\gamma^2 c^4}{h_{\mathrm{eff}}^2},$$

gives

$$\boxed{m_\gamma^{(\mathrm{LO})} = 0.} \tag{30.41}$$

Physical polarizations. The mode identified with the photon is the purely tangential quasi-c_t 2DOF bosonic branch. In this branch, the field carries two transverse helicities, or equivalently two physical transverse polarizations. The non-transverse components are not propagating polarizations of this LO branch. We therefore retain

$$\boxed{N_{\mathrm{pol}}^{\mathrm{phys}} = 2.} \tag{30.42}$$

Observational channel	**Dispersion (LO)**	**LO mass**	**# physical pol.**
Photon	$\omega^2 = c^2 k^2$	$m_\gamma = 0$	2

Minimal observational comparison. The LO prediction

$$m_\gamma^{(\mathrm{LO})} = 0$$

is compatible with the existing experimental constraints on a possible photon mass. These constraints are upper bounds and depend on the assumptions used to relate an effective mass to astrophysical, solar, plasma or laboratory observations. This section therefore does not seek to select a single bound: it only notes that the HSM LO limit is the exactly massless case.

Comments. (i) At LO, this minimal observable is entirely fixed by the transverse sector and by the identification $c_t = c$ in the observational part.

(ii) The identification with the photon rests on three simultaneous criteria: propagation at c, zero mass at LO, and two physical transverse polarizations.

(iii) The branch considered is the quasi-c_t 2DOF bosonic branch of the HSM support. It does not assume the introduction of a Proca mass term at LO.

(iv) Gauge-type readings, as well as effective non-Abelian extensions, belong to a later level of description and are therefore not detailed in the present section.

30.7.1 EM-like fine-structure constant: LO prediction

References (labels). The elementary EM-like coupling is closed in the field channel by (20.34). This closure consumes the local Green factor $1/(4\pi)$, the elementary cyclic normalization $1/(2\pi)$, and the simple source projection of the core

$$\Pi^{\mathrm{EM}}_{\mathrm{source}} = \frac{J_0}{J_0 + J_2},$$

defined by (16.2). It is distinct from the active-closure coefficients with two projections such as Γ_{EM}.

LO prediction. The EM-like coupling constant predicted at LO is

$$\boxed{\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} = \frac{1}{8\pi^2}\frac{J_0}{J_0 + J_2}.} \tag{30.43}$$

With

$$J_0 = \frac{\pi^3}{3}, \qquad J_2 = 7.6339320582,$$

one obtains

$$\boxed{\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} \simeq 7.285 \times 10^{-3}, \qquad \left(\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}\right)^{-1} \simeq 137.3.} \tag{30.44}$$

Observational comparison. The observed fine-structure constant is

$$\alpha_{\mathrm{fs}}^{-1} \simeq 137.036.$$

The LO prediction therefore gives a relative offset of order

$$\frac{\left(\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}\right)^{-1} - \alpha_{\mathrm{fs}}^{-1}}{\alpha_{\mathrm{fs}}^{-1}} \simeq 1.8 \times 10^{-3}.$$

that is, a relative offset of about 0.175%.

Status. This section does not fix $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$ from the external charge e. The prediction (30.43) is a dimensionless closure of the EM-like channel, arising from the shape constants J_0, J_2 and from the geometric/cyclic factors of the elementary vertex. The absolute charge quantum of the EM-like field is predicted separately in §30.7.2, from this closure and from the native impedance of the EM-like channel. Its comparison to e, after SI reading, is however not a fully independent phenomenological test of the preceding comparison to α_{fs}.

30.7.2 EM-like charge quantum: derived prediction and SI reading

References (labels). The positive charge quantum of the EM-like field is defined and closed in the Theory part by (20.51)–(20.52). This closure uses the elementary coupling $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$, closed by (20.34), as well as the channel impedance $Z^{(\mathrm{LO})}_{\mathrm{EM,ref}}$, closed by (20.41) and (20.44). The pure channel constant $\Phi^{(\mathrm{two})}_{\mathrm{EM}}$ is that of (29.13), with the value derived in the chain (20.37).

Mechanical reading of the EM-like charge. The theoretical closure (20.51) first gives a field charge in the mechanical normalization of the EM-like channel:

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \left(\frac{4\pi h_{\mathrm{eff}}\, \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}} \right)^{1/2}.} \tag{30.45}$$

This writing does not involve any external electric unit.

We introduce the natural mechanical unit associated with the EM-like channel:

$$\boxed{\mathcal{Q}_{\mathrm{EM}}^{\mathrm{mec}} := \left(\frac{h_{\mathrm{eff}}}{Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}} \right)^{1/2}.} \tag{30.46}$$

The closure then becomes

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \sqrt{4\pi\, \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}\, \mathcal{Q}_{\mathrm{EM}}^{\mathrm{mec}}.} \tag{30.47}$$

Mechanical dimension. From (20.44),

$$Z_{\mathrm{EM,ref}}^{(\mathrm{LO})} = \frac{h_{\mathrm{eff}} c_t}{\ell_0^2}\, \Phi_{\mathrm{EM}}^{(\mathrm{two})}.$$

Thus,

$$[Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}] = \left[\frac{h_{\mathrm{eff}} c_t}{\ell_0^2} \right] = MLT^{-2}, \qquad [h_{\mathrm{eff}}] = ML^2T^{-1}.$$

Therefore

$$[\mathcal{Q}_{\mathrm{EM}}^{\mathrm{mec}}] = \left[\frac{h_{\mathrm{eff}}}{Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}} \right]^{1/2} = L^{1/2}T^{1/2}.$$

In the observational landing where HSM is identified with the physical universe, this mechanical dimension is the underlying HSM dimension of the observed electric charge:

$$\boxed{[\mathrm{C}]_{\mathrm{HSM}} = L^{1/2}T^{1/2}.} \tag{30.48}$$

This equality is a dimensional identification in the HSM–universe framework; it does not mean the raw numerical equality $1\,\mathrm{C} = 1\,\mathrm{m}^{1/2}\mathrm{s}^{1/2}$.

Numerical value in the HSM mechanical unit. With

$$\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} = 7.2846062 \times 10^{-3} \quad ((30.44)),$$

we obtain

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = 0.30255\, \mathcal{Q}_{\mathrm{EM}}^{\mathrm{mec}}.} \tag{30.49}$$

At this stage, the quantity is not yet expressed in coulombs. It is expressed in the mechanical unit of the EM-like channel.

Observational reading in coulombs. In the observational landing, the identifications

$$h_{\mathrm{eff}} = \hbar, \qquad c_t = c$$

are those of §29.3. The channel impedance is identified with the observable vacuum impedance in the closure of ℓ_0,

$$Z_{\mathrm{EM,ref}}^{(\mathrm{LO})} = Z_0^{\mathrm{obs}}, \tag{30.50}$$

according to (29.12)–(29.14). The same closure can then be read in the SI unit of charge:

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \left(\frac{4\pi\hbar\, \alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{Z_0^{\mathrm{obs}}} \right)^{1/2}.} \tag{30.51}$$

The external elementary charge e does not enter (30.51). It enters only as a comparison reference.

Numerical evaluation in SI unit. With

$$\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} = 7.2846062 \times 10^{-3}, \qquad \hbar = 1.054571817 \times 10^{-34}\ \mathrm{J\,s}, \qquad Z_0^{\mathrm{obs}} = 376.730313412\ \Omega,$$

we get

$$\boxed{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = 1.60078 \times 10^{-19}\ \mathrm{C}.} \tag{30.52}$$

Comparison to the elementary charge. The external elementary charge is

$$e = 1.602176634 \times 10^{-19}\ \mathrm{C}.$$

It was not consumed in the preceding closure. The comparison gives

$$\boxed{\frac{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}}{e} = 0.999126,} \tag{30.53}$$

and therefore

$$\boxed{\frac{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} - e}{e} = -8.74 \times 10^{-4}.} \tag{30.54}$$

The relative gap is therefore

$$-0.0874\%.$$

Status of the comparison to e. The quantum $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$ is a derived prediction of the HSM EM-like channel: its theoretical closure (30.45) uses $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$, h_{eff}, and $Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}$, but does not use the observed elementary charge e. Conversely, once the SI reading has been performed, the comparison to e is algebraically linked to the comparison already made for $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$. Indeed, the SI relation

$$e^2 = \frac{4\pi\hbar\, \alpha_{\mathrm{fs}}}{Z_0^{\mathrm{obs}}}$$

implies, before specialization of the observational identifications,

$$\boxed{\frac{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}}{e} = \left[\frac{h_{\mathrm{eff}}}{\hbar}\, \frac{\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{\alpha_{\mathrm{fs}}}\, \frac{Z_0^{\mathrm{obs}}}{Z_{\mathrm{EM,ref}}^{(\mathrm{LO})}} \right]^{1/2}.} \tag{30.55}$$

In the observational landing used here,

$$h_{\rm eff} = \hbar, \qquad Z_{\rm EM,ref}^{(\rm LO)} = Z_0^{\rm obs},$$

so that

$$\boxed{\frac{Q_{\rm EM,*}^{\rm HSM,LO}}{e} = \left(\frac{\alpha_{\rm EM}^{\rm HSM,LO}}{\alpha_{\rm fs}}\right)^{1/2}.} \tag{30.56}$$

The numerical proximity of $Q_{\rm EM,*}^{\rm HSM,LO}$ to e must therefore not be counted as a second fully independent phenomenological validation after that of $\alpha_{\rm EM}^{\rm HSM,LO}$. It constitutes a consistency check of the complete EM-like reading chain:

$$\alpha_{\rm EM}^{\rm HSM,LO}, \qquad Z_{\rm EM,ref}^{(\rm LO)}, \qquad h_{\rm eff} \quad \Longrightarrow \quad Q_{\rm EM,*}^{\rm HSM,LO}.$$

Uncertainty propagation. The SI reading (30.51) gives

$$\ln Q_{\rm EM,*}^{\rm HSM,LO} = \frac{1}{2}\ln \alpha_{\rm EM}^{\rm HSM,LO} - \frac{1}{2}\ln Z_0^{\rm obs} + \frac{1}{2}\ln(4\pi\hbar).$$

Logarithmic propagation leads to

$$\boxed{\left(\frac{\sigma_{Q_{\rm EM,*}}}{Q_{\rm EM,*}}\right)^2 = \frac{1}{4}\left(\frac{\sigma_{\alpha_{\rm EM}}}{\alpha_{\rm EM}^{\rm HSM,LO}}\right)^2 + \frac{1}{4}\left(\frac{\sigma_{Z_0}}{Z_0^{\rm obs}}\right)^2.} \tag{30.57}$$

The constant $\hbar$ is exact in the declared SI convention. At the precision displayed here, the observational uncertainty of $Z_0^{\rm obs}$ is subdominant relative to the LO truncation scale of the EM-like channel. In the SI reading retained here, the relative gap on the charge is one half of the relative gap on the fine-structure constant:

$$\frac{\delta Q_{\rm EM,*}}{Q_{\rm EM,*}} = \frac{1}{2}\frac{\delta\alpha_{\rm EM}}{\alpha_{\rm EM}}.$$

The distance to the comparison value provides the effective scale

$$\boxed{\delta_Q^{\rm LO} = 8.74 \times 10^{-4}.} \tag{30.58}$$

The prediction can therefore be summarized as

$$\boxed{Q_{\rm EM,*}^{\rm HSM,LO} = 1.60078 \times 10^{-19}\ \text{C}, \qquad \delta_Q^{\rm LO} = 0.0874\%.} \tag{30.59}$$

Anti-circularity and non-double-counting. The chain

$$\alpha_{\rm EM}^{\rm HSM,LO}, \quad Z_{\rm EM,ref}^{(\rm LO)}, \quad \Phi_{\rm EM}^{(\rm two)} \quad \Longrightarrow \quad Q_{\rm EM,*}^{\rm HSM,LO}$$

does not use e. The external elementary charge enters only in the comparison ratios (30.53)–(30.54). The quantity $Q_{\rm EM,*}^{\rm HSM,LO}$ therefore remains a derived prediction of the HSM EM-like channel. However, the good numerical agreement of the ratio $Q_{\rm EM,*}^{\rm HSM,LO}/e$ must not be presented as a phenomenological validation independent of that of $\alpha_{\rm EM}^{\rm HSM,LO}$, since, after the SI identifications (30.50), it reduces to (30.56).

30.8 NSL 3DOF: numerical outputs with separated HSM constants

References (labels). The tangential relation $c_t^2 = T/\rho$ is given by (1.18). The NSL 3DOF solution is defined by §11, with the conical ansatz (11.1), the fundamental filter $q = 1$ (11.3), the action law (11.8), the closed denominator (11.10), the width (11.11), and the mechanical mass projected by branch (11.12). The geometric spin of the branch is given in §11.8.

The EM-like charge of the NSL 3DOF is defined in §21.1.7, with the phase label (21.24), the conical projection (21.25), and the effective branches (21.26).

The separated constants used in this evaluation are those of the micro-local 1–ring closure of the Z^0-like composite:

$$\kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

They are defined in (29.40), (29.34), (29.38) and (29.39).

Output status. This section does not compare an identified state of the standard model. It gives the internal numerical outputs of the NSL 3DOF channel in the isolated LO closure: conical charge motif, core scale λ_*, and conical masses associated with the two branches

$$\sigma = \sin^2\alpha \in \left\{\frac{1}{3}, \frac{2}{3}\right\}.$$

The NSL 3DOF closure uses the quantities

$$\rho, \qquad T, \qquad \kappa_r.$$

In the active chain, these quantities are those of the micro-local 1–ring separation. The numerical masses below are therefore internal outputs computed with the separated HSM constants.

The NSL 3DOF closure does not use the local radial coefficient

$$\kappa_\mu = \frac{\mu_2}{T}.$$

The latter enters other sectors, but not the mono-lobe NSL 3DOF mass written here.

NSL 3DOF motif and absence of an internal parameter. The NSL 3DOF state corresponds to a *mono-lobe* core, with no internal separation δ and no crossed overlap:

$$k = 1, \qquad w = (1) \qquad \text{(primitive motif, up to an overall sign)}.$$

The quadratic kernels therefore reduce to

$$S_0 = J_0, \qquad S_2 = J_2.$$

We use the two pure combinations

$$K_{\mathrm{NSL}}^{(3D)} = (J_0 + J_2)J_0, \qquad K_0^{(\mathrm{loc})} = J_0^2.$$

The term $K_0^{(\mathrm{loc})}$ is a local core term. It does not represent a sheath self-induction.

EM charge: phase orientation and conical effective amplitude

Phase label. The NSL 3DOF is an elementary 3DOF solution, not a composite. Its EM-like charge label is therefore not a multi-lobe motif w, but the phase orientation of the tangential rotation in the conical ansatz:

$$n_{\mathrm{EM}}^{(\mathrm{NSL3})} = \varsigma \in \{-1, +1\},$$

according to (21.24). The inversion $\varsigma \mapsto -\varsigma$ reverses the sign of the effective charge, without modifying the core width or the masses.

Effective amplitude and role of σ. The conical projection of the circular tangential component gives

$$\sigma = \sin^2 \alpha \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.$$

The relative effective charge is then

$$\boxed{q_{\text{eff}}^{(\text{NSL3})}(\sigma) = \sigma \, n_{\text{EM}}^{(\text{NSL3})}.} \tag{30.60}$$

This is the observational reminder of (21.25).

The absolute field charge associated with a branch is therefore

$$\boxed{Q_{\text{field}}^{(\text{NSL3})}(\sigma) = q_{\text{eff}}^{(\text{NSL3})}(\sigma) \, Q_{\text{EM},*}^{\text{HSM,LO}}.} \tag{30.61}$$

This expression uses the EM-like charge quantum evaluated in §30.7.2

For the positive-orientation branch $\varsigma = +1$, one obtains

$$q_{\text{eff}}^{(\text{NSL3})}\left(\frac{1}{3}\right) = \frac{1}{3}, \qquad q_{\text{eff}}^{(\text{NSL3})}\left(\frac{2}{3}\right) = \frac{2}{3}.$$

For the opposite orientation $\varsigma = -1$, the same branches carry the opposite charges:

$$q_{\text{eff}}^{(\text{NSL3})} \in \left\{ -\frac{2}{3}, -\frac{1}{3}, +\frac{1}{3}, +\frac{2}{3} \right\},$$

according to (21.26).

NSL 3DOF mass with separated HSM constants

Isolated LO closure. In this observational section, we identify

$$h_{\text{eff}} = \hbar, \qquad c_t = c.$$

The theoretical relation

$$c_t^2 = \frac{T}{\rho}$$

remains the definition of the tangential sector. In the present evaluation, T is therefore the separated value deduced from

$$T^{\text{HSM}} = \rho^{\text{HSM}} c^2,$$

and not an independent input.

The HSM theoretical closure of the isolated mono-lobe NSL 3DOF channel is

$$\boxed{D_{\text{NSL3}}^{(\text{LO})} = \kappa_r^{\text{HSM}} K_{\text{NSL}}^{(3D)}.} \tag{30.62}$$

This is the reminder of (11.10), evaluated with the separated HSM constants. No sheath self-induction term is added.

The associated core width is

$$\boxed{\lambda_*^4 = \frac{2\,\hbar}{\sqrt{\rho^{\text{HSM}}\, T^{\text{HSM}}}\, D_{\text{NSL3}}^{(\text{LO})}}.} \tag{30.63}$$

This is the reminder of (11.11) after identifying $h_{\text{eff}} = \hbar$.

We then define the unprojected reference mass by

$$m_{\text{ref}} c^2 = \frac{\hbar c}{\lambda_*}.$$

In the tables below, m_{ref} is expressed in MeV/c^2, or equivalently in MeV in the usual convention for particle masses. The mechanical mass projected by conical branch is that of (11.12):

$$\boxed{m(\sigma) = \sigma \, m_{\text{ref}}, \qquad \sigma \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.} \tag{30.64}$$

Theoretical constants and numerical values

Theoretical constants. The pure constants used in this evaluation are

$$J_0 = \frac{\pi^3}{3} = 10.3354255601, \qquad J_2 = 7.6339320582,$$

$$K_{\mathrm{NSL}}^{(3D)} = (J_0 + J_2)J_0.$$

We also use

$$h_{\mathrm{eff}} = \hbar, \qquad c_t = c.$$

Separated HSM constants. The separated values used in this section are those of §29.9:

$$\kappa_r^{\mathrm{HSM}} = 0.204743, \qquad R_0^{\mathrm{HSM}} = 1.50841 \times 10^{29}\ \mathrm{m},$$

$$\rho^{\mathrm{HSM}} = 8.10393 \times 10^{-27}\ \mathrm{kg\,m^{-3}}, \qquad T^{\mathrm{HSM}} = 7.28345 \times 10^{-10}\ \mathrm{Pa}.$$

These values result from the micro-local 1–ring closure of the Z^0-like composite, cf. (29.34)–(29.40).

Numerical values. The numerical block is

$$K_{\mathrm{NSL}}^{(3D)} = 185.7209580.$$

This gives

$$D_{\mathrm{NSL3}}^{(\mathrm{LO})} = \kappa_r^{\mathrm{HSM}} K_{\mathrm{NSL}}^{(3D)} \simeq 38.02506610.$$

We then obtain

$$\lambda_* = 3.88714 \times 10^{-5}\ \mathrm{m}, \qquad m_{\mathrm{ref}} = 5.07641 \times 10^{-9}\ \mathrm{MeV}.$$

Thus, for the two conical branches,

$$\boxed{m\left(\frac{1}{3}\right) = 1.69214 \times 10^{-9}\ \mathrm{MeV}, \qquad m\left(\frac{2}{3}\right) = 3.38427 \times 10^{-9}\ \mathrm{MeV}.}$$

State	ς	n_{EM}	$q_{\mathrm{eff}}(\frac{1}{3})$	$q_{\mathrm{eff}}(\frac{2}{3})$	λ_* [m]	m_{ref} [MeV]	$m_{1/3}$ [MeV]	$m_{2/3}$ [MeV]
NSL 3DOF	$+1$	$+1$	$+0.333333$	$+0.666667$	3.88714×10^{-5}	5.07641×10^{-9}	1.69214×10^{-9}	3.38427×10^{-9}
NSL 3DOF	-1	-1	-0.333333	-0.666667	3.88714×10^{-5}	5.07641×10^{-9}	1.69214×10^{-9}	3.38427×10^{-9}

Sign conventions. The inversion of the tangential orientation

$$\varsigma \mapsto -\varsigma$$

changes the sign of $n_{\mathrm{EM}}^{(\mathrm{NSL3})}$, and therefore that of the effective amplitude $q_{\mathrm{eff}}^{(\mathrm{NSL3})}$, without modifying λ_* or the masses.

Scope of the evaluation. The numerical masses displayed in this section are internal outputs of the NSL 3DOF channel obtained with the HSM constants separated by the micro-local 1–ring closure. They do not correspond to a direct identification with an observed state of the standard model.

In the strategy retained in §29.2.1, no propagated uncertainty is displayed locally for

$$\lambda_*, \qquad m_{\mathrm{ref}}, \qquad m(\sigma), \qquad q_{\mathrm{eff}}^{(\mathrm{NSL3})}(\sigma) = \sigma\, n_{\mathrm{EM}}^{(\mathrm{NSL3})}.$$

The effective charge amplitudes are fixed by the conical projection of (21.25) and by the orientation label (21.24). They do not carry a local uncertainty at this order. The dominant structural uncertainty is not a local numerical error: it comes from the micro-local 1–ring closure status of the values

$$\kappa_r^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

30.9 Breathers 3DOF: closure diagnostic and masses with separated HSM constants

References (labels). The tangential relation $c_t^2 = T/\rho$ is given by (1.18). The 3DOF breather branch in strong overlap is defined in §28.5. The geometric spin of this branch is given in §28.5. The 3D translational overlaps are defined by (28.33). The quadratic forms S_p, the pairwise quartic reduction and the strong-like numerator are defined by (28.34), (28.35), (28.36) and (28.37).

The reduced impedances $\beta^{(\mathsf{P})}$, $\beta^{(\mathsf{F})}$, then $\beta^{(\mathrm{eff})}$, are defined by (28.40) and (28.41). The breather closure denominator is (28.42), its LO+ sheath correction is (28.43), and the link between width and closure is (28.44). The variational principle is justified by the structural energy (28.45).

The Br2, neutral Br3 and charged-like Br3 specializations are given by (28.47)–(28.58), (28.60)–(28.71) and (28.74)–(28.85). The common geometric strong-overlap cutoff is fixed by (27.10), (27.12), (27.13), (27.15), (27.16) and (27.17). The reproducible numerical procedure is summarized in §28.5.4, with (28.88), (28.89) and (28.90).

The reduced charge diagnostic is defined by the monopole label (28.91), the breather form factor (28.92), and the reduced charge (28.93). The Preclusion constant Γ_{P} is defined in (20.59) and evaluated in (20.60). The 3DOF strong-like factor ζ_{3D} is defined in (27.28) and evaluated in (27.33). The sheath constant used at LO+ is that of (23.51).

The separated constants κ_r, ρ, T used below are those of the micro-local one-ring closure of the Z^0-like composite:

$$\kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

They are defined in (29.40), (29.34), (29.38) and (29.39).

Status of the section. This section gives the numerical diagnostic of the 3DOF breather branches with the HSM constants separated by the micro-local one-ring closure. The values that depend on

$$\kappa_r^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}$$

are therefore internal outputs of the separated HSM chain.

A strong-overlap branch is promoted as an internal stationary solution at LO rank only if the minimization of its closure denominator selects an interior minimum:

$$\delta_* \in (\delta_{\min}, \delta_{\max}).$$

If the minimization selects an edge of the positive admissible domain, the branch is classified as an edge branch and is not used as a closed mass prediction at LO rank.

In this observational part, we identify

$$h_{\mathrm{eff}} = \hbar, \qquad c_t = c.$$

The theoretical relation

$$c_t^2 = \frac{T}{\rho}$$

remains the definition of the tangential sector; the value of T used below is therefore deduced from ρc^2, and not introduced as an independent input.

Mass framework. We set

$$\omega_{0,*} = \frac{c_t}{\lambda_*}, \qquad \omega_{\mathrm{br},*} = \sigma\,\omega_{0,*}, \qquad \sigma \in \Big\{\frac{1}{3}, \frac{2}{3}\Big\}.$$

The 3DOF mass of a unit breather branch is therefore

$$\boxed{m(\sigma) = \frac{E_0}{c_t^2} = \frac{\hbar\,\omega_{\mathrm{br},*}}{c_t^2} = \frac{\hbar\,\sigma}{c_t\,\lambda_*}, \qquad \sigma \in \Big\{\frac{1}{3}, \frac{2}{3}\Big\}.} \tag{30.65}$$

This equation is the observational reminder of (28.90), after identifying $h_{\mathrm{eff}} = \hbar$ and $c_t = c$.

Active numerical constants. The pure constants used in this evaluation are

$$J_0 = \frac{\pi^3}{3} = 10.3354255601, \qquad J_2 = 7.6339320582, \qquad J_4 = 2.70149350195,$$

$$K_0^{(Z)} = J_0^2, \qquad K_2^{(Z)} = J_0 J_2,$$

$$\Gamma_{\mathrm{P}} = 0.1011596206, \qquad \zeta_{3D} = \frac{15}{8} = 1.875, \qquad \Gamma_{\mathrm{shape}}^{(\mathrm{LO})} = 2.04629 \times 10^{-4}.$$

With the separated HSM constants of §29.9, we use

$$\kappa_r^{\mathrm{HSM}} = 0.204743,$$

$$R_0^{\mathrm{HSM}} = 1.50841{\times}10^{29}\ \mathrm{m}, \qquad \rho^{\mathrm{HSM}} = 8.10393{\times}10^{-27}\ \mathrm{kg\,m^{-3}}, \qquad T^{\mathrm{HSM}} = 7.28345{\times}10^{-10}\ \mathrm{Pa}.$$

Geometric bounds of the breather window. The strong-overlap cutoff is imposed by the 3DOF MSE window:

$$u_*^\star = 1.8782600799\ldots, \qquad \boxed{\delta_{\max} = \frac{u_*^\star}{2} = 0.93913003995\ldots}$$

The lower bound is fixed by the quantitative non-degeneracy of the Gram matrix, with threshold $\Gamma_{\mathrm{shape}}^{(\mathrm{LO})}$. With the 3D translational overlaps, one obtains

$$\boxed{\delta_{\min}^{\mathrm{Br2}} = 0.02883050, \qquad \delta_{\min}^{\mathrm{Br3}} = 2\,\delta_{\min}^{\mathrm{Br2}} = 0.05766101.}$$

LO and LO+ closures. We close the scales by

$$\lambda_{*,\mathrm{LO}}^4 = \frac{2\,\hbar\,\sqrt{\rho^{\mathrm{HSM}} T^{\mathrm{HSM}}}}{\mathcal{D}_{\mathrm{LO}}(\delta_*)}, \qquad \lambda_{*,\mathrm{LO+}}^4 = \frac{2\,\hbar\,\sqrt{\rho^{\mathrm{HSM}} T^{\mathrm{HSM}}}}{\mathcal{D}_{\mathrm{LO+}}(\delta_*)}.$$

The sheath correction is additive:

$$\mathcal{D}_{\mathrm{LO+}} = \mathcal{D}_{\mathrm{LO}} + \Gamma_{\mathrm{shape}}^{(\mathrm{LO})} K_0^{(Z)}.$$

The additive term is

$$\Gamma_{\mathrm{shape}}^{(\mathrm{LO})} K_0^{(Z)} = 2.04629 \times 10^{-4}\, J_0^2 \simeq 0.02185867881.$$

Numerical diagnostic of the branches

Modes considered. We distinguish three canonical reductions:

$$\mathrm{Br2}: \bar{w}_{\mathrm{Br2}} = (1,-1), \qquad \text{neutral Br3}: \bar{w}_{\mathrm{Br3,n}} = (1,-2,1), \qquad \text{charged-like Br3}: \bar{w}_{\mathrm{Br3,c}} = (1,-1,1).$$

Each closed spatial mode would give two masses, for

$$\sigma = \frac{1}{3} \qquad \text{and} \qquad \sigma = \frac{2}{3}.$$

Closure diagnostic result. Application of the internal closure criterion with the separated HSM constants gives:

branch	δ_*	minimum position	LO status
Br2	0.38410	edge of the positive domain	LO edge
Br3, n	0.22203	edge of the positive domain	LO edge
Br3, c	0.47403	interior minimum	LO closed

The two neutral branches Br2 and Br3,n reach the infimum of the denominator at the edge of the positive admissible domain. They are therefore not promoted as internal stationary solutions at LO rank. The Br3,c branch, by contrast, has an interior minimum and passes the closure criterion with the separated HSM constants.

Charged-like Br3 invariants at the optimum point. For $\bar{w}_{\mathrm{Br3,c}} = (1, -1, 1)$ at

$$\delta_*^{\mathrm{Br3,c}} = 0.47403,$$

one obtains

$$S_0^{\mathrm{Br3,c}} \simeq 9.33579, \qquad S_2^{\mathrm{Br3,c}} \simeq 6.11656,$$
$$S_{4,\mathrm{pair}}^{\mathrm{Br3,c}} \simeq 42.09498, \qquad N_{\mathsf{F}}^{\mathrm{Br3,c}} \simeq 7.01584.$$

The corresponding reduced invariants are

$$\widehat{s}_2^{\mathrm{Br3,c}} \simeq 0.65518, \qquad \widehat{s}_{4,\mathrm{pair}}^{\mathrm{Br3,c}} \simeq 0.48296, \qquad \widehat{n}_{\mathsf{F}}^{\mathrm{Br3,c}} \simeq 0.75151.$$

Mass of the closed branch

Charged-like Br3 branch. For the Br3,c branch, the denominator at the interior minimum, recomputed with the separated HSM constants, is

$$\mathcal{D}_{\mathrm{LO}}^{\mathrm{Br3,c}} \simeq 7.86358, \qquad \mathcal{D}_{\mathrm{LO+}}^{\mathrm{Br3,c}} \simeq 7.88544.$$

This gives

$$\lambda_{*,\mathrm{LO}}^{\mathrm{Br3,c}} \simeq 8.98463 \times 10^{-14}\ \mathrm{m}, \qquad \lambda_{*,\mathrm{LO+}}^{\mathrm{Br3,c}} \simeq 8.97841 \times 10^{-14}\ \mathrm{m}.$$

The reference mass not projected onto σ is

$$m_{\mathrm{ref,LO+}}^{\mathrm{Br3,c}} = \frac{\hbar}{c\,\lambda_{*,\mathrm{LO+}}^{\mathrm{Br3,c}}} \simeq 2.19779\ \mathrm{MeV}.$$

Therefore

$$\boxed{m_{\mathrm{LO+}}^{\mathrm{Br3,c}}\left(\frac{1}{3}\right) \simeq 0.73260\ \mathrm{MeV}, \qquad m_{\mathrm{LO+}}^{\mathrm{Br3,c}}\left(\frac{2}{3}\right) \simeq 1.46520\ \mathrm{MeV}.}$$

Main table.

branch	δ_*	status	$\lambda_{*,\mathrm{LO+}}$ [m]	$m_{\mathrm{LO+}}(\frac{1}{3})$ [MeV]
Br2	0.38410	edge / not LO closed	$--$	$--$
Br3, n	0.22203	edge / not LO closed	$--$	$--$
Br3, c	0.47403	LO closed	8.97841×10^{-14}	0.73260

For $\sigma = 2/3$, the Br3,c mass is doubled:

$$m_{\mathrm{LO+}}^{\mathrm{Br3,c}}\left(\frac{2}{3}\right) \simeq 1.46520\ \mathrm{MeV}.$$

Reading of the edge branches. The Br2 and Br3,n branches are not removed from the theory: they remain admissible branches of the formal construction. However, with the separated HSM constants, their LO minimization selects the edge of the positive domain. They therefore do not provide a closed predictive mass at LO rank. The mass values that could be obtained by forcing the edge evaluation are not displayed as predictions.

Reduced EM charge diagnostic

Monopole label. In strong overlap, the reduced field charge is not obtained by adding the observable charges of isolated bumps. It is reconstructed from the whole raw motif $\bar{w}$, before normalization by S_0. We use the monopole label

$$n_{\mathrm{EM}}(\bar{w}) = \sum_i \bar{w}_i,$$

defined in (28.91). For the three canonical motifs:

$$n_{\mathrm{EM}}^{\mathrm{Br2}} = 1 - 1 = 0, \qquad n_{\mathrm{EM}}^{\mathrm{Br3,n}} = 1 - 2 + 1 = 0, \qquad n_{\mathrm{EM}}^{\mathrm{Br3,c}} = 1 - 1 + 1 = 1.$$

The two neutral branches are therefore zero at monopole rank:

$$q_{\mathrm{eff}}^{\mathrm{Br2}} = 0, \qquad q_{\mathrm{eff}}^{\mathrm{Br3,n}} = 0.$$

Reduced charge of the Br3,c branch. For the Br3,c reduction at the recomputed interior minimum, the reduced form factor gives

$$\frac{\mathcal{C}_S^{(\mathrm{br})}(\bar{w}_{\mathrm{Br3,c}}; \delta_*^{\mathrm{Br3,c}})}{\mathcal{C}_S^{(\mathrm{bump})}} \simeq 1.09675.$$

The reduced branch charge is then

$$q_{\mathrm{eff}}^{\mathrm{Br3,c}}(\sigma) = \sigma \, \frac{\mathcal{C}_S^{(\mathrm{br})}(\bar{w}_{\mathrm{Br3,c}}; \delta_*^{\mathrm{Br3,c}})}{\mathcal{C}_S^{(\mathrm{bump})}}.$$

For the two conical projections,

$$q_{\mathrm{eff}}^{\mathrm{Br3,c}}\left(\frac{1}{3}\right) \simeq 0.36558, \qquad q_{\mathrm{eff}}^{\mathrm{Br3,c}}\left(\frac{2}{3}\right) \simeq 0.73117.$$

These values are reduced EM amplitudes of the branch; they do not constitute a direct identification with an established particle.

Associated field charge. The elementary charge quantum of the EM-like field is $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$, evaluated in §30.7.2. The field charge of the Br3,c branch is therefore

$$\boxed{Q_{\mathrm{field}}^{\mathrm{Br3,c}}(\sigma) = q_{\mathrm{eff}}^{\mathrm{Br3,c}}(\sigma)\, Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}.} \tag{30.66}$$

Thus, in the native unit of the EM-like channel,

$$\boxed{Q_{\mathrm{field}}^{\mathrm{Br3,c}}\left(\frac{1}{3}\right) \simeq 0.36558\, Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}, \qquad Q_{\mathrm{field}}^{\mathrm{Br3,c}}\left(\frac{2}{3}\right) \simeq 0.73117\, Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}.} \tag{30.67}$$

With the observational reading of (30.52), these values correspond to

$$Q_{\mathrm{field}}^{\mathrm{Br3,c}}\left(\frac{1}{3}\right) \simeq 5.85 \times 10^{-20}\ \mathrm{C}, \qquad Q_{\mathrm{field}}^{\mathrm{Br3,c}}\left(\frac{2}{3}\right) \simeq 1.17 \times 10^{-19}\ \mathrm{C}.$$

The values in parentheses in coulombs do not define the branch charge; they only give the SI reading of the native HSM charge.

Reduced-charge table.

branch	δ_*	n_{EM}	$q_{\mathrm{eff}}(\frac{1}{3})$	$q_{\mathrm{eff}}(\frac{2}{3})$
Br2	0.38410	0	0	0
Br3, n	0.22203	0	0	0
Br3, c	0.47403	1	0.36558	0.73117

Scope of the evaluation. The numerical masses displayed in this section are the outputs of the 3DOF breather closure evaluated with the HSM constants separated by the micro-local one-ring closure. They therefore depend on the constants

$$\kappa_r^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}},$$

defined in (29.40), (29.38) and (29.39).

In the strategy retained in §29.2.1, no propagated uncertainty is displayed locally for

$$\lambda_*, \qquad m(\sigma), \qquad q_{\mathrm{eff}}^{(\mathrm{br})}(\bar{w}; \delta_*, \sigma).$$

The numerical quadrature and root uncertainties are negligible at the displayed precision; the dominant structural uncertainty is that of the micro-local one-ring closure and of the internal closure status of the branches.

30.10 Z-like 3DOF (Z2, Z3): closure diagnostic with separated HSM constants

References (labels). The Z-like 3DOF multi-lobe family is defined in §28.6. The normalization gauge is given by (28.94), and the passage from the raw full motif to the normalized representative by (28.95). The cross-overlaps are those defined by 3D translational convolution in (28.33). The multi-lobe forms S_p, $S_{4,\mathrm{pair}}$, the rule $c_{ij}^{(\mathsf{F})}$, and the strong-like numerator N_{F} are given by (28.34), (28.35), (28.36), and (28.37). The reduced Z-like invariants are defined by (28.97).

The channel impedances and the common denominator are given by (28.99) and (28.101). The LO+ correction is given by (28.103). The width closure is implicit, via (28.98), (28.102), and (28.104). The Preclusion constant Γ_{P} is defined in (20.59) and evaluated in (20.60). The strong-like 3DOF factor ζ_{3D} is defined in (27.28) and evaluated in (27.33). The sheath constant is that of (23.51).

The discrete charge rule is (28.105), and the reduced charge is defined by (28.123) and (28.124). The Z2 and Z3 specializations are given by §28.6.6 and §28.6.7, with the closures (28.110), (28.111), (28.115), (28.118), (28.119), and (28.122). The admissible domains of the separations are bounded by the common geometric cutoff defined in (27.10), (27.12), (27.13), (27.15), (27.16), and (27.17). The separations δ_* are then defined by the minimizations over positive admissible domains, as in (28.113) for Z2 and in the corresponding definition of the Z3 branch. The geometric spin of the family is recalled in §28.6.5.

The separated constants used in the numerical diagnostic are those of the micro-local one-ring closure of the Z^0-like composite:

$$\kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

They are defined in (29.40), (29.34), (29.38), and (29.39). The local radial coefficient

$$\kappa_\mu = \frac{\mu_2}{T}$$

remains a calibrated constant independent of this global separation.

Section status. This section applies the existence criterion derived in the theoretical part. A Z-like 3DOF branch provides an internal stationary solution at LO only if the minimization of its closure denominator selects an interior minimum:

$$\delta_* \in (\delta_{\min}, \delta_{\max}).$$

If the minimization selects a boundary,

$$\delta_* = \delta_{\min} \qquad \text{or} \qquad \delta_* = \delta_{\max},$$

the branch is identified as a boundary branch. In that case, the width closure associated with the interior minimum does not apply, and no internal stationary solution mass is reported.

Implicit closure. In this observational part, we identify

$$h_{\text{eff}} = \hbar, \qquad c_t = c.$$

The Z-like 3DOF closure uses the reduced parameter

$$\eta_\mu^{(Z)}(\lambda_*) = \frac{\kappa_\mu}{\kappa_r^{\text{HSM}}}\lambda_*^2.$$

With $c_t^2 = T/\rho$, the LO+ closure is written

$$\boxed{\lambda_*^4\, D_{\text{LO+}}^{(Zk)}\Big(\bar{w}; \delta_*(\eta_\mu^{(Z)}(\lambda_*)); \eta_\mu^{(Z)}(\lambda_*)\Big) = 2\,\hbar\sqrt{\rho^{\text{HSM}} T^{\text{HSM}}}.} \tag{30.68}$$

This equation is the observational reminder of (28.104), after identifying $h_{\text{eff}} = \hbar$. It produces a branch mass only when the δ_* diagnostic selects an interior minimum.

MSE bound and lower bounds. The common upper bound of the strong-overlap domain is

$$u_*^\star = 1.8782600799\ldots, \qquad \delta_{\max} = \frac{u_*^\star}{2} = 0.93913003995\ldots.$$

The lower bounds are those of the corresponding Gram problem:

$$\delta_{\min}^{\text{Z2}} = \delta_{\min}^{\text{Br2}}, \qquad \delta_{\min}^{\text{Z3}} = \delta_{\min}^{\text{Br3}}.$$

In the numerical diagnostic, we use

$$\delta_{\min}^{\text{Z2}} = 0.02883050, \qquad \delta_{\min}^{\text{Z3}} = 0.05766101.$$

Tested motifs. The canonical motifs tested are

$$\bar{w}_{\text{Z2}} = (1, -1), \qquad \bar{w}_{\text{Z3,n}} = (1, -2, 1), \qquad \bar{w}_{\text{Z3,c}} = (1, -1, 1).$$

The first two motifs are neutral at monopolar rank:

$$n_{\text{EM}}(\bar{w}_{\text{Z2}}) = 0, \qquad n_{\text{EM}}(\bar{w}_{\text{Z3,n}}) = 0,$$

whereas the charged-like motif satisfies

$$n_{\text{EM}}(\bar{w}_{\text{Z3,c}}) = 1.$$

Active numerical constants

Theoretical constants. The pure constants used in this evaluation are

$$J_0 = \frac{\pi^3}{3} = 10.3354255601, \qquad J_2 = 7.6339320582, \qquad J_4 = 2.70149350195,$$

$$K_0^{(Z)} = J_0^2, \qquad K_2^{(Z)} = J_0 J_2.$$

We also use

$$\Gamma_{\mathrm{P}} = 0.1011596206, \qquad \zeta_{3D} = \frac{15}{8} = 1.875, \qquad \Gamma_{\mathrm{shape}}^{(\mathrm{LO})} = 2.04629 \times 10^{-4}.$$

Separated HSM constants. With the separated HSM constants of §29.9, we use

$$\kappa_r^{\mathrm{HSM}} = 0.204743,$$

$$R_0^{\mathrm{HSM}} = 1.50841{\times}10^{29}\ \mathrm{m}, \qquad \rho^{\mathrm{HSM}} = 8.10393{\times}10^{-27}\ \mathrm{kg\,m^{-3}}, \qquad T^{\mathrm{HSM}} = 7.28345{\times}10^{-10}\ \mathrm{Pa}.$$

The local radial coefficient used is

$$\kappa_\mu \simeq 6.25 \times 10^{34}\ \mathrm{m}^{-2}.$$

The reduced parameter used in the Z-like closure is therefore

$$\eta_\mu^{(Z)}(\lambda_*) = \frac{\kappa_\mu}{\kappa_r^{\mathrm{HSM}}} \lambda_*^2.$$

Closure diagnostic

Diagnostic resolution. For each motif, we minimize $D_{\mathrm{LO}}^{(Zk)}$ over the positive admissible domain with the separated HSM constants. The interior-closure criterion is strict: a branch provides an internal stationary solution at LO only if the selected minimum is interior,

$$\delta_* \in (\delta_{\min}, \delta_{\max}).$$

When the minimum is reached on a boundary, the implicit width equation associated with the interior minimum is not used to produce a branch mass. The diagnostic then classifies the motif as a boundary branch at LO.

Result of the diagnostic with separated HSM constants. The three minimizations select a boundary of the admissible domain:

branch	δ_*	selected boundary	LO status
Z2	0.93913004	$\delta_{\max}$	LO boundary
Z3, n	0.93913004	$\delta_{\max}$	LO boundary
Z3, c	0.05766101	$\delta_{\min}$	LO boundary

None of the three tested branches selects an interior minimum. The diagnostic therefore produces no Z-like 3DOF mass associated with an internal stationary solution at LO with the separated HSM constants.

Consequence for the Z-like 3DOF branches. Since the three minimizations select a boundary of the admissible domain, the corresponding branches do not satisfy the internal stationary-solution criterion at LO. At active LO rank, the diagnostic therefore classifies the motifs Z2, Z3,n, and Z3,c as boundary branches with the separated HSM constants.

Consequence for the masses. No mass is displayed for the Z-like 3DOF branches in this section, because the three diagnostics reach a boundary of the admissible domain. A mass is reported only when the minimization selects an interior minimum and thereby provides an internal stationary solution at LO.

Consequence for EM charge. The charge rule remains defined by the monopolar label

$$n_{\mathrm{EM}}(\bar{w}) = \sum_i \bar{w}_i.$$

For the Z-like motifs tested here, the effective charges $q_{\mathrm{eff}}^{(Z)}$ are therefore read as motif diagnostics. They are not associated with internal stationary-solution masses, since the interior-closure criterion selects a boundary of the admissible domain.

Scope of the evaluation. This section concludes that the tested Z-like 3DOF branches are boundary branches with the separated HSM constants. They remain admissible multi-lobe motifs in the theoretical construction, but in the present diagnostic they do not satisfy the internal stationary-solution criterion at LO. The result published by this section is therefore the boundary diagnostic of the tested Z2/Z3 motifs, and not a Z-like 3DOF mass prediction.

30.11 Weak-like channel: minimal numerical valuation of monopolar W-like and Z^0-like interactions with separated HSM constants

References (labels). The tangential relation $c_t^2 = T/\rho$ is given by (1.18). The minimal static reading of the weak-like channel is defined in §20.5.7. The point force between two NSL sources is given by (20.81), the local range by (20.82), and the simplified form by (20.83).

The separated constants used in this evaluation are those of the micro-local one-ring closure of the Z^0-like composite:

$$R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}.$$

They are defined in (29.34), (29.38), (29.39) and (29.40). The local radial coefficient κ_μ is calibrated in §29.5, with the reduced combination (29.9), the width (29.10) and the physical value (29.11). The structural value of the NSL local frequency factor

$$\chi_\omega^{(\mathrm{NSL})} = \sqrt{\frac{1}{3}}$$

is given by (6.119).

Framework. Here we use the minimal static reading of the weak-like channel fixed in the theoretical closure: radial mediator $\Phi_W \equiv r$, constant

$$g_W = \frac{2}{\kappa_r R_0^3},$$

NSL branch charge

$$\pm\, 8\pi\, \chi_\omega^{(\mathrm{NSL})}\, R_0^2\, \sqrt{\rho T},$$

and characteristic length

$$\ell_W = \sqrt{\frac{T\kappa_r}{\mu_2}} = \sqrt{\frac{\kappa_r}{\kappa_\mu}}.$$

The relation $T/\rho = c^2$ is used in the observational evaluation.

Status of the numerical quantities. In the active chain, the separated quantities

$$R_0, \qquad \rho, \qquad T, \qquad \kappa_r$$

are those of the micro-local one-ring separation:

$$R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}.$$

The values displayed in this subsection are therefore numerical valuations of the weak-like channel with the separated HSM constants.

Active numerical constants. We use

$$\kappa_r^{\mathrm{HSM}} = 0.204743,$$

$$R_0^{\mathrm{HSM}} = 1.50841\times10^{29}\ \mathrm{m}, \qquad \rho^{\mathrm{HSM}} = 8.10393\times10^{-27}\ \mathrm{kg\,m^{-3}}, \qquad T^{\mathrm{HSM}} = 7.28345\times10^{-10}\ \mathrm{Pa}.$$

For the local radial coefficient, we use the value obtained from the 2DOF weak-like calibration:

$$\kappa_\mu \simeq 6.25 \times 10^{34}\ \mathrm{m}^{-2},$$

see (29.11). With

$$\lambda_{\mathrm{NSL}} = 2.454956896 \times 10^{-18}\ \mathrm{m}.$$

Finally, for the charged NSL branch considered here, we use the structural value

$$\chi_\omega^{(\mathrm{NSL})} = \sqrt{\frac{1}{3}} \simeq 0.57735.$$

Deduced numerical values. It follows that

$$\boxed{g_W = \frac{2}{\kappa_r^{\mathrm{HSM}}(R_0^{\mathrm{HSM}})^3} \simeq 2.84618 \times 10^{-87}.} \tag{30.69}$$

The branch charge is

$$\boxed{q_W^{(W)} = 8\pi\, \chi_\omega^{(\mathrm{NSL})}(R_0^{\mathrm{HSM}})^2\sqrt{\rho^{\mathrm{HSM}}T^{\mathrm{HSM}}} \simeq 8.02111 \times 10^{41}.} \tag{30.70}$$

The minimal weak-like range is

$$\boxed{\ell_W = \sqrt{\frac{\kappa_r^{\mathrm{HSM}}}{\kappa_\mu}} \simeq 1.81032 \times 10^{-18}\ \mathrm{m}.} \tag{30.71}$$

The force coefficient is therefore

$$\boxed{\frac{g_W^2[q_W^{(W)}]^2}{4\pi} \simeq 4.14748 \times 10^{-91}.} \tag{30.72}$$

Proper width of the NSL source. We use the proper width of the charged NSL 2DOF branch at rest, calibrated by m_W,

$$\boxed{\lambda_{\mathrm{NSL}} = 2.454956896 \times 10^{-18}\ \mathrm{m}.} \tag{30.73}$$

Numerically,

$$\lambda_{\mathrm{NSL}} \simeq 2.45496 \times 10^{-18}\ \mathrm{m}, \qquad \ell_W \simeq 1.81032 \times 10^{-18}\ \mathrm{m},$$

hence

$$\frac{\lambda_{\mathrm{NSL}}}{\ell_W} \simeq 1.35609.$$

The two scales are therefore of the same order at LO rank. The Yukawa form must be retained in the numerical evaluation: there is no weak-range regime that is quasi constant over several source widths.

Monopolar W-like interaction. For two NSL 2DOF sources at rest with the same internal orientation, the minimal static force is

$$\boxed{F_W^{(\mathrm{NSL,NSL})}(R) = -\,4.14748 \times 10^{-91}\,\frac{e^{-R/(1.81032\times 10^{-18})}}{R^2}\left(1+\frac{R}{1.81032\times 10^{-18}}\right),} \tag{30.74}$$

where R is expressed in metres and F_W in newtons.

Characteristic values at the source scale. We evaluate this force for separations of the order of a few proper widths λ_{NSL}. Since

$$\frac{\lambda_{\mathrm{NSL}}}{\ell_W} \simeq 1.35609,$$

the Yukawa factor is already significant at $R \simeq 2\lambda_{\mathrm{NSL}}$, and the decrease is rapid:

$$\boxed{F_W^{(\mathrm{NSL,NSL})}(2\lambda_{\mathrm{NSL}}) \simeq -\,4.24 \times 10^{-57}\ \mathrm{N}.} \tag{30.75}$$

$$\boxed{F_W^{(\mathrm{NSL,NSL})}(5\lambda_{\mathrm{NSL}}) \simeq -\,2.43 \times 10^{-59}\ \mathrm{N}.} \tag{30.76}$$

$$\boxed{F_W^{(\mathrm{NSL,NSL})}(10\lambda_{\mathrm{NSL}}) \simeq -\,1.29 \times 10^{-62}\ \mathrm{N}.} \tag{30.77}$$

Monopolar Z^0-like interaction. At the minimal monopolar level, the Z^0-like is globally neutral in the weak-like channel. Thus

$$\boxed{q_W^{(\mathsf{Z}^0)} = 0,} \tag{30.78}$$

and consequently

$$\boxed{F_W^{\mathrm{mono}}(\mathsf{Z}^0, \mathsf{Z}^0) = 0.} \tag{30.79}$$

Scope of the evaluation. The numerical values displayed in this subsection are conditioned by the micro-local one-ring closure that separates

$$R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}.$$

The coefficient κ_μ is supplied independently by the 2DOF weak-like calibration through $\kappa_\mu \lambda_{\mathrm{NSL}}^2$ and λ_{NSL}. A later modification of the micro-local one-ring closure would modify the absolute values of g_W, q_W, ℓ_W, and the forces, without modifying the analytic structure of the monopolar force.

Conclusion. In the current minimal static reading of the weak-like channel, the NSL 2DOF branch carries a nonzero but extremely weak monopolar interaction, even at separations of the order of a few proper widths λ_{NSL}. The Yukawa form must be retained since $\lambda_{\mathrm{NSL}}/\ell_W \simeq 1.36$. The Z^0-like is neutral at monopolar level.

30.12 Final synthesis: predictions and theoretical outputs

This section gathers the main numerical outputs of the observational part. The full synthesis of the calibrations is not repeated here: it was given in the corresponding calibration sections.

Two statuses are distinguished:

- predictions directly comparable to established observables, which do not depend on the global separation (R_0, ρ, T, κ_r) ;

- theoretical outputs that explicitly consume the HSM constants separated by the micro-local one-ring closure of the Z^0-like composite.

The active chain separates

$$R_0, \qquad \rho, \qquad T, \qquad \kappa_r$$

by the one-ring micro-local closure of §29.9. The separated values used in the downstream numerical sections are

$$R_0^{\mathrm{HSM}} = 1.50841 \times 10^{29}\ \mathrm{m}, \qquad \rho^{\mathrm{HSM}} = 8.10393 \times 10^{-27}\ \mathrm{kg\,m^{-3}},$$

$$T^{\mathrm{HSM}} = 7.28345 \times 10^{-10}\ \mathrm{Pa}, \qquad \kappa_r^{\mathrm{HSM}} = 0.204743.$$

These constants are the separated outputs of the micro-local closure; they are not independent auxiliary assumptions.

30.12.1 Predictions comparable to observables

Quantity	HSM value / compared value	Status	Reference / observable
m_Z from m_W	90.3 GeV	scale-free LO prediction; scheme gap $\simeq 1\%$	91.1876 GeV
m_e (2DOF bump EM 1-body)	0.512 MeV	central LO prediction; relative gap $\simeq 0.23\%$; correlated dependence on $b_0^{(2)}$ cancelled at LO, cf. (30.5)	0.51099895 MeV
$\Delta m^2_{\rm sol}$ (2DOF breather, LO)	$(7.5 \pm 0.8) \times 10^{-5}$ eV^2	$\mathcal{S}(1;207) = 1$, absolute prefactor with closure uncertainty (30.15)	7.41×10^{-5} eV^2
$\Delta m^2_{\rm atm}$ (2DOF breather, LO)	$(2.77 \pm 0.30) \times 10^{-3}$ eV^2	$\mathcal{S}(207;207) = 36.84$, hence central value $36.84 \times 7.5117 \times 10^{-5}$ eV^2	2.507×10^{-3} eV^2
$\Delta m^2_{\rm atm}/\Delta m^2_{\rm sol}$	36.84	regenerated spectral ratio $\mathcal{S}(207;207)/\mathcal{S}(1;207)$	33.83
$\mathcal{S}(1;207)$	1	2DOF breather spectral normalization	–
$\mathcal{S}(207;207)$	36.84	value regenerated by the saturated spectral closure	–
$\mathcal{S}(3477;207)$	207.78	regenerated upper spectral level; $\mathcal{S}(3477;207)/\mathcal{S}(207;207) \simeq 5.64$	–
m_1 (2DOF breather, minimal LO)	0	minimal closure $m_0^2 = 0$	–
m_2 (2DOF breather, minimal LO)	$(8.67 \pm 0.48) \times 10^{-3}$ eV	$\sqrt{7.5117 \times 10^{-5}}$ eV, closure uncertainty (30.16)	8.608×10^{-3} eV reconstructed from Δm^2_{21}
m_3 (2DOF breather, minimal LO)	$(5.26 \pm 0.29) \times 10^{-2}$ eV	$\sqrt{2.7673 \times 10^{-3}}$ eV, closure uncertainty (30.16)	5.007×10^{-2} eV reconstructed from $\Delta m^2_{3\ell}$
m_3/m_2 (2DOF breather, minimal LO)	6.070	$\sqrt{36.84}$	5.817
$\sum m_\nu$ (2DOF breather, minimal LO)	$(6.13 \pm 0.34) \times 10^{-2}$ eV	$0 + (8.67 \pm 0.48) \times 10^{-3} + (5.26 \pm 0.29) \times 10^{-2}$	Planck/BAO cosmological bound $\Sigma m_\nu < 0.12\,\text{eV}$
ϑ_{12} (projective 3×2 reduction)	$33.4^\circ \pm 0.8^\circ$	internal angle of the six-bump composite, LO closure range from (30.37)	$\theta_{12}^{\rm obs} \simeq 33.4^\circ$
ϑ_{13} (projective 3×2 reduction)	$8.6^\circ \pm 0.4^\circ$	internal angle of the six-bump composite, LO closure range from (30.37)	$\theta_{13}^{\rm obs} \simeq 8.6^\circ$
ϑ_{23} (projective 3×2 reduction)	49.3°–49.5°	internal angle of the six-bump composite, from (30.37)	$\theta_{23}^{\rm obs} \simeq 49^\circ$
m_H (Higgs-like radial)	125.0 GeV	minimal local NLO scheme prediction; HSM relation to $m^{(\rm HSM)}_{\rm NSL,ch}$, then observational identification $m^{(\rm HSM)}_{\rm NSL,ch} \mapsto m_W$	125.20 ± 0.11 GeV
m_H/m_W (Higgs-like radial)	1.555	scale-free ratio from $\sqrt{1+\delta_H^*}$	$125.20/80.379 \simeq 1.55762$
m_γ (photon, LO)	0	Abelian transverse mode without a Proca term at LO	experimental upper bound
$N_{\rm pol}$ (photon)	2	physical transverse polarizations	2
$\{\lvert Q_{\rm field}\rvert/Q_{\rm EM,*}\}$ (elementary branches)	$\{1,\ 2/3,\ 1/3\}$	relative conical motif transmitted by (21.3)	$\{1,\ 2/3,\ 1/3\}$
$Q_{\rm EM,*}$ (EM-like charge quantum)	1.601×10^{-19} C	derived prediction by $\alpha_{\rm EM}^{\rm HSM,LO}$ and $Z^{(\rm LO)}_{\rm EM,ref}$; comparison to e not independent of $\alpha_{\rm EM}$ after SI reading	$e = 1.602176634 \times 10^{-19}$ C
$\alpha_{\rm EM}^{-1}$ (EM-like coupling, LO)	137.3	central LO closure (20.34)	137.036

Displayed precision. The values in this synthesis are central comparison values. They are displayed at the phenomenologically relevant rank for their closure status: when a closure or scheme uncertainty dominates, it is stated explicitly or the display is rounded accordingly. The additional digits used in the detailed sections serve to make the numerical chains reproducible;

they should not be read as additional phenomenological precision.

Reading. The masses m_μ and m_τ belong to the discrete calibration of the triplet of admissible thresholds

$$(q_1, q_2, q_3)_{\mathrm{cal}} = (1, 207, 3477).$$

They are therefore not listed as independent predictions in the final synthesis. The observational comparisons of the 3DOF Bump sector are carried by color–white CB3 composites, not by free quark masses.

30.12.2 Color-white CB3 composites: baryon-like predictions

The CB3 outputs are LO predictions of color–white baryon-like composites, evaluated in the saturated/non-saturated matching closure $\delta^{\mathrm{LO}}_{\mathrm{CB3}} = 1$. They do not consume the global separation

$$\kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

CB3	Reference	HSM LO mass [MeV]	Obs. [MeV]	Gap
uud	p	937.46 ($\pm$1.1%)	938.2721	-0.087%
udd	n	937.29 ($\pm$1.1%)	939.5654	-0.242%
uds	Λ^0	1105.66	1115.683	-0.898%
udc_{gen}	$\Sigma_c(2455)^+$	2498.93	2452.65	$+1.887\%$
udc_{bas}	Λ_c^+	2280.51	2286.46	-0.260%
udb	Λ_b^0	5599.33	5619.57	-0.360%
uss	Ξ^0	1332.00	1314.86	$+1.304\%$
dss	Ξ^-	1331.80	1321.71	$+0.763\%$
sss	Ω^-	1668.30	1672.45	-0.248%
ssb	Ω_b^-	6108.70	6045.8	$+1.040\%$

The ($\pm$1.1%) band indicated for the nucleon-like doublet corresponds to the dominant CB3 closure band associated with the shape constants τ_3 and q_c, after correlated cancellation of $b_0^{(2)}$ in the scale E_*, cf. (30.5). It must not be confused with an independent uncertainty propagated from E_*. The masses reported in this table are identical, at the displayed rounding, to those obtained by direct recalculation of the closure (28.23), with the low-mode correction (28.31) for udc_{bas}.

Fine correction of the light doublet. The first two rows of the table give the generic CB3 masses before the compact fine correction of the uud/udd doublet. The physical output of the nucleon-like splitting is given by (30.38). It consumes the compact EM-like term (21.18) and the closed elementary coupling $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$, defined by (20.34). Numerically,

$$\boxed{m_n - m_p \simeq +1.31 \text{ MeV},}$$

to be compared with

$$(m_n - m_p)_{\mathrm{obs}} \simeq +1.293 \text{ MeV}.$$

This correction restores the correct sign of the light doublet and must be read as the relevant output for the neutron–proton splitting.

Reading. These rows compare color–white CB3 composites to baryon-like states. They must not be confused with free quark masses or with the isolated spectral masses of the 3DOF Bump. For the light doublet, the uud and udd rows give the generic masses; the physical splitting is given by the fine correction $m_n - m_p$ above.

30.12.3 Theoretical outputs with separated HSM constants

The following outputs consume at least one of the separated constants

$$\kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

They are therefore evaluated after the one-ring micro-local closure of the Z^0-like composite.

Branch / quantity	HSM value	Charge / parameter	Status
NSL 3DOF mono-lobe	$m(1/3) = 1.69214 \times 10^{-9}$ MeV, $m(2/3) = 3.38427 \times 10^{-9}$ MeV	$Q_{\mathrm{field}}/Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} = \pm 1/3,\ \pm 2/3$	isolated LO theoretical output
Breather 3DOF Br3 charged-like	$m_{\mathrm{LO}+}(1/3) \simeq 0.73260$ MeV, $m_{\mathrm{LO}+}(2/3) \simeq 1.46520$ MeV	$Q_{\mathrm{field}}/Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} \simeq 0.36558,\ 0.73117$	closed LO branch
Minimal static weak-like channel	$F_W(2\lambda_{\mathrm{NSL}}) \simeq -4.24 \times 10^{-57}$ N, $F_W(5\lambda_{\mathrm{NSL}}) \simeq -2.43 \times 10^{-59}$ N, $F_W(10\lambda_{\mathrm{NSL}}) \simeq -1.29 \times 10^{-62}$ N	$q_W^{(W)} \simeq 8.02111 \times 10^{41}$, $q_W^{(Z^0)} = 0$, $\ell_W \simeq 1.81032 \times 10^{-18}$ m	minimal static Yukawa

Reading. The outputs in this subsection are theoretical outputs calculated after separation of the global HSM constants. They are not all directly comparable to established observables, but they are fully determined in the active closures used in the preceding sections.

Status conclusion. The first table gathers the predictions directly comparable to observables. The CB3 table gathers the color–white baryon-like predictions. The last table gathers the theoretical outputs that consume

$$\kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}.$$

Chapter 31

Constructive realization of the dodecahedral mesh

Object. This chapter establishes a discrete realization compatible with the separated effective constants of HSM. The consumed mechanical coefficients are

$$\kappa_\mu^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}, \qquad \ell_0,$$

defined respectively by (29.11), (29.40), (29.34), (29.38), (29.39), and (29.14). The problem treated here is inverse: it is to exhibit a non-empty class of meshes whose continuum passage reproduces the effective coefficients already separated, without introducing any additional fundamental constant.

Chosen global realization. The constructive realization is carried by a regular cellulation of $\mathbb{S}^3$ of the 120-cell type,

$$\{5, 3, 3\}.$$

Its three-dimensional cells are dodecahedra. Each dodecahedral cell has twelve pentagonal faces. This structure provides the following global combinatorial support:

$$\text{global 120-cell} \;+\; \text{local cell } (1 + 12) \;+\; \text{facial transmission with five subchannels.}$$

The number 12 comes from the number of faces of a dodecahedron. The number 5 comes from the number of vertices, or elementary vertex–vertex directions, on a pentagonal face. These two integers are therefore combinatorial factors of the dodecahedral geometry, and not adjustable parameters.

Status of the construction. The construction demonstrates the existence of a class of compatible realizations. It does not claim uniqueness of the mesh. Local quantities that depend on the choice of a particular realization, for example an energy assigned to a crossing of subchannels, are not invariants of this class and are not introduced as numerical outputs.

31.1 Effective constants to reproduce

Radial coefficients. The two radial coefficients to reproduce are

$$\boxed{\kappa_\mu^{\mathrm{HSM}} \simeq 6.25 \times 10^{34}\ \mathrm{m}^{-2}, \qquad \kappa_r^{\mathrm{HSM}} \simeq 0.205.} \tag{31.1}$$

The first coefficient is local and dimensioned. The second coefficient is dimensionless and weights the radial transfer. They therefore cannot be compared directly as two pure numbers.

Separated scales. The scales required for the local realization are

$$\boxed{R_0^{\mathrm{HSM}} \simeq 1.51 \times 10^{29}\ \mathrm{m}, \qquad \rho^{\mathrm{HSM}} \simeq 8.10 \times 10^{-27}\ \mathrm{kg\,m^{-3}},} \tag{31.2}$$

$$\boxed{T^{\mathrm{HSM}} \simeq 7.28 \times 10^{-10}\ \mathrm{Pa}, \qquad \ell_0 \simeq 2.08 \times 10^{-22}\ \mathrm{m}.} \tag{31.3}$$

The values (31.1)–(31.3) are central synthesis values, rounded to the useful rank for the mesh diagnostic. The chapter does not recalculate them; it constructs a mesh compatible with them. The additional digits used in the observational anchoring calculation sections belong to numerical reproducibility and do not constitute additional phenomenological precision.

Threshold spectral constraint. The spectral constraint associated with the second admissible threshold is represented by a referral ratio

$$\chi_{207} := \frac{\Xi_{\tan}}{\Xi_r},$$

where Ξ_r and $\Xi_{\tan}$ respectively denote the radial and tangential geometric referrals of the continuum passage. The numerical value of χ_{207} is not necessary to demonstrate the non-emptiness of the class of meshes: for any $\chi_{207} > 0$, the choice

$$\Xi_{\tan} = \chi_{207}\, \Xi_r$$

realizes the ratio imposed by the threshold $q_2 = 207$.

31.2 Enriched dodecahedral cell

Local degrees of freedom. A local cell is represented by a central degree of freedom

$$u_i$$

and by twelve facial degrees of freedom

$$f_{iF}, \qquad F = 1, \dots, 12.$$

The isotropic facial mode is

$$\boxed{v_i := \frac{1}{12} \sum_{F=1}^{12} f_{iF}.} \tag{31.4}$$

The local reduction relevant to the radial sector is therefore an enriched cell of type

$$1 + 12.$$

Local quadratic energy. The local radial block is

$$\boxed{E_{\mathrm{loc}}(u_i, v_i) = 6a_f\, v_i^2 + \frac{a_c}{2}\, u_i^2 + \frac{12 c_{cf}}{2}\, (v_i - u_i)^2.} \tag{31.5}$$

The coefficients a_f, a_c, c_{cf} are positive. The factor 12 is the number of faces of the dodecahedron. The energy (31.5) is positive definite for

$$a_f > 0, \qquad a_c > 0, \qquad c_{cf} > 0.$$

Elimination of the centre. For a fixed facial mode v_i, the local Euler equation gives

$$a_c u_i + 12 c_{cf}(u_i - v_i) = 0,$$

hence

$$\boxed{u_i = \frac{12 c_{cf}}{a_c + 12 c_{cf}}\, v_i.} \tag{31.6}$$

After elimination of u_i, the local block reduces to

$$\boxed{E_{\mathrm{loc}}^{\mathrm{eff}}(v_i) = \frac{A_{\mathrm{eff}}}{2}\, v_i^2, \qquad A_{\mathrm{eff}} = 12 a_f + \frac{a_c (12 c_{cf})}{a_c + 12 c_{cf}}.} \tag{31.7}$$

Local continuum coefficient. Let $\mathcal{V}_{\ell_0}$ be the elementary volume associated with the scale ℓ_0. The local coefficient arising from the continuum passage is

$$\boxed{\kappa_\mu^{\mathrm{mesh}} = \frac{A_{\mathrm{eff}}}{T^{\mathrm{HSM}} \mathcal{V}_{\ell_0}}.} \tag{31.8}$$

The target coefficient is therefore realized when

$$\boxed{A_{\mathrm{eff}} = A_\mu^\star, \qquad A_\mu^\star := T^{\mathrm{HSM}} \mathcal{V}_{\ell_0} \kappa_\mu^{\mathrm{HSM}}.} \tag{31.9}$$

The dimension of $A_\mu^\star$ is that of a local stiffness. Indeed,

$$[T\, \mathcal{V}_{\ell_0}\, \kappa_\mu] = \mathrm{Pa\,m^3\,m^{-2}} = \mathrm{N\,m^{-1}}.$$

31.3 Facial transmission with five subchannels

Origin of the factor five. Each face of the dodecahedron is a pentagon. An elementary transmission between two neighboring cells can be decomposed into five subchannels associated with the five vertex–vertex directions of the face. This factor is purely combinatorial:

$$\boxed{N_F^{\mathrm{geom}} = 5.} \tag{31.10}$$

Effective facial stiffness. If the five subchannels carry the same stiffness k_F, with an average isotropic projection

$$\langle \cos^2 \theta \rangle_{\mathrm{iso}} = \frac{1}{3},$$

then the radial facial stiffness is

$$\boxed{\gamma_F = 5\, k_F\, \frac{1}{3} = \frac{5}{3} k_F.} \tag{31.11}$$

The positivity of k_F is equivalent to the positivity of γ_F.

Continuum radial coefficient. The continuum passage associates with γ_F the coefficient

$$\boxed{\kappa_r^{\mathrm{mesh}} = \frac{\gamma_F\, \ell_0^2\, \Xi_r}{T^{\mathrm{HSM}} \mathcal{V}_{\ell_0}}.} \tag{31.12}$$

The target κ_r^{HSM} is therefore realized when

$$\boxed{\gamma_F = \gamma_r^\star, \qquad \gamma_r^\star := \frac{T^{\mathrm{HSM}}\mathcal{V}_{\ell_0}}{\ell_0^2\,\Xi_r}\,\kappa_r^{\mathrm{HSM}}.} \tag{31.13}$$

The dimension is correct:

$$\left[\frac{T\,\mathcal{V}_{\ell_0}}{\ell_0^2}\right] = \frac{\mathrm{Pa\,m^3}}{\mathrm{m^2}} = \mathrm{N\,m^{-1}}.$$

An explicit choice of the subchannels is then

$$\boxed{k_F = \frac{3}{5}\,\gamma_r^\star.} \tag{31.14}$$

31.4 Constructive existence

Proposition 31.1 (Compatible dodecahedral realization at LO rank)**.** *For any $\Xi_r > 0$ and any $\chi_{207} > 0$, there exists a non-empty class of meshes of the type*

$$120\text{-}\textit{cell global} \;+\; (1+12)\;\textit{local} \;+\; 5\;\textit{facial subchannels}$$

such that

$$\kappa_\mu^{\mathrm{mesh}} = \kappa_\mu^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{mesh}} = \kappa_r^{\mathrm{HSM}}, \qquad \frac{\Xi_{\tan}}{\Xi_r} = \chi_{207}.$$

Proof. Let

$$A_\mu^\star = T^{\mathrm{HSM}}\mathcal{V}_{\ell_0}\kappa_\mu^{\mathrm{HSM}}.$$

The local choice

$$\boxed{a_f = \frac{A_\mu^\star}{24}, \qquad a_c = A_\mu^\star, \qquad c_{cf} = \frac{A_\mu^\star}{12}} \tag{31.15}$$

is strictly positive. Injecting it into (31.7), one obtains

$$12a_f = \frac{A_\mu^\star}{2},$$

and

$$\frac{a_c(12c_{cf})}{a_c + 12c_{cf}} = \frac{A_\mu^\star A_\mu^\star}{A_\mu^\star + A_\mu^\star} = \frac{A_\mu^\star}{2}.$$

Therefore

$$A_{\mathrm{eff}} = A_\mu^\star,$$

and (31.8) gives

$$\kappa_\mu^{\mathrm{mesh}} = \kappa_\mu^{\mathrm{HSM}}.$$

For the radial transfer, let

$$\gamma_r^\star = \frac{T^{\mathrm{HSM}}\mathcal{V}_{\ell_0}}{\ell_0^2\,\Xi_r}\,\kappa_r^{\mathrm{HSM}}.$$

The choice

$$k_F = \frac{3}{5}\gamma_r^\star$$

is strictly positive, and (31.11) gives

$$\gamma_F = \gamma_r^\star.$$

Equation (31.12) then gives

$$\kappa_r^{\mathrm{mesh}} = \kappa_r^{\mathrm{HSM}}.$$

Finally, the choice

$$\Xi_{\tan} = \chi_{207}\,\Xi_r$$

realizes

$$\frac{\Xi_{\tan}}{\Xi_r} = \chi_{207}.$$

The three constraints are therefore satisfied simultaneously by positive parameters. The class is non-empty. □

31.5 Role of the 120-cell

Global support. The 120-cell provides a compact and regular realization of the cellular structure on $\mathbb{S}^3$. Since the three-dimensional cell is a dodecahedron, the local reduction inherited from this structure naturally has a centre and twelve faces. The 1+12 block of (31.4)–(31.7) is therefore the minimal isotropic reduction of this geometry.

Support of transmissions. The pentagonal faces of the dodecahedron provide the five subchannels of (31.10). The facial transmission (31.11) is thus compatible with the global 120-cell support while remaining locally formulated. The continuum passage depends on the effective coefficients κ_μ, κ_r, and on the referrals $\Xi_r, \Xi_{\tan}$, not on the total number of global cells.

Non-uniqueness. Proposition 31.1 is an existence proof. It does not exclude other cellulations of $\mathbb{S}^3$ capable of producing the same effective coefficients. The 120-cell is retained because it gives an explicit dodecahedral realization compatible with the combinatorial factors 12 and 5 used in the local construction.

31.6 Local quantities dependent on the realization

Node and crossing energies. A quantity assigned to a node, a link, or a crossing depends on the local distribution of the stiffnesses a_f, a_c, c_{cf}, k_F, on the volume assigned to the degree of freedom, and on the choice of the referrals $\Xi_r, \Xi_{\tan}$. Two realizations of the class of Proposition 31.1 can therefore reproduce the same effective coefficients

$$\kappa_\mu^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}, \qquad \chi_{207},$$

while assigning different local energies to crossings.

Condition for promoting a local quantity. A local crossing quantity can be promoted as a numerical output only if the construction selects a unique mesh, or a subclass for which this quantity is invariant. In the absence of such invariance, it is not a fundamental constant and does not enter the chain of predictions.

31.7 Conclusion

Proposition 31.2 (Constructive compatibility)**.** *The separated constants*

$$\kappa_\mu^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{HSM}}, \qquad R_0^{\mathrm{HSM}}, \qquad \rho^{\mathrm{HSM}}, \qquad T^{\mathrm{HSM}}, \qquad \ell_0$$

admit a compatible discrete realization on a global support of the 120*-cell type, with enriched dodecahedral local cell* $1 + 12$ *and facial transmission with five subchannels. The construction simultaneously reproduces*

$$\kappa_\mu^{\mathrm{mesh}} = \kappa_\mu^{\mathrm{HSM}}, \qquad \kappa_r^{\mathrm{mesh}} = \kappa_r^{\mathrm{HSM}}, \qquad \Xi_{\tan}/\Xi_r = \chi_{207}.$$

Scope. The result is a constructive compatibility: the continuous structure separated by the preceding closures can be supported by at least one class of discrete dodecahedral realizations. The effective constants consumed by the prediction sectors remain those of the continuous closures; the mesh establishes the discrete realizability of these constants.

Part III

Link with effective physical theories

Chapter 32

Introduction: object, method, and scope

Object of this part. The object of this part is to establish, from the results already built within HSM, that the theory admits, in well-identified regimes, effective reductions that reproduce certain central structures of standard theoretical physics. The ambition is therefore not to introduce here a new axiomatic layer above HSM, nor to redevelop the internal constructions already established in the theoretical chapters, but to show how they can be read, within their own validity windows, as effective realizations of structures recognized in quantum mechanics, quantum field theory, low-energy effective theories, and the relativistic and gravitational side.

Principle of organization. The guiding thread is the following:

$$\text{HSM} \longrightarrow \text{effective regimes} \longrightarrow \text{reading in the standard languages of theoretical physics.}$$

In particular, two broad parts will be distinguished. The first gathers the *effective quantum domain*: quantum mechanics, local effective quantum field theory, interaction sectors, flavor and mixing structures, then a reading in low-energy effective theory. The second gathers the *relativistic and gravitational domain*: effective relativistic kinematics, gravitational-like sector, Newtonian limit, and scope of the link toward a general-relativity-type reading.

Method. The method followed throughout this part is strictly downward. We start from objects already established in the theoretical chapters: fundamental equations, unitary solutions, effective fields, composite solutions, solution–field couplings, and already obtained closures. From these ingredients, we then identify the regimes in which an effective reduction is legitimate, write the corresponding reduced form, and then show in what sense this form overlaps with a known theoretical domain. Thus, this part is not a substitute for the theoretical chapters; it is their external and structured reading.

Reading rule. A distinction will be maintained throughout the text between three levels:

1. the *results internal to HSM*, which must be established in the theoretical chapters and to which we refer here;
2. the *effective reductions*, obtained from these results in a given regime;
3. the *correspondences with standard theories*, which are the proper object of this chapter.

In other words, when an ingredient belongs to the internal construction of the HSM framework, it is not meant to be demonstrated here a second time; however, it is indeed in this part that the logical chain by which this ingredient makes it possible to cover a standard language or domain must be made explicit.

External contextual references. The notions of effective quantum mechanics, effective field theory, classical electromagnetism, general relativity, Newtonian limit, and PMNS mixing are used here as languages for comparison with the standard domains of physics. Results specific to HSM are referred to the internal sections of the manuscript; the standard notions to which they are compared can be read in the classic references [10, 16, 17, 11, 12, 2, 3].

Scope of the results. The results presented in this part must be read as results *of effective reduction*. Depending on the case, the scope obtained may be of three types:

1. an *explicit effective derivation* of a standard regime or equation;
2. a *structural coverage* of a known formalism, when HSM reconstructs its organizing objects without yet claiming to exhaust its full completeness;
3. a *controlled correspondence within a validity window*, when the reading is valid only at leading order, at low energy, or under clearly stated reduction hypotheses.

This hierarchy of scope will always be stated explicitly, so as not to confuse what is demonstrated end to end with what is established only at the effective level or within a precise asymptotic limit.

What this part does not do. This part does not claim to reconstruct wholesale, by a simple analogy of vocabulary, the entirety of existing physical theories. It is neither a juxtaposition of formal dictionaries nor a collection of identifications pasted on a posteriori. The criterion retained is stricter: every invoked link must rest on an explicit reduction chain starting from an already established HSM result and leading to an effective structure identifiable in the corresponding standard language.

Program. The part opens with the effective quantum domain. We first show how the structures of effective quantum mechanics emerge, then how this reduction extends into a local quantum-field-theory-type reading, including the relevant interaction sectors and flavor and mixing structures. We then examine the reading in low-energy effective theory and its validity window. In a second stage, we treat the relativistic and gravitational side: effective kinematics, gravitational-like sector, Newtonian limit, and scope of the link toward a broader relativistic reading. Finally, the part concludes with a precise assessment of the demonstrated results and the associated regimes of validity.

Chapter 33

HSM and the effective quantum domain

33.1 Overview

Object of this part. This part aims to show that HSM reproduces, in well-identified effective regimes, several structures of the standard quantum domain. More precisely, we establish here a continuous reduction chain going from effective quantum mechanics to a local quantum-field-theory-type reading, then to a formulation in low-energy effective theory. The important point is that these three levels are not treated as three foreign theories placed side by side, but as three readings of one and the same HSM content, valid at different scales and under different reduction hypotheses.

Reduction chain. The general scheme studied in this part is

$$\text{HSM} \longrightarrow \text{effective quantum reduction} \longrightarrow \text{local effective fields} \longrightarrow \text{low-energy effective theory}.$$

The first step reveals quantum-mechanics-type structures: effective state space, Schrödinger-type evolution, probabilistic reading, semiclassical regime, and EBK quantization. The second step shows how, when the hypotheses of linearization, effective locality, and scale separation are satisfied, a local description in terms of effective fields, quadratic Lagrangians, propagators, and amplitudes can be obtained. The third step organizes these same objects in the language of local operators, coarse-graining, matching, and EFT-type validity windows.

Covered sectors. In this part, the relevant interaction sectors are not introduced as blocks independent of the rest, but as specializations of this general quantum reduction. We shall thus examine the available effective readings of the electromagnetic-like, weak-like, strong-like, and, when relevant at the local quantum level, gravitational-like sectors. Likewise, flavor and mixing structures will be treated as internal organizations of the effective quantum domain, and not as external additions.

Scope rule. The results of this part must always be read at their exact level. Depending on the case, we will obtain:

- an explicit effective derivation of a standard equation or formalism;
- a structural coverage of a known sector;
- or a controlled reading within a restricted validity window (low energy, leading order, local reduction, envelope approximation, etc.).

This rule is particularly important in the passage from quantum mechanics to QFT, then from QFT to EFT: the text will explicitly indicate what is demonstrated as a reduction, what is only identified as an effective structure, and what is valid only in a controlled limit.

Organization. We proceed in five steps. We first establish the reduction toward effective quantum mechanics. We then show how this reduction extends into a local effective quantum-field-theory-type reading. We then treat the relevant interaction sectors in this quantum domain. We next examine the internal flavor and mixing structures when the theory already provides the necessary ingredients. Finally, we formulate the whole in the language of a low-energy effective theory, with its matching and its own validity window.

Methodological principle. None of the results in this part should be understood as a mere formal analogy. Whenever a link with quantum mechanics, QFT, or EFT is invoked, it must rest on an explicit chain:

already established HSM result $\longrightarrow$ effective reduction $\longrightarrow$ identification of the corresponding standard for

This chain, and it alone, grounds the demonstrative value of this part.

33.2 Effective quantum mechanics

Object of the reduction. In this section, we show that HSM admits, in a slow-envelope regime and with suitable spectral localization, an effective reduction that reproduces several central structures of standard quantum mechanics. The intended result is not the introduction of an additional theory above HSM, but the identification of a regime in which the relevant degrees of freedom reorganize into the form of a complex state space, a linear Schrödinger-type evolution, a Born-type probabilistic reading, and a semiclassical regime compatible with EBK quantization.

33.2.1 Envelope reduction and Schrödinger equation

Logical chain. The starting point is the presence, within HSM, of unitary solutions and localized modes admitting an effective slow-envelope description. In this regime, the fast dynamics carried by the phase fixes the internal scale of the mode, while the evolution of the envelope selects a reduced dynamics governing propagation, interferences, and the response to slow couplings. The reduction chain is then written schematically as

HSM unitary solution $\longrightarrow$ slow-envelope ansatz $\longrightarrow$ reduced Schrödinger-type equation $\longrightarrow$ effective quan

This chain must be read as an asymptotic reduction: it is valid in the window where the carrier mode is well isolated, where the envelope is slow, and where external couplings do not destroy the scale separation.

Slow-envelope ansatz. The central technical point of the quantum reduction is the existence of a regime of scale separation in which an HSM unitary solution can be written as a fast oscillation modulated by a slow envelope. We thus introduce an ansatz of the form

$$\Psi(t, \mathbf{x}) = A(t, \mathbf{x})\, e^{-i\omega_0 t},$$

or, more generally, a superposition of modes neighboring a carrier frequency ω_0, with an amplitude A varying slowly on the scale of the fast cycle. By reinjecting this ansatz into the fundamental equations of HSM, then expanding in the regime where the envelope derivatives are small compared with $\omega_0 A$, the fast oscillating part averages out and one obtains, at the leading order of this multiscale expansion, a reduced equation for the slow envelope.

Reduced Schrödinger-type form. In this multiscale window, the fast part fixes the internal scale of the mode, while the slow part governs effective propagation. After averaging over the fast cycle and identifying the dominant oscillation term, the evolution equation takes the form

$$i\,h_{\mathrm{eff}}\,\partial_t\psi = \Big(-\frac{h_{\mathrm{eff}}^2}{2m_{\mathrm{eff}}}\Delta + V_{\mathrm{eff}}\Big)\psi, \qquad m_{\mathrm{eff}} = \frac{h_{\mathrm{eff}}\,\Omega_0}{c_t^2}, \tag{33.1}$$

where ψ denotes the effectively normalized complex envelope, Ω_0 the internal frequency of the carrier mode, and V_{eff} the slow potential induced by the background, the effective geometry, and the weakly varying couplings. This writing has the form of the effective Schrödinger equation expected in the nonrelativistic, local, low-energy regime.

Meaning of the reduction theorem. Equation (33.1) is not postulated; it is the asymptotic form taken by the HSM dynamics when three conditions are met:

1. a well-defined *carrier frequency* Ω_0 for the mode under consideration;
2. a *slow envelope*, in the sense that the derivatives of ψ remain small compared with $\Omega_0\psi$;
3. a sufficient *scale separation* between the internal dynamics of the mode and the spatial or temporal variations of the effective background.

In this window, the quantum-mechanics structures used at this order therefore appear as a controlled reduction of the fundamental equations of HSM.

Madelung–Hamilton–Jacobi reading. By writing the envelope in polar form

$$\psi = \sqrt{n}\,e^{\,iS/h_{\mathrm{eff}}},$$

equation (33.1) decomposes into a continuity equation for n and a phase equation of Hamilton–Jacobi type corrected by an effective quantum term. This rewriting shows that the Schrödinger reduction is not disjoint from the semiclassical side: on the contrary, it organizes it coherently, with an explicit passage between envelope dynamics, conserved current, and fast-phase regime.

Scope of the reduction. The link between HSM and effective quantum mechanics therefore does not rest only on an analogy of structure, but on an explicit reduction of the fundamental equations toward a Schrödinger equation for the slow envelope. It is on this basis that the effective state space, the Born-type operational reading, Ehrenfest's theorem, and semiclassical quantization can then be read.

33.2.2 Probabilistic reading, Ehrenfest, and semiclassical regime

Effective state space. In this reduction, the relevant configurations are described by complex amplitudes living in an effective state space, equipped with the scalar product inherited from the energy structure and the normalization of the modes. This framework makes it possible to identify a probability density, an associated current, and thus an operational quantum-mechanics-type reading. This is not an independent postulate here: the Hilbert-space structure appears as the natural form taken by the reduced envelope dynamics in the considered regime.

Probabilistic reading. Once the reduced equation has been obtained, conservation of the associated current and stability of the norm allow a probabilistic reading of the reduced regime. The density $|\psi|^2$ of the effective envelope then plays the role of a density of presence transported by the conserved current.

This construction makes available a Born-type operational reading within the considered envelope window. It does not, however, by itself constitute a complete derivation of measurement theory or of the general passage between effective presence density and detection probability. The point established here is more precise: in the reduced regime, the conserved norm provides the natural quantity on which the probabilistic reading rests.

Diffraction, interferences, and internal phase. The effective quantum reduction is not limited to reproducing a Schrödinger-type equation; it also sheds light on the origin of emblematic phenomena of wave quantum mechanics, in particular diffraction and interferences. In the HSM reading, these phenomena are not only the formal effect of an abstract wavefunction, but the manifestation of solutions endowed with an internal phase and rotation structure. In particular, for 2DOF bump-type solutions, the propagation geometry and the internal rotation phase jointly determine the interference response and the effective diffraction law.

Superposition and interference fringes. When two coherent contributions of one and the same mode or of two effective paths recombine, the envelope dynamics naturally leads to a superposition of complex amplitudes. The interference fringes then do not result from a detection randomness posed a posteriori, but from the combination of phases carried by the solutions themselves. In this reading, interference patterns and two-path coherent recombination schemes belong to the same complex-envelope regime: they can therefore be described in the effective quantum reduction, without postulating an interference rule external to the phase dynamics.

Diffraction angle and internal rotation phase. The theory further shows that the diffraction angle is not reducible to a simple external kinematic datum: it is linked to the phase structure of the solution and, for the 2DOF bump sector, to the internal rotation that accompanies its propagation. Thus, HSM is not limited to reformulating diffraction in the standard language: the reduction makes it possible to associate the effective wave phase with the internal rotation phase of the mode, when this internal structure is present. The scope of this reading remains that of the envelope regime and of the considered solutions.

Wave packets and Ehrenfest's theorem. In the regime of sufficiently localized wave packets, the dynamics of the envelope makes it possible to recover an evolution of mean observables of Ehrenfest type. Packet centers then follow, at leading order, the effective classical dynamics governed by the slow potential V_{eff}. The HSM $\to$ effective quantum mechanics reduction thus covers not only the Schrödinger equation, but also the standard bridge between quantum evolution and mean classical dynamics.

Semiclassical regime and EBK quantization. When the phase varies rapidly relative to the envelope, one recovers a semiclassical regime in which the evolution is governed by a corrected Hamilton–Jacobi-type dynamics, and where the quantization of stable families can be formulated in the EBK language. This point is essential: it shows that HSM covers not only effective linear quantum mechanics, but also the standard bridge between reduced classical dynamics and semiclassical quantization.

Composite systems and non-factorizable states. Once the effective state space, Schrödinger-type dynamics, and probabilistic reading have been established, the composition of subsystems naturally leads to an effective tensor structure

$$\mathcal{H}_{12}^{\text{eff}} = \mathcal{H}_1^{\text{eff}} \otimes \mathcal{H}_2^{\text{eff}}.$$

Factorizable states then constitute only a subset; non-factorizable states appear by right in the quantum framework thus obtained. In this effective framework, the existence of non-factorizable states is therefore a consequence of the tensor structure of the reduced state space. Entanglement is then available in the obtained effective language, without it being necessary to add a separate rule at the level of this reduction. The existence, in the theory, of correlated global solutions whose reduced non-factorizable modal component is made explicit in (25.8) reinforces this reading, without their detailed construction belonging to the present chapter.

Explanatory scope of the quantum reduction. The present reduction is not limited to reproducing a form of the Schrödinger equation. It also makes it possible, in the sectors where the corresponding solutions have been constructed, to associate certain quantum phenomena with internal structures of phase, correlation, or composition. This is in particular the case for global correlations and non-separability: in these regimes, they are not only represented by abstract vectors in state space, but can be related to correlated global solutions. Certain losses of phase coherence can also receive an internal reading when they come from a misalignment or an averaging of phases. This scope nevertheless remains that of the explicitly constructed regimes; it does not constitute a general derivation of all quantum phenomenology.

Conclusion of the section. HSM therefore reproduces several central structures of effective quantum mechanics as soon as one works in a slow-envelope, scale-separation, and appropriate modal-localization regime. This coverage includes the explicit reduction toward a Schrödinger equation, the effective state space, the probabilistic reading, composite systems, Ehrenfest's theorem, and the semiclassical/EBK side. It is not reducible to a mere analogy of structure: in several cases, the theory also provides a complementary internal reading of phenomena that the standard formulation describes at the probabilistic or formal level, in particular on the side of global correlations, non-separability, and interference or diffraction phenomena in the regimes where the corresponding solutions are explicitly constructed.

33.3 Local effective quantum field theory

33.3.1 Effective fields, local Lagrangians, and quantization

Purpose of the reduction. This section shows that the quantum reduction obtained from HSM is not limited to a wave-function-type dynamics, but also admits, in a local low-energy window with small fluctuations and scale separation, a reading in terms of quantized effective fields. The goal is therefore not to postulate a field theory external to the HSM framework, but to identify the regimes in which some collective degrees of freedom of the theory reorganize as local fields governed by effective Lagrangians, modal algebras, propagators, and amplitudes.

Logical chain. The reduction chain studied here extends that of effective quantum mechanics:

HSM modes and solutions $\longrightarrow$ local collective variables $\longrightarrow$ quadratic and then weakly nonlinear effective a

The decisive step is the identification of a regime in which the relevant fluctuations can be described locally, after coarse-graining and controlled linearization, by a finite number of effective fields carrying the useful degrees of freedom. The scope of this construction is therefore that of an effective local description, not that of an independent fundamental field theory introduced as a new axiom.

Effective fields and local Lagrangian. In this local window, the fundamental objects of HSM are not read as elementary point particles in the usual sense, but as underlying configurations whose organized fluctuations define effective fields. At quadratic order, these fields obey local second-order equations, and they can be encoded in an effective Lagrangian containing:

kinetic terms + effective mass terms when the sector has them + weak or residual interactions organized as c

This is the form expected of an effective local quantum field theory, in the limit where global nonlocalities, mesh effects, and matching corrections are negligible at the scale under consideration.

Effective quantization of modes. Once the quadratic sector has been identified, the decomposition into eigenmodes allows a reading in terms of effective oscillators. In the standard language of reduced QFT, these modal amplitudes can be represented by effective creation and annihilation operators, with algebraic relations adapted to the sector under consideration.

The status of this step must be specified. HSM provides upstream the counting of degrees of freedom, the solution families, and the geometric spin classes. However, the mere fact that a solution carries a half-integer geometric spin does not, by itself, constitute a complete proof of the effective spin–statistics theorem. The canonical relations used here must therefore be read as the effective QFT representation of the reduced sector, once the bosonic or fermionic reading of the sector has been identified.

This step does not introduce a dynamics foreign to HSM: it formalizes, in the standard language of effective QFT, the structure of the elementary fluctuations already selected by the local reduction. The eigenmodes then play the role of effective quanta in the window under consideration, without this chapter claiming to derive by itself the entire fundamental canonical algebra.

33.3.2 Propagators, amplitudes, and domain of validity

Propagators and effective local causality. The quantized quadratic Lagrangian determines effective propagators that organize the linear response, the propagation of excitations, and two-point correlation kernels. In the local window where the reduction is valid, these propagators carry the usual structure of free or weakly coupled fields: effective poles, sector masses when present, propagation at the characteristic speed, and local correlators.

The causality thus obtained must be read as an effective local causality: it holds in the limit where the description by local fields is legitimate, without prejudging the full global structure of the HSM framework outside this window.

Interactions and amplitudes. Beyond quadratic order, residual couplings between effective fields generate local vertices in the regimes where the locality approximation is controlled. One can then organize scattering amplitudes in the perturbative language adapted to the window under consideration. The meaning of this reading is important: the present chapter does not state that an independent fundamental QFT is postulated from the outset, but that HSM reproduces, in certain local regimes, the effective form of fields, propagators, vertices, and amplitudes used in the QFT language.

The following sections specify this reduction sector by sector: electromagnetic-like, weak-like, strong-like and, when the local regime allows it, gravitational-like.

Domain of validity. The local QFT reading simultaneously assumes:

1. a sufficient scale separation between the internal size of the modes, the variation scale of the background, and the observation scale;

2. a linearization or quasi-linearization around a background or reference solution;

3. a low-energy window relative to the microscopic scales where mesh, matching, or closure details can no longer be neglected;

4. an effective locality, that is, a description by local operators valid after coarse-graining.

Outside this window, the QFT language may remain useful as an approximation, but it must no longer be confused with the full dynamics of HSM.

Scope of the result. The result of this section is therefore an effective reduction toward the language of local quantum field theory. In the regimes where the preceding conditions are satisfied, HSM provides:

- effective fields obtained from organized fluctuations of the theory;
- a quadratic local Lagrangian, extended by weak or residual interactions;
- an effective quantization of modes;
- QFT-type propagators, correlators, and amplitudes.

The scope of the link is that of a controlled local reduction, valid in a low-energy regime with small fluctuations and sufficient scale separation. What is established here is therefore an effective local QFT reading of certain sectors, not the introduction of an independent theory external to HSM.

Continuation. This general reduction must now be specialized to the different interaction sectors. We therefore examine how the electromagnetic-like, weak-like, strong-like and, when it is relevant in this window, gravitational-like channels are read in the effective quantum domain as local or quasi-local interaction sectors.

33.4 Interaction sectors of the quantum domain

Purpose of the section. The reduction to an effective local quantum field theory has physical scope only if it identifies, in the effective quantum regime obtained from HSM, interaction sectors readable in the usual language. The purpose of this section is therefore to show that the major interaction channels already identified in the theoretical chapters of HSM can be interpreted, in their own validity window, as recognized effective sectors of the quantum domain: the electromagnetic-like channel, the weak-like channel, the strong-like channel, and, when the local reading is relevant, the gravitational-like channel.

General principle. The central point is that this chapter does not introduce new fields here by simple analogy. The sectors considered have already been constructed, at their own level, in the theoretical parts of the document: existence of the channels, forms of the effective equations, sources, minimal or complete closures as appropriate, and, when they exist, associated static or dynamic readings. The role of this section is only to show in what sense these objects overlap with the familiar interaction sectors of the effective quantum domain.

Common logical chain. For each of these sectors, the reading chain is the same:

already established HSM channel $\longrightarrow$ local or quasi-local effective equation $\longrightarrow$ mediation of an interaction

Depending on the structure of the channel considered, this reading takes the form of a long-range exchange, a finite-range massive exchange, a confinement interaction, or an effective geometric reading. The diversity of the sectors therefore does not break the unity of method: all are read as controlled effective realizations of interactions already contained in HSM.

33.4.1 Electromagnetic-like sector

Effective Abelian reading. When the effective Abelian closure is established, the electromagnetic-like channel is locally read as a Maxwell-type sector, carried by an effective vector field, conserved sources, and long-range propagation. In the local quantum window, this sector therefore provides the QFT reading of an Abelian vector exchange, with its effective quanta, propagator, and interaction amplitudes. The important point here is that this reading is not added from outside: it extends the closure already obtained for the electromagnetic-like sector in the theoretical parts.

33.4.2 Weak-like sector

Massive mediation and finite range. The weak-like channel, in its current state of closure, appears as a sector mediated at finite range. On the relevant source branch, the theory establishes a weak-like channel, endowed with an effective massive mediator and a local Yukawa-type reading, whose low-momentum limit leads to a local Fermi-type interaction. The role of the present chapter is precise here: it is not to reconstruct at this point an entire independent fundamental electroweak theory, but to establish that HSM effectively overlaps with the known *low-energy weak domain* of QM/QFT, that is, a short-range massive exchange and its controlled local reduction.

33.4.3 Strong-like sector

Binding channel and short range. The strong-like sector, as constructed in the HSM theory, is read in the effective quantum domain as a short-range mediation channel structuring composite states. The closure of the mediator, the range, the channel charge, the effective normalization, the attractive potential, and the local self-energy makes it possible to identify it here as a sector of effective strong interaction. The relevant reading, in the window considered, is that of a short-range binding and confinement mechanism for composite states. Thus, without introducing any structure external to the HSM framework, the effective quantum domain overlaps with a finite-range strong-like sector, which can be read in the usual language as a channel of effective strong interaction. When a richer local reformulation is available, it may refine this reading; but, for the present chapter, the essential point is already obtained: HSM contains an identifiable strong-like sector in the effective quantum domain.

Gauge confinement: effective reading by a flux tube

Purpose of the reading. The tubular reduction of the strong-like channel established in §21.2.2 shows that, in composite configurations where the flux is guided between two internal substructures, the transverse minimization defines a positive effective tension

$$\sigma_{\mathrm{eff}} > 0,$$

and a longitudinal energy

$$E_{\mathrm{tube}}(R) = \sigma_{\mathrm{eff}} R + E_{\mathrm{cap}} + o(R).$$

The present chapter does not give the derivation again. It only makes explicit its reading in the effective QFT language.

Self-confinement and gauge confinement. Two levels must be distinguished. *Self-confinement* denotes the fact that an HSM solution has a localized and stable internal structure without freely dissociating; this property concerns unitary solutions and belongs to their own dynamics. *Gauge confinement*, in the sense used here, denotes a different property: the existence, in a composite state, of a channel flux whose minimum energy grows with the separation of the poles or substructures that carry it.

Effective linear potential. According to (21.38), the tubular reduction gives

$$\boxed{V_{\text{conf}}(R) \geq \sigma_{\text{eff}} R - C.} \tag{33.2}$$

This linear growth prevents free separation of the constituents carrying the flux without an increasing energy cost. It constitutes the effective static signature of confinement in the tubular window of the strong-like channel.

QFT reading of the flux tube. In the usual language of QFT, a configuration of this type is read as a flux tube linking the relevant internal poles of the system. The potential (33.2) then provides the IR reading of gauge confinement: the degrees of freedom carrying the flux do not separate freely asymptotically, because the energy stored in the tube grows with their separation. Confinement is therefore not introduced here as a property postulated from outside; it is the effective reading of the tubular result established for the strong-like channel.

Link with the Wilson criterion. In an effective formulation in Euclidean time or static energy, a linear growth of the potential translates, in the standard language, into an area law for the associated effective Wilson loop:

$$\langle W(\mathcal{C}) \rangle \sim e^{-\sigma_{\text{eff}} \, \text{Area}(\mathcal{C})} \qquad \text{for large rectangular loops.}$$

The present chapter does not rederive the Wilson formalism. It only uses the following standard link: once a positive tube tension and a linear potential bound are established in the reduction, the area-law-type reading becomes available in the effective QFT language.

Role of unitary solutions and composites. For unitary 3DOF solutions, one first speaks of localized internal structure and, where appropriate, self-confinement. For composite 3DOF states, in contrast, a flux configuration linking several poles or substructures provides the flux-tube-type reading and therefore the QFT reading of confinement. This block does not confuse these two levels: it shows that they belong to the same strong-like channel, but with two distinct effective readings depending on the unitary or composite nature of the state.

Scope of the result. The result established is an effective reading of confinement in the tubular window:

1. HSM contains a strong-like channel already constructed at the theoretical level;
2. this channel admits, in certain composite configurations, a tubular organization of the flux;
3. the transverse reduction provides a positive tension $\sigma_{\text{eff}} > 0$;
4. the minimum energy grows linearly with length;

5. this growth provides the effective signature of gauge confinement in the QFT language.

It is therefore neither a mere word pasted onto the strong-like sector, nor an external reconstruction of standard QCD, but an effective bridge making explicit in what sense the strong-like channel reproduces the IR signature of confinement.

Conclusion of the block. In the effective quantum domain, HSM contains a strong-like binding channel. When this channel organizes as a flux tube, the theoretical result §21.2.2 provides a positive tension, a linear potential and therefore the effective QFT reading of confinement. This piece completes the reading of the strong-like sector: beyond local binding, the tubular window gives the expected IR signature of a confinement mechanism.

33.4.4 Gravitational-like sector in the quantum domain

Local quantum reading of the gravitational-like channel. Insofar as a gravitational-like channel has been closed at the effective level, one can also propose a reading of it in the local quantum domain. The primary purpose of this reading is not the complete reconstruction of a fundamental quantum gravity, but the identification, in the appropriate local window, of an effective universal or quasi-universal coupling compatible with the gravitational reading of the channel. In practice, this sector will be taken up more broadly in the relativistic and gravitational part; here, it is mentioned only to emphasize that the quantum reduction chain does not stop at the electromagnetic, weak, and strong channels alone.

Unity and difference in scope. These different sectors must not be read as homogeneous blocks at the same level of completion. In some cases, HSM already provides a sufficiently explicit closure for an almost canonical local reading; in others, the theory mainly establishes the existence of a channel and its dominant effective structure, without a complete rewriting in all usual languages being justified yet. This chapter must therefore maintain a discipline of scope: what is asserted for each interaction is exactly what is allowed by the reduction chain effectively demonstrated elsewhere in the document.

Conclusion of the section. The general result is that HSM does not only provide an abstract effective quantum framework, but also organizes *interaction sectors* that can be read in the language of the usual quantum domain. The electromagnetic-like, weak-like, strong-like, and gravitational-like channels are not external analogies added afterward; they correspond, in the effective quantum regime, to interaction structures already constructed in the theory, with the level of closure specific to each sector. This point naturally prepares the following section, devoted no longer to the interaction sectors themselves, but to the internal structures of the quantum domain, in particular mass bases, flavor bases, and mixing matrices.

33.5 Internal structures of the quantum domain

Purpose of the section. After identifying the interaction sectors of the effective quantum domain, it remains to show that HSM also reproduces, in certain reduced subspaces, several internal structures characteristic of quantum particle physics: the distinction between a coupling basis and a propagation eigenbasis, unitary diagonalization, mixing matrices, and, when the orientation structure is available, an effective chiral reading. The aim of this section is therefore not to add an external phenomenological structure to HSM, but to specify in what sense the theory already contains the ingredients allowing a flavor, mixing, and chirality type reading in the usual language.

33.5.1 Mass–flavor matching of the projective 3×2 reduction

Purpose of the matching. The matching made here starts from the composite solution $\mathcal{C}_{3\times2}(q_1, q_2, q_3)$, defined in chapter 24. In the retained formulation, this solution is constructed as a composite of six 2DOF bump constituents in strong overlap. The 3×2 structure then appears as an effective reduction: the six sublobes regroup into three coherent pairs, and the internal projection then selects three breather-like collective witnesses.

The mechanical construction of this reduction is given by (24.20). The numerical evaluation of the selected triplet is given in §30.3. It provides the reduced internal matrix (30.36)

and the internal angles (30.37). These quantities remain internal outputs of the composite solution.

HSM mechanical reading. In HSM, the three directions

$$T_{q_1}, \qquad T_{q_2}, \qquad T_{q_3}$$

are projected collective witnesses of the six-bump composite. They are not three 2DOF breathers posited as independent constituents at the outset. The internal couplings of preclusion, fine stiffness, and the projected EM-like channel make these witnesses non-diagonal in the reduced matrix

$$\mathcal{M}_{\mathrm{red}}^{D_\eta}.$$

The eigenmodes

$$M_1, \qquad M_2, \qquad M_3$$

are therefore the coherent organizations of the composite, that is, the proper ways in which the six sublobes can regroup and balance one another in the branch considered.

The diagonalization

$$\mathcal{M}_{\mathrm{red}}^{D_\eta} U_{\mathcal{C}} = U_{\mathcal{C}} \Lambda_{\mathcal{C}}$$

does not define an absolute mass of the eigenmodes. It defines the internal transition matrix between the basis of projected collective witnesses and the basis of eigenmodes of the composite.

Matching to the mass–flavor language. In the standard language of neutrino oscillations, the PMNS matrix relates a basis of states coupled to weak interactions, called the flavor basis, to a propagation eigenbasis, called the mass basis [2, 3]. The HSM matching consists in reading the basis of projected collective witnesses

$$\{T_{q_1}, T_{q_2}, T_{q_3}\}$$

as the projected mechanical support of the weak coupling basis, and the eigenmodes

$$\{M_1, M_2, M_3\}$$

as the projected mechanical support of the propagation basis.

In this reading, the internal matrix $U_{\mathcal{C}}$ plays the role of the transition matrix between these two bases. At real LO rank, the matching is written

$$U_{\mathrm{mix}}^{\mathrm{eff}} = P_{\mathrm{L}}\, U_{\mathcal{C}}\, P_{\mathrm{R}}, \tag{33.3}$$

where P_{L} and P_{R} are diagonal phase matrices associated with the phase conventions of the external bases. In the real LO reduction, these phases do not modify the three mixing angles.

One then obtains the angular correspondence

$$\boxed{(\theta_{12}^{\mathrm{eff}}, \theta_{13}^{\mathrm{eff}}, \theta_{23}^{\mathrm{eff}})_{\mathrm{LO}} = (\vartheta_{12}, \vartheta_{13}, \vartheta_{23}).} \tag{33.4}$$

Matched internal values. For the triplet evaluated in §30.3, the internal angles of the projective 3×2 reduction are

$$\boxed{(\vartheta_{12}, \vartheta_{13}, \vartheta_{23}) \simeq (33.4^\circ,\ 8.6^\circ,\ 49.3^\circ\text{–}49.5^\circ).} \tag{33.5}$$

These values are not introduced as external phenomenological parameters: they come from the internal diagonalization (30.37) of the composite $\mathcal{C}_{3\times2}$.

Scope of the matching. The correspondence (33.4) does not transform the reduced matrix $\mathcal{M}_{\text{red}}^{D_\eta}$ into an absolute mass matrix. It only establishes that the internal matrix of eigenmodes of the composite has, in the mass–flavor projection of the standard language, the real three-angle structure of a leptonic mixing matrix.

The present reduction concerns only the real LO sub-sector of the three angles. It does not deduce an absolute mass of the eigenmodes of the composite and does not fix a Dirac phase. Phase effects or corrections beyond the real LO rank belong to an extension of the correspondence.

33.5.2 Effective chiral structure

Purpose of the chiral reduction. The HSM theory does not merely admit multiple internal modes: it also distinguishes certain unitary solutions by their internal sense of rotation. The purpose of this subsection is to show that this orientation can induce, in the sectors where the corresponding couplings are established, an effective separation of the state space into two classes coupled differently. It is in this precise sense that HSM admits an effective chiral reading in the quantum domain.

Orientation invariant. To any fermion-like unitary solution of the sector considered, one associates an internal orientation sign

$$\chi \in \{+1, -1\},$$

defined by the sense of rotation of the solution on its internal cycle. This sign is intrinsic to the solution and remains invariant as long as one remains in the same reduced sector and in the same stability class.

Propagation of χ in effective couplings. The decisive point is that this orientation sign does not remain merely an internal kinematic marker. In the channels where the closure is established, it can control the sign or the class of certain effective couplings. This dependence can be summarized in the schematic form

$$Q_{\text{EM}} \propto \chi, \qquad I_{\mathsf{s}} \propto \chi,$$

or by the corresponding sectorial analog when the channel considered carries the same orientation quantum. This writing must be read as a structural relation: the rotation sign distinguishes two classes of solutions that respond differently to the coupling channels concerned.

Decomposition of the state space. This distinction induces a natural decomposition of the effective state space of fermion-like states:

$$\mathcal{H}_{\text{fermion}} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where $\mathcal{H}_+$ groups the states of orientation sign $\chi = +1$ and $\mathcal{H}_-$ those of sign $\chi = -1$. This is not an arbitrary decomposition: it is imposed by the different response of the two orientation classes to the effective channels that inherit the sign χ.

Effective chiral reading. In the effective QFT reading, this decomposition can be put in correspondence with a chiral decomposition:

$$\mathcal{H}_{\text{fermion}} = \mathcal{H}_L \oplus \mathcal{H}_R,$$

where the subspaces $\mathcal{H}_L$ and $\mathcal{H}_R$ correspond to the two classes of states distinguished by their internal orientation and by their effective couplings. This identification is an effective reading: it does not postulate an external chiral structure, but translates into the usual language an asymmetry already present in the unitary solutions.

Physical meaning of chirality. Chirality is therefore not introduced here as an external concept grafted onto HSM. It appears as the reading, in the effective quantum language, of an internal asymmetry of rotation. The transition HSM $\to$ QFT is thus made in three steps:

internal orientation $\chi \longrightarrow$ distinguished effective couplings $\longrightarrow$ effective chiral decomposition.

The scope of this reading is that of the sectors where the χ-dependence of the couplings has been established.

Spin projections and effective QFT reading. The HSM framework does not only fix the geometric class of the spin (integer or half-integer), but also provides, depending on the families, an effective projection structure. For the half-integer branches, the projection along the propagation axis or the selected internal axis takes the form

$$S_{\parallel} = \pm\frac{1}{2}\, h_{\text{eff}}.$$

For the purely transverse integer branches, the leading rank carries two helicities

$$S_{\parallel} = \pm h_{\text{eff}},$$

without this generally implying the existence of a complete triplet of projections. The radially pure scalar branch finally satisfies

$$S = 0.$$

These results belong to the internal structure of HSM; QFT provides here their usual effective reading, not their foundation.

Scope of the result. The result established here is an *effective chirality* in the following sense:

1. fermion-like solutions split into two oriented classes;
2. this orientation can govern the sign or the class of certain effective couplings;
3. the state space then decomposes into two physically distinguishable subspaces;
4. this decomposition provides the bridge toward a chiral reading in the QFT language.

What is established is therefore not a vocabulary analogy, but an effective transition structure between HSM and the usual chiral distinction of quantum states.

Conclusion of the subsection. HSM thus admits an effective chiral structure in the quantum domain: the internal sense of rotation of unitary solutions induces, in the sectors where the corresponding couplings are established, a separation of the state space into two classes coupled differently. The left/right distinction is then not added externally to the formalism; it is the effective QFT reading of an orientation structure already contained in the HSM solutions.

The result must nevertheless be limited to what is actually established here: the subsection provides a bridge toward a chiral reading of the reduced sectors, but does not yet derive the complete form of maximally chiral weak couplings of the $V - A$ type. Such an identification would require an additional sectorial closure explicitly relating the HSM orientation projectors to the effective weak couplings.

33.6 Low-energy EFT reading

33.6.1 Coarse-graining, operators, and matching

Purpose of the section. The preceding reductions have shown that HSM admits, in appropriate windows, an effective quantum-mechanical reading, then a local reading in terms of effective quantum fields. It remains to organize these same results in the more synthetic language of a low-energy effective theory. The purpose of this section is therefore to show that, at the scale of long wavelengths and low momenta, HSM can be organized as an EFT in the usual sense: relevant degrees of freedom, truncated local action, hierarchy of operators, coefficient matching, and controlled domain of validity.

Reduction chain. The logic followed is the following:

HSM $\longrightarrow$ coarse-graining / scale separation $\longrightarrow$ local effective action $\longrightarrow$ operator organization $\longrightarrow$ low-energy EFT reading.

The central idea is that, once the fast degrees of freedom have been integrated out or frozen, the remaining slow excitations are described by a finite number of effective fields and by a local action expanded in powers of the fields and of their derivatives. This passage is not an external hypothesis: it corresponds to the natural form taken by the theory once a scale separation is effectively available.

Local action and operator hierarchy. In this regime, the effective action can be organized as a sum of local operators classified by effective dimension or by number of derivatives. At the first rank appear the quadratic terms and the dominant local couplings already identified in the QFT reading. Higher-order contributions then appear as corrections suppressed by an effective cutoff scale Λ_{EFT}. This scale is not universal in principle: it depends on the reduced sector, the integrated modes, and the retained validity window. It is nevertheless bounded by the internal scales of HSM and must not be treated as an independent phenomenological parameter. Thus, the EFT reading does not add new dynamical content: it reorganizes, in a low-energy language, the local structures already obtained by reduction.

Matching. The point specific to an EFT is not only the writing of local operators, but the controlled identification of their coefficients. In the present case, matching consists in expressing these coefficients as functions of the fundamental constants of HSM, of the channel structures already closed, and of pure numbers of internal geometry. In other words, the effective parameters of the EFT are not left floating in principle: at least at leading rank and in the sectors effectively closed, they inherit a determined origin in the underlying theory. The EFT reading of HSM is therefore an EFT with internal anchoring, and not a mere independent phenomenological parameterization.

Covered sectors. This EFT organization applies first to the sectors already readable in the local quantum reduction. The electromagnetic-like abelian sector provides the most directly closed example. The weak-like sector admits a controlled EFT reading within its low-energy scope, where a massive finite-range exchange reduces to an effective local interaction. The strong-like sector can be integrated into this reading in the windows where the closure of its channel is established at the theoretical level, in particular for the binding or flux-tube configurations already constructed. The gravitational-like sector, finally, enters this organization only insofar as a controlled local writing is legitimate in the window considered. In each of these cases, the EFT reading consists in retaining the relevant degrees of freedom, integrating the faster or more massive structures, and then encoding their effects in local and quasi-local operators ordered by importance.

Scope and consistency. The scope of this reading must be stated carefully. An EFT is not a fundamental theory competing with HSM; it is a low-energy rewriting of content already present in HSM. Its validity requires scale separation, a controlled momentum window, and a consistent truncation of the effective action. When these hypotheses are satisfied, the usual EFT tools become available: derivative expansion, sectorial matching, low-energy amplitudes, effective non-Abelian reading when the internal structure lends itself to it, and, when the hypotheses of analyticity, effective unitarity, and scale separation are satisfied, positivity constraints and dispersion relations. The following two sections specify precisely these last two points, which strengthen the demonstrative scope of the EFT reading without extending it beyond its validity window.

Validity window. The domain covered by this reading is the one in which the wavelengths considered are large compared with the micro-scales of the theory, the heavy modes are not explicitly excited, and nonlocal or strongly nonperturbative corrections remain subdominant. In this window, the description by hierarchically organized local operators is legitimate; outside it, one must return either to the more complete QFT reduction, or directly to the HSM structures of which the EFT was only a controlled approximation.

Conclusion of the section. HSM therefore also admits, in the low-energy domain, a reading in terms of a local effective theory. The essential structure is the following: scale separation, coarse-graining, ordered local action, coefficients anchored in the theory, and then explicitly controlled validity domain. Thus, the EFT reading is not an external methodological addition; it constitutes a reduced and organized form of the HSM content in the window where the slow degrees of freedom dominate. The following sections specify two important consequences of this: the appearance, in certain reduced subspaces, of an effective internal algebra compatible with a non-Abelian reading, and then the positivity and dispersion constraints that frame the low-energy operators.

33.6.2 Effective internal algebra and non-Abelian reading

Purpose of the section. The low-energy EFT reading is not limited, in HSM, to abelian sectors or to interactions simply parameterized by isolated local operators. When the relevant internal degrees of freedom reorganize into multiplets stable under the action of closed effective operators, the theory also admits an effective non-Abelian reading. The purpose of this section is to show that such a reading is not added from outside, but can be founded on an internal structure already present in HSM.

Effective internal subspace. Consider an effective subspace $\mathcal{V}_{\mathrm{int}}$ generated by the relevant internal modes of a reduced sector, in particular in the 3DOF configurations where structured

non-equivalent components appear. On this space, the internal operations induced by the coupling kernels and by the channel projectors define effective endomorphisms acting on the retained multiplets. The decisive point is that, in the reduction window where this action is well defined, these endomorphisms can be organized as a finite family of effective generators.

Commutators and closure. Let $\{T^a\}$ be a family of effective generators acting on $\mathcal{V}_{\mathrm{int}}$. Since these objects are operators on an invariant subspace, their commutator

$$[T^a, T^b] := T^a T^b - T^b T^a$$

defines again an endomorphism of the same space. When the chosen family is closed under commutation, one can write

$$[T^a, T^b] = i\, f^{abc}\, T^c, \tag{33.6}$$

where the f^{abc} are effective structure constants. The antisymmetry of the commutator immediately implies

$$f^{abc} = -f^{bac}.$$

Thus, the non-Abelian closure is not postulated: it appears as the algebraic form taken by the effective internal action on the retained subspace.

Jacobi identity. Once the T^a are read as operators on $\mathcal{V}_{\mathrm{int}}$, the Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$$

is automatically satisfied as a general identity of the algebra of endomorphisms. Reported on the structure constants, it gives the usual condition

$$f^{ade} f^{bce} + f^{bde} f^{cae} + f^{cde} f^{abe} = 0. \tag{33.7}$$

In other words, as soon as a closed family of effective generators is extracted from the internal dynamics of HSM, the corresponding Lie structure is acquired at the level of the reduction.

Hermitian reading and effective compactness. In the window where the effective generators are represented by Hermitian operators on the internal space equipped with its effective scalar product, the convention

$$[T^a, T^b] = i\, f^{abc} T^c$$

is the usual convention of compact algebras in a unitary representation. In an effectively isolated simple subsector, the quadratic form

$$\mathcal{K}^{ab} = f^{acd} f^{bcd} \tag{33.8}$$

is then read as the natural metric on the effective generators. This point should be formulated with discipline: what is established here is an *effective compact reading* of the internal algebra in the reduced representation considered; it is not an unqualified assertion of a complete global classification of all possible internal structures of HSM.

Non-Abelian scope. This internal closure is nevertheless sufficient for the proper role of the present chapter. It shows that, when HSM provides an invariant subspace and a closed family of effective generators, the non-Abelian reading of the reduced sectors is founded on a real internal structure: generators, commutators, structure constants, Jacobi identity, and effective compact reading. Thus, the local Yang–Mills forms invoked in the EFT/QFT reduction are not introduced here as an external dressing; they correspond to the rewriting, in the usual language, of an internal organization already contained in the theory.

Level of generality. The result must nevertheless not be overinterpreted. HSM provides here the prototype of an effective non-Abelian reading on the subspaces where generator closure is effectively realized. Depending on the sectors and multiplets considered, this reading may remain minimal or, on the contrary, unfold further. The point established in the present chapter is therefore the *internal availability* of an effective Lie structure compatible with a compact reading, and not the premature statement of a complete classification of all realizable algebras.

In particular, this section does not yet identify the specific algebras of the Standard Model, such as $SU(2)_L$, $SU(3)_c$, or $U(1)_Y$. It establishes only the general mechanism by which an effective Lie algebra can emerge from an invariant internal subspace equipped with generators closed under commutation.

Conclusion of the section. In the low-energy EFT domain, HSM therefore admits an effective non-Abelian reading in the subspaces where the internal degrees of freedom close into a family of generators acting on an invariant subspace. The corresponding Lie structure — commutator, structure constants, Jacobi identity, and effective compact reading — provides the conceptual support needed for the Yang–Mills rewriting of the sectors concerned. This piece completes the non-Abelian side of the EFT reading and naturally prepares the examination of positivity and dispersion constraints.

33.6.3 Positivity, dispersion, and EFT constraints

Purpose of the section. The low-energy EFT reading is complete only if the retained local operators also satisfy the usual consistency constraints: unitarity, effective causality, and positivity of certain combinations of coefficients. The purpose of this section is to show that, in the window where the EFT derived from HSM is legitimate, these constraints are not merely recalled as external conditions, but can be related to the very structure of the reduction.

Forward amplitude and dispersion relation. Consider a $2 \to 2$ scattering amplitude, $\mathcal{M}(s,t)$, in the low-energy regime and in forward scattering ($t = 0$). Under the usual hypotheses of local analyticity and effective unitarity in the band considered, the second derivative of the amplitude at the expansion point satisfies a dispersion relation of the form

$$\mathcal{M}''(0) = \frac{2}{\pi} \int_{s_{\rm th}}^{\infty} ds \, \frac{\Im \mathcal{M}(s,0)}{s^3}, \tag{33.9}$$

where $s_{\rm th}$ denotes the relevant threshold. By the optical theorem, one has

$$\Im \mathcal{M}(s,0) \propto s \, \sigma_{\rm tot}(s) \geq 0,$$

so that the right-hand side is non-negative in the validity window of the effective description.

Consequence for the EFT. At low energy, the forward amplitude is expanded in powers of s. The first contributions directly constrained by the second derivative in the forward limit are those carrying the quadratic behavior in energy, typically the dimension-eight operators in a usual EFT organization. One writes schematically

$$\mathcal{M}(s,0) = \mathcal{M}_{\rm canon}(s) + \frac{s^2}{\Lambda_{\rm EFT}^4} \sum_i \alpha_i \, c_i^{(8)} + \cdots,$$

where the $c_i^{(8)}$ are the effective coefficients of the retained dimension-eight operators, and where the α_i are kinematic factors depending on the choice of channel, polarization, and, where relevant, internal structure. Equation (33.9) then imposes the non-negativity of the combinations effectively probed:

$$\sum_i \alpha_i \, c_i^{(8)} \geq 0. \tag{33.10}$$

In particular channels, this constraint may specialize into simpler bounds on certain coefficients taken separately; this point nevertheless depends on the chosen EFT basis, the retained input states, and the analyticity hypotheses used.

Anchoring in HSM. The point specific to HSM is that this dispersive positivity is not read in a vacuum. In the low-energy reduction, the effective quadratic energy comes from a positive kernel operator on $\mathbb{S}^3$, with the measure and projection windows already fixed by the theory. This positivity of the quadratic sector implies, in the window where integration of the fast modes remains consistent, the absence of ghosts, the non-negativity of the relevant spectral densities, and the convexity of the dominant contributions to the reduced amplitude. Thus, the constraints (33.10) are compatible with the HSM anchoring of the EFT coefficients.

Scope of the result. The scope of the result must be stated precisely. What is established here is an *effective positivity constraint* on the combinations of EFT coefficients accessible in the forward limit and in the window where local analyticity, effective unitarity, and scale separation are valid. It is therefore neither an out-of-domain assertion nor an exhaustive classification of all possible deformations of the EFT; it is an additional consistency criterion imposed on the operators derived from HSM in the low-energy regime.

Conclusion of the section. HSM therefore admits not only a local low-energy EFT reading, but also, in the windows where the dispersion hypotheses are satisfied, part of the associated consistency constraints. In the forward limit and under the validity hypotheses of the reduction, the combination of effective unitarity, local analyticity, and positivity of the quadratic sector inherited from HSM imposes positivity bounds on the combinations of coefficients of the dimension-eight operators probed in the forward limit. This piece strengthens the demonstrative scope of the EFT reading: it is not only a convenient rewriting, but a reduction subject to internal and testable consistency constraints.

Chapter 34

HSM and the relativistic and gravitational domain

34.1 Effective relativistic kinematics

Purpose of the section. After showing that HSM admits, in the appropriate regimes, a reading of the effective quantum domain, it remains to establish that it also admits a reading of the relativistic domain in the kinematic sense. The point here is not to reintroduce relativity as an external postulate, but to show that certain structures already contained in the theory lead, in the appropriate window, to the usual relations between energy, momentum, limiting speed, and effective dynamics of propagating states.

General principle. The intended bridge is the following:

HSM propagating solution $\longrightarrow$ effective dispersion relation $\longrightarrow$ relativistic kinematics.

In this reading, relativity does not appear as an independent axiomatic overlay, but as the effective form taken by the dynamics of excitations when they are described at a scale where the limiting propagation speed, the spectral organization of the modes, and the expression of energy can be read in the usual relativistic language.

Limiting speed and propagation structure. The theory fixes a characteristic propagation speed

$$c_t = \sqrt{T/\rho},$$

which plays the role of the internal limiting speed of excitations in the regime considered. Once the observed reading identifies this propagating sector with the relevant physical regime, this limiting speed provides the kinematic support for the relativistic reduction. The decisive point is that this speed does not result from an external convention: it is already contained in the fundamental structure of HSM.

Energy, momentum, and propagating modes. When propagating solutions or quasi-solutions are described by modes of frequency ω and wave vector k, the theory provides a dispersion relation whose form, in the appropriate effective regime, is read as a relativistic relation between energy and momentum. This reading makes it possible to identify, depending on the sectors and reductions considered, the effective analogs of the relations

$$E^2 = p^2c^2 + m^2c^4, \qquad E = pc \quad \text{or} \quad E \approx mc^2 + \frac{p^2}{2m},$$

not as postulates brought in from outside, but as limiting or reduced forms already contained in the theory.

Explicit realization on moving branches. The effective relativistic kinematics mentioned above is not merely an abstract reading frame. It is already realized explicitly, in the theory, by the moving branches of localized and propagating solutions. In particular, in the quadratic framework on $\mathbb{S}^3$, the boost laws established in the corresponding chapter give

$$E(u) = \Gamma\, E_0, \qquad P(u) = \Gamma\, \frac{u}{c_*^2}\, E_0, \qquad \Gamma = \frac{1}{\sqrt{1 - u^2/c_*^2}}, \tag{34.1}$$

where c_* denotes the effective limiting speed of the sector considered. In the standard propagating reading one takes $c_* = c_t$. In the radial quadratic framework, when the latter is the relevant kinematic sector, one may have $c_* = c_r$. The notation c_* is therefore only a local notation for this section, used to avoid confusing the different kinematic sectors.

Inertial consequence. The low-speed expansion of (34.1) gives

$$E(u) = E_0 + \frac{1}{2}\, \frac{E_0}{c_*^2}\, u^2 + \mathcal{O}(u^4),$$

so that the effective inertial mass of the moving branch is

$$M_{\mathrm{inert}} = \frac{E_0}{c_*^2}. \tag{34.2}$$

In other words, localized HSM solutions do not merely inherit a general relativistic reading at the level of dispersion: their moving branches already realize explicitly the correspondence between rest energy and inertial mass in the appropriate kinematic window.

Reading for HSM solutions. This point will be used implicitly throughout the rest of the discussion whenever a localized or quasi-localized solution is treated as an effective moving entity: the relation between rest energy, momentum, and inertia is not added from outside, but inherited from the boost law specific to the branch considered.

Scope of the relativistic reduction. The result targeted in this section is therefore an *effective relativistic kinematics*. This means that HSM covers, in the window considered, the essential kinematic structures of relativity: limiting speed, energy–momentum relation, distinction between massive and massless sectors, and the usual limiting regimes. This section does not yet treat gravitation proper; it only prepares the ground by establishing that the relativistic language of propagation and dispersion is already available in HSM even before the introduction of the gravitational-like sector.

Organization. We will begin by making explicit the dispersion structure and its kinematic interpretation. We will then examine the reading of the massive and massless sectors, and then the coherence of the nonrelativistic and ultrarelativistic limits. This basis will then make it possible to address, in the following sections, the gravitational-like sector, the Newtonian limit, and the scope of the link with a general-relativity-type reading.

Opening conclusion. The question posed in this part is therefore no longer the existence of an effective quantum framework, but the reading of the relativistic and gravitational domain. The first step consists in showing that HSM excitations already admit, in their effective propagation regime, a coherent relativistic kinematic reading.

34.2 Effective gravitation and Newtonian limit

Purpose of the section. Once the relativistic kinematic reading of HSM excitations has been established, it remains to show that the theory also admits an effective gravitational reading. The issue is not to introduce an additional interaction added from outside, but to show that a gravitational-like channel already constructed in the theory is read, in the appropriate regime, as a universalizing attractive interaction, and that it admits a Newtonian limit in the usual sense.

Reduction chain. The intended bridge is the following:

already established gravitational-like channel $\longrightarrow$ long-range effective potential $\longrightarrow$ Newtonian acceleration

In other words, gravitation is not postulated separately: it appears as the reading, in the usual language, of a mediation structure already contained in HSM and closed on its fundamental constants.

Gravitational-like channel and effective universality. In the theory, the gravitational-like sector is distinguished from the other channels by its extended range and by the fact that its effective source is linked to the energy or energy density of the structures considered. This property is decisive for the gravitational reading: the interaction does not select only a particular sectorial charge, but effectively couples, at the relevant order, to what plays the role of the inertial or energy content of the system. This universalizing character is what founds the bridge to the gravitational language.

Long-range potential. In the static or quasi-static regime, the gravitational-like channel induces a long-range effective potential, whose local form is read, in the appropriate window, as an attractive central potential. At distances large compared with the internal microscales, the structure obtained reduces to a law of the type

$$\Phi_{\mathrm{grav}}(r) \sim -\frac{G_{\mathrm{eff}} M}{r},$$

where G_{eff} denotes the effective gravitational constant resulting from the closure of the gravitational-like channel, and M the effective inertial mass or energy of the source. The important point for the present section is that this form is not an external hypothesis: it results from the long-range reduction of the gravitational-like channel already constructed.

Newtonian limit. Once an effective $1/r$-type potential is established, the dynamics of slow bodies is read, to first order, in the language of Newtonian mechanics. If Φ_{grav} is read as the gravitational potential per unit mass, the acceleration of a test body takes the form

$$\ddot{\mathbf{x}} = -\nabla \Phi_{\mathrm{grav}},$$

or, equivalently,

$$m\,\ddot{\mathbf{x}} = -m\,\nabla \Phi_{\mathrm{grav}}.$$

One thus recovers an inverse-square attraction law between point-reduced sources. Newtonian gravitation then appears as the low-speed, weak-field, long-distance limit of the effective gravitational mediation of HSM.

Inertial and gravitational equivalence. In this reading, the coincidence between inertial mass and gravitational mass does not constitute an independent axiom, but the expression of the fact that the gravitational-like channel couples to the same effective content that already determines the inertial response of the system. Thus, the structure of the potential and the dynamics of the test body are read within a scheme where the source of the channel and the inertial content originate from the same theoretical background. This property is essential for the coherence of the Newtonian limit.

Scope of the result. The result established here is therefore the following: HSM admits an effective gravitation in the long-range regime, and this gravitation possesses a well-defined Newtonian limit. What is demonstrated at this stage is not yet the entire geometric formalism of general relativity, but the availability, in HSM, of a universalizing attractive channel whose static and weak-field reduction reproduces the usual domain of Newtonian gravitation.

Articulation with the rest. This section therefore provides the minimal base of the gravitational link: existence of a gravitational-like channel, long-range potential, Newtonian limit, and inertial coherence. On this basis, the next section can examine whether, and in what sense, this reading extends toward a more general general-relativity-type formulation.

34.3 General-relativity-type reading

Purpose of the section. After establishing the existence of an effective relativistic kinematics and of a gravitational-like sector admitting a Newtonian limit, it remains to examine whether HSM also admits, in an appropriate window, a general-relativity-type reading. The issue is not to paste an external geometric formalism onto the theory, but to show that, at the scale where gravitation is read as an effective deformation of propagation, the structures already constructed in HSM can be reorganized in the language of a dynamic geometry.

General principle. The intended bridge is the following:

gravitational-like channel and effective dynamics $\longrightarrow$ deformation of propagation $\longrightarrow$ effective metric readi

In other words, geometrization is not postulated a priori: it appears when the propagation of excitations and the dynamics of probes can be rewritten as taking place in an effective metric structure depending on the energy content of the system.

Effective metric and propagation. In the regime where the gravitational-like sector acts as a slow deformation of the propagation background, the effect of the channel on trajectories, effective proper times, and dispersion relations can be encoded in an effective metric $g^{\mathrm{eff}}_{\mu\nu}$. This metric is not a fundamental object added to HSM; it is the rewriting, in the usual relativistic language, of a dynamics already contained in the theory. Gravitation then ceases to be read as a simple external force in a fixed space, and becomes the effective manifestation of a deformed propagation geometry.

Weak-field regime. In the weak-field and low-speed limit, the effective metric reading must match the Newtonian reading of the previous section. The minimal weak-field form compatible with this matching is written

$$g^{\mathrm{eff}}_{00} \simeq -\Big(1+\frac{2\Phi}{c^2}\Big), \qquad g^{\mathrm{eff}}_{ij} \simeq \delta_{ij}\Big(1-\frac{2\Phi}{c^2}\Big),$$

or an equivalent form depending on the conventions used, where Φ is the effective gravitational potential derived from the long-range channel.

The status of this writing is that of an effective weak-field matching: it imposes compatibility with the Newtonian limit and with effective relativistic propagation, but does not by itself constitute a complete derivation of Einstein's equations from HSM. It shows that the geometric reading can extend the previous reduction without breaking its coherence.

34.3.1 Strong regime: effective trapping surfaces and regularity

Purpose of the block. The general-relativity-type reading should not be limited to the weak-field regime alone. It is useful to identify what HSM already makes it possible to state when a strong concentration of energy deforms the gravitational-like channel. The upstream result available is not a complete global black-hole-type solution, but a more precise building block: localized HSM sources of finite energy produce a regular response of the gravitational-like channel, and the notion of a trapping surface can be defined kinematically by the closure of the outgoing channels,

$$c_{\mathrm{eff}}^{(+)}\big|_{\Sigma} = 0.$$

The present block therefore gives the effective relativistic reading of this building block, without overstating a complete theory of HSM black holes.

Effective trapping surface. In this reading, a surface is said to be trapping when the outgoing component of effective propagation vanishes. The inner region is said to be trapped when this component becomes directed inward, in the convention where the positive sign denotes outgoing propagation. Such a surface plays the role of a kinematic analog of an apparent horizon: it marks the effective closure of the outgoing propagation channels, but does not, by itself, constitute a singularity of the theory.

Strong regime and status of the effective horizon. When the energy concentration becomes sufficient to strongly modify propagation, the effective metric reading can describe a boundary between an external propagating region and a trapped region. This boundary can be read as an effective horizon. However, in the current state of the closure, three levels must be distinguished:

regular gravitational-like response, effective trapping surface, complete global black-hole-type solutio

The first level is established by the regularity of the gravitational-like channel for finite sources; the second is a natural kinematic definition; the third remains a later development, which would require the complete nonlinear closure of the strong regime.

Regularity and absence of a fundamental singularity. The robust point is the following: in HSM, the underlying mechanical variables of the gravitational-like channel are not replaced by a fundamental metric singularity. A possible divergence of the effective metric description, if it appeared by extension beyond its domain of validity, should therefore not be interpreted as a divergence of the HSM variables themselves. The effective horizon is a propagation surface; the relevant regularity is that of finite HSM sources and of the response of the gravitational-like channel.

Physical reading. In the usual language, this structure allows a regular trapping-zone-type reading: outgoing propagation can be closed by the gravitational-like configuration, without this imposing a fundamental central singularity. This reading provides a bridge toward compact black-hole-type objects, but it should not be confused with a complete global black-hole

solution. The questions of closed effective radius, area law, accretion dynamics, and interior–exterior matching belong to a specific later development.

Scope of the result. The result established at this stage is therefore the following:

1. localized HSM sources of finite energy produce a regular gravitational-like response;
2. an effective trapping surface can be defined by the closure of the outgoing propagation component;
3. this surface admits a relativistic reading as an effective horizon;
4. no fundamental metric singularity is imposed by this reading, since the effective metric is not an independent fundamental variable of HSM.

This result founds a regularized reading of the strong regime, but does not yet constitute a complete black-hole-type solution.

Conclusion of the block. HSM therefore provides a first regular building block for the reading of the strong gravitational regime: finite sources of the gravitational-like channel do not produce a fundamental singularity, and the closure of the outgoing channels defines the notion of an effective trapping surface. In the present chapter, this structure is naturally read as a regular effective horizon. The construction of a complete global black-hole-type HSM solution nevertheless remains a distinct project, requiring a dedicated nonlinear closure of the strong regime.

Energy content and effective equations. In this reading, the energy and inertial content of HSM configurations plays the role of source of the effective geometry. The essential point is not to copy Einstein's equation formally, but to show that the theory admits an organization in which:

1. the effective geometry depends on the energy content;
2. the propagation of probes is read geodesically at the relevant order;
3. the conservation and coherence identities impose a tensorial structure compatible with a general-relativity-type dynamics.

Thus, general relativity is not read here as an independent axiom, but as an effective form of closure between energy, propagation, and geometry.

Scope of the reading. The scope of the result must be formulated precisely. What is targeted here is a *general-relativity-type reading*: HSM covers, in the appropriate regime, the essential structural elements of a relativistic geometric gravitation, namely an effective metric, coupling to energy content, geodesic propagation at the relevant order, and correct matching to the Newtonian limit. This is not to assert at this stage that the whole formalism of general relativity is reconstructed without remainder in all regimes, but rather that its effective geometric core belongs to the domain covered by HSM.

Coherence with the usual observations. Insofar as the effective metric reading is available, the phenomena classically associated with general relativity — trajectory deflection, propagation delays, gravitational shifts, and post-Newtonian corrections — become readable as manifestations of this effective geometry. The present chapter does not have to recalculate all these effects one by one here; its role is to show that the framework needed for their reading already exists in HSM, once the correct relativistic and gravitational reduction is adopted.

Conclusion of the section. HSM therefore covers, beyond Newtonian gravitation alone, a general-relativity-type reading once the gravitational-like sector is reread as an effective deformation of the geometry of propagation. The effective metric, the coupling to energy content, the geodesic reading, and the matching to the weak-field limit provide the minimal structure needed for a geometric relativistic interpretation. In the strong regime, the theory already provides a regularized building block: finite HSM sources do not correspond to fundamental singularities of the gravitational-like channel, and the closure of the outgoing channels makes it possible to define effective trapping surfaces. This reading opens the way to a description of compact objects of the effective horizon type, without yet constituting a complete global black-hole solution. The next section specifies the exact scope of this link, as well as its limits.

34.4 Scope of the gravitational link

Purpose of the section. The previous sections have established that HSM admits an effective relativistic kinematics, a gravitational-like sector admitting a Newtonian limit, and then a general-relativity-type reading in an appropriate window. It remains to specify the exact scope of this link. The purpose of the present section is therefore to distinguish what is effectively demonstrated, what is established in the form of a controlled reduction, and what must not be overinterpreted beyond the domain of validity reached.

Three levels of result. The gravitational link obtained in HSM must be read according to three distinct levels.

1. At the first level, the theory provides a long-range *effective gravitation*, with attractive potential and Newtonian limit.
2. At the second level, it admits an *effective geometric reading*, in which the propagation of probes and excitations reorganizes in the form of an effective metric depending on the energy content.
3. At the third level, it covers, in a controlled window, the *structural core* of a geometric relativistic gravitation.

These three levels must not be confused: they do not correspond to the same degree of generality or to the same degree of demonstration.

What is established. What is effectively established by the previous reduction chain is the following fact: HSM contains a gravitational-like channel whose long-range reduction produces a universalizing attractive interaction, then a Newtonian limit, and finally an effective metric reading coherent with this first level. In this sense, the usual gravitational domain is not merely approximated qualitatively: it is effectively covered in its fundamental low-energy, weak-field, and effective relativistic propagation structures. In the strong regime, the currently available result is more limited: HSM provides regularity of finite sources and a kinematic definition of effective trapping surfaces, but not yet a complete global black-hole-type solution.

What is established under reduction hypotheses. The general-relativity-type reading obtained here nevertheless remains an *effective reading*. It rests on precise hypotheses: scale separation, effective closure of the gravitational-like channel, moderate or controlled field regime, and the possibility of rewriting propagation in the language of an effective metric. For the strong regime, an additional hypothesis would be needed to go beyond the current building block: one would have to close explicitly the relation between energy concentration, the outgoing effective speed $c_{\mathrm{eff}}^{(+)}$, and the existence of a global trapping surface. In other words, what

is demonstrated is not the raw statement of a geometric theory posed independently of HSM, but the possibility, in a well-defined domain, of recovering the language and relevant structures of relativistic gravitation.

What must not be overstated. A stronger claim than the result obtained should therefore not be formulated here. The present part does not aim to declare without remainder that all of general relativity, in all its regimes and all its reformulations, is already reconstructed point by point from HSM without additional work. Rather, it establishes that HSM covers the *relevant relativistic gravitational domain*, with its Newtonian limit, its effective metric reading, and its coherence structures, as long as one remains within the validity window of the reduction. It also authorizes a cautious reading of effective trapping surfaces in the strong regime, but it must not be presented as a complete reconstruction of black-hole physics as long as the corresponding nonlinear closure has not been established.

Consequence for the role of HSM. The meaning of the link is then clear. HSM is not merely compatible with relativistic gravitation; the theory provides an internal effective support for it, from its own fundamental objects. Usual gravitation appears as a derived reading, and not as an independent block juxtaposed with the HSM framework. It is in this sense that one may say that the relativistic gravitational domain belongs to the family of effective readings derived from HSM.

Summary of the relativistic and gravitational domain. The present part has therefore established a chain of increasing scope:

HSM propagation $\longrightarrow$ effective relativistic kinematics $\longrightarrow$ long-range gravitation $\longrightarrow$ Newtonian limit —

To this chain is added, for the strong regime, a more limited but important building block:

finite HSM source $\longrightarrow$ regular gravitational-like response $\longrightarrow$ effective trapping surface.

This chain is sufficient to found the gravitational bridge at the level required by the present chapter: to show that the usual relativistic and gravitational domain is not external to HSM, but constitutes an effective reading of it in the right regime, without overstating a complete closure of black-hole-type objects.

Transition toward the general conclusion. It then becomes possible to conclude the chapter as a whole. The quantum part and the relativistic-gravitational part converge toward the same result: HSM admits, in well-identified regimes, effective readings of the major languages of usual theoretical physics. The general conclusion will gather these results, distinguish what is established from what remains conditional or open, and fix the exact scope of the link constructed in this chapter.

Chapter 35

Conclusion: scope and limits of the link

Purpose of the chapter. The purpose of the present chapter was to show that HSM should not be read as an isolated theory, external to the major languages of usual theoretical physics, but that it admits, on the contrary, in well-identified regimes, effective readings that reproduce their essential structures. The question was therefore not that of a simple formal compatibility, but that of a *demonstrative bridge*: in what sense do quantum mechanics, quantum field theory, low-energy effective theories, and the relativistic and gravitational domain already belong to the readable content of HSM?

Overall result. The answer obtained is positive. In the effective quantum window, HSM admits a Schrödinger-type quantum-mechanical reading, equipped with an effective state space, a probabilistic reading, a semiclassical regime, and a structure of composite systems allowing nonfactorizable states. This reduction is not limited to reproducing the usual formalism: it also shows that certain phenomena that standard quantum mechanics treats mostly at the probabilistic or statistical level — notably on the side of global correlations, nonseparability, and certain forms of phase decoherence — admit, in the theory, a more explicit dynamic and geometric support. The same theory then admits, in a suitable local reduction, a quantum-field-theory-type reading: effective fields, local Lagrangians, effective canonical quantization, propagators, amplitudes, interaction sectors, flavor structures, mixing matrices, and effective chirality. This canonical quantization must be understood as the effective procedure associated with the quadratic reduction and with the modal structure of the sectors considered; by itself, it does not constitute a complete derivation of the theory of measurement or of the general ontological status of quantum operators. Finally, in the low-energy regime, these same structures organize into an EFT endowed with an internal matching, an effective non-Abelian reading, and positivity and dispersion constraints in the windows where their hypotheses are satisfied.

Relativistic and gravitational side. In the relativistic and gravitational window, HSM also admits a relativistic-type kinematics founded on its internal limiting speed, a long-range gravitational-like channel, a Newtonian limit, and then an effective geometric general-relativity-type reading. This reading is not necessarily limited to weak field: it also possesses a first cautious extension toward the strong regime, where finite HSM sources give a regular response of the gravitational-like channel and where the closure of the outgoing channels makes it possible to define effective trapping surfaces. This structure authorizes a regularized effective-horizon-type reading, without yet constituting a complete global black-hole solution. Here again, the link obtained is not a simple parallelism of vocabulary, but an effective reduction: the usual structures of relativistic gravitation appear as a controlled reorganization of the content already

constructed in the theory.

Exact scope of the link. The scope of the present chapter must nevertheless be formulated precisely. What is established here is not, in every case, a complete axiomatic reconstruction of each target theory taken separately in its full generality. The result is both more rigorous and more relevant: HSM effectively covers, in the correct validity windows, the *organizing structures* of these theories. Depending on the case, this link takes the form:

1. of an *explicit reduction* toward a usual equation or formalism;
2. of a *structural coverage* of a known theoretical language;
3. of a *controlled effective reading* in a precise scale regime.

This distinction prevents confusing what is demonstrated as an effective passage with what would be only an external analogy.

What is established in the window of the chapter. At the end of the chapter, the following may be considered established, each at its own level of scope:

- the reduction toward Schrödinger-type effective quantum mechanics;
- the probabilistic reading, wave packets, and Ehrenfest theorem;
- the structure of composite systems and the belonging of entanglement to the quantum domain thus obtained;
- the local QFT reading at the level of effective fields, quantization, propagators, and amplitudes;
- the availability of effective interaction sectors readable as electromagnetic-like, weak-like, strong-like, and gravitational-like sectors;
- the flavor, mixing, and effective chirality structure;
- the low-energy EFT reading, with matching, effective internal algebra, positivity, and dispersion when the corresponding hypotheses are satisfied;
- effective gravitation, the Newtonian limit, and the metric general-relativity-type reading in the appropriate validity window;
- the gravitational-like regularity of finite HSM sources and the kinematic definition of effective trapping surfaces in the strong regime.

What remains effective in nature. Several results of the chapter must nevertheless remain read at their exact level of reduction. This is the case in particular:

- of certain non-Abelian readings, which hold at the effective level of the actually closed internal generators;
- of the confinement-type reading, which rests on the effective formation of flux tubes and on an IR reading of the strong-like sector;
- of the geometric gravitational reading, which holds as a controlled effective reduction and not as a regime-free assertion of all possible forms of general relativity;
- of the reading of trapping surfaces as effective horizons, which must not yet be confused with a complete global HSM black-hole solution.

This precision does not weaken the result: on the contrary, it fixes its true scope.

Limits of the chapter. The purpose of the present chapter was not to rewrite in detail all the internal constructions of the theory or to exhaust each sector in all its possible variants. Its assumed limit is that of a *linking* chapter: it organizes the passages, establishes the reduction chains, specifies the covered domains, and distinguishes the levels of scope. The internal closure proofs, the detailed constructions of the solutions, and the calculations specific to each channel belong to the theory itself; the present chapter only shows in what sense these results fit into the major already recognized frameworks of theoretical physics.

General conclusion. The final result of the chapter can therefore be formulated simply: *the major effective languages of usual theoretical physics are not external to HSM.* Quantum mechanics, quantum field theory, low-energy EFT, effective relativity, and geometric gravitation belong, in their respective domains, to the effective readings derived from the theory. Thus, HSM is not added to existing theories as a juxtaposed competing proposal; it covers them as effective readings derived from its fundamental structure.

General conclusion

What has been established. The manuscript has developed an autonomous mechanical construction on $\mathbb{S}^3$. Its starting point is not the juxtaposition of quantum mechanics, relativity, gravitation, and the usual interactions, but the study of a meshed hyperspherical support carrying tangential degrees of freedom, radial responses, internal rotations, global matchings, and localized solutions.

From this mechanics, HSM constructs families of solutions with two and three degrees of freedom, massive and non-massive solutions, bound states, composite structures, effective interaction channels, and readings toward the usual physical domains. The coherence of the framework rests on a continuous chain:

$$\text{geometry} \rightarrow \text{dynamics} \rightarrow \text{solutions} \rightarrow \text{effective fields} \rightarrow \text{closures} \rightarrow \text{comparisons}.$$

Structural results. The analysis of the solutions reveals a discrete architecture that is not postulated as initial data. The model selects families of solutions, internal degrees of freedom, conical projections, reduced charge patterns, neutral and charged solutions, as well as composite branches. The patterns 1, 2/3, and 1/3 are not introduced as external charges: they appear as reduced field charge coefficients, normalized by the elementary quantum of the EM-like channel and transmitted by the source–field pairing (21.3).

The manuscript also reveals four identifiable interaction channels, in the appropriate regimes, with the gravitational, electromagnetic, weak, and strong sectors. These channels are not added as external forces: they are constructed as effective responses of the same mechanics, with their sources, field charges, ranges, constants, and domains of validity.

The electromagnetic sector plays a particular role. It first closes a dimensionless fine-structure constant,

$$\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}} \simeq 7.285 \times 10^{-3}, \qquad (\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}})^{-1} \simeq 137.3,$$

cf. (30.44), then an elementary quantum of field charge. In the HSM mechanical normalization, this charge has dimension

$$[Q_{\mathrm{EM},*}]_{\mathrm{HSM}} = L^{1/2}T^{1/2},$$

which leads, in the observational landing, to the dimensional reading

$$[\mathrm{C}]_{\mathrm{HSM}} = L^{1/2}T^{1/2},$$

cf. (30.48). After observational reading, the elementary quantum of field charge is

$$Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}} \simeq 1.601 \times 10^{-19}\ \mathrm{C},$$

to be compared with the elementary charge e, with a ratio

$$Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}/e \simeq 0.999,$$

cf. (30.52) and (30.53). This comparison does not consume e in the closure of the charge quantum: $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$ remains a prediction derived from the HSM EM-like channel. In contrast,

after the SI identifications, the numerical agreement with e is not independent of the agreement on $\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}$, since

$$\frac{Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}}{e} = \left(\frac{\alpha_{\mathrm{EM}}^{\mathrm{HSM,LO}}}{\alpha_{\mathrm{fs}}}\right)^{1/2}.$$

It must therefore be read as a consistency check of the complete EM-like chain, and not as a second independent validation of the fine-structure constant.

The weak-like sector provides another structure of its own: the manuscript constructs a channel expression, a charge associated with the relevant branches, a range, and an effective force between sources. This sector is therefore not merely named by analogy; it has an internal mechanical expression arising from HSM solutions and their couplings.

The photon-like sector provides a complementary structural result. It appears as a non-massive transverse mode, with two physical polarizations, compatible at leading order with a Maxwell-type reading. In the same sector, the compactness of $\mathbb{S}^3$ allows two-center global solutions. The manuscript thus formulates a CHSH-type theoretical experiment, based on a single global composite solution rather than on two independent objects.

Set of fundamental constants and global closure. The HSM closure reduces the description of the framework to a restricted number of fundamental mechanical constants:

$$\ell_0, \quad T, \quad \rho, \quad R_0, \quad \kappa_r, \quad \kappa_\mu, \quad \chi.$$

From this set of constants, the manuscript reconstructs the local and global scales of the model: limiting speed, effective action constant, families of solutions, masses, field charges, couplings, non-massive modes, and composite structures.

The important point is that h_{eff} is not introduced as a fundamental constant independent of the support. It appears as an effective action constant calculated in the theory, then identified with the observed action constant in the calibration part.

A central result of this closure is the determination of a global scale of the hypersphere. In the micro-local 1-ring closure, this scale is evaluated as

$$R_0^{\mathrm{HSM}} \simeq 1.50841 \times 10^{29}\ \mathrm{m},$$

cf. (29.34). HSM therefore provides not only a local description of the solutions associated with particles; it relates these solutions to a global scale of the support, from the microscopic mesh to the radius of the hypersphere.

In the reading where HSM describes the physical universe, this quantity must be read as the global radius of the HSM hyperspherical support. It does not, by itself, constitute a direct observational measurement of the size of the visible universe, nor an identification with the Hubble radius. The local solutions, effective constants, and interaction channels are then inscribed in the same hypersphere of radius R_0^{HSM}, subject to the calibration chain and the cosmological correspondence made explicit in the relevant sections.

Quantitative results. The numerical comparison rests on a limited number of calibration inputs, declared in the relevant chains: the identification of h_{eff} with the observed action constant, the identification of c_t with the observed limiting speed, the mass of the W boson, the vacuum impedance, the gravitation constant, and the hierarchy ratios used for the selection of the leptonic levels.

Once these inputs are declared, the closed chains produce outputs comparable to the observed reference values. They include in particular the mass of the solution associated with the electron, the mass of the neutral solution associated with the Z^0, the mass of the radial sector associated with the Higgs boson, the masses of HSM composites identifiable with observed

baryons, the mass splittings of the sector associated with neutrinos, the internal angles of the tri–breather composite, the matching of these angles to the PMNS structure in the correspondence part, the EM-like fine-structure constant, the elementary quantum of EM-like charge, and the properties of the solution associated with the photon.

In several main chains, the central values obtained are close to the observed reference values. Some outputs are at the percent level, while others inherit explicit closure uncertainties, in particular when regularized LO closures are involved. The important result is not only the numerical proximity obtained, but the fact that the masses, ratios, reduced charges, internal angles, correspondence matchings, and compared structures are produced by declared calculation chains, in which the calibration inputs, theoretical outputs, closures used, and comparisons are distinguished.

Charges of solutions and electromagnetic quantum. The theory distinguishes three levels of charge. The first is the internal Noether current carried by a solution. The second is the field charge Q_{field}, obtained by the source–field dictionary of the electromagnetic channel, cf. (21.3). The third is the elementary quantum $Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}$, closed by the electromagnetic channel itself.

The charges of solutions are expressed in native form as

$$Q_{\mathrm{field}}(\mathcal{S}) = q_{\mathrm{eff}}(\mathcal{S})\, Q_{\mathrm{EM},*}^{\mathrm{HSM,LO}}.$$

For the elementary branches, the reduced coefficients give the patterns

$$q_{\mathrm{eff}} = \pm 1, \qquad q_{\mathrm{eff}} = \pm\frac{2}{3}, \qquad q_{\mathrm{eff}} = \pm\frac{1}{3}.$$

The composite branches also carry shape corrections specific to their closure. The numerical values associated with each family are given in the corresponding prediction sections.

Beyond the inventory of the Standard Model. The mechanical catalogue of HSM is broader than the currently identified inventory of the Standard Model. Some families of solutions find a natural correspondence with observed particles or sectors. Others constitute closed theoretical branches of the HSM catalogue: they have masses, reduced charges, and couplings determined by the model, without being associated with an identified observed particle.

This point is important: HSM does not merely associate a known name with each solution. The theory produces a space of admissible solutions, with allowed families, excluded families, spectral levels, and coupling constraints. The correspondences with the Standard Model then appear as one part of the mechanical catalogue, not as its unique starting point.

Quantum, relativistic, and gravitational readings. The last part of the manuscript shows that HSM does not merely produce solutions and masses. In the appropriate windows, the theory admits an effective quantum-mechanical reading: state space, envelope reduction, Schrödinger equation, probabilistic reading, semiclassical regime, composite systems, and non-factorizable states.

It also admits a local reading of the effective field theory type: effective fields, local Lagrangians, propagators, amplitudes, interaction sectors, mass–flavour matchings, effective mixing matrices, and effective chirality. These elements are not postulated as independent axioms; they appear as reduced descriptions of the underlying mechanical dynamics.

On the relativistic and gravitational side, HSM reconstructs a kinematics with an internal limiting speed, a long-range gravitational-like channel, a Newtonian limit, and an effective geometric reading of the general-relativity type. The effective metric, the coupling to the energy content, and geodesic propagation then appear as an effective organization of the mechanical content, and not as separate postulates.

What HSM contributes. In the usual formulations of the Standard Model, many masses and coupling constants appear as experimentally determined parameters. HSM proposes another organization: a small number of inputs fixes the observational landing scales and certain structural thresholds, then the masses, reduced field charges, couplings, and properties of the identifiable solutions are calculated from the internal dynamics of the model.

The contribution of HSM is therefore not only to reproduce certain numbers. It is to propose a single mechanical framework in which the following appear simultaneously: effective inertia of relativistic type, a common action constant, geometric spin, oriented field charges, non-massive modes, bound states, composite structures, interaction channels, effective quantum, relativistic, and gravitational readings, as well as a global scale of the hypersphere.

This unity of construction is the central result of the manuscript. HSM does not start from a primary probabilistic law or a postulated relativistic geometry: in the regimes studied, these structures appear as effective readings of an underlying deterministic mechanical dynamics.

Falsifiability and limits. The model is falsifiable because it is not limited to qualitative vocabulary. It produces masses, ratios, admissible levels, allowed or excluded families, coupling structures, effective constants, reduced field charges, and spectral constraints. An observation incompatible with these outputs, or the evidence of a branch excluded by the closures of the model, would directly contradict the corresponding chains.

This falsifiability rests on traceability. Each claimed result must be relatable to its hypotheses, equations, internal constants, approximation rank, and any calibration inputs used. This is why the manuscript systematically distinguishes calibration inputs, theoretical outputs, predictions comparable to observables, observational readings, and branch-selection diagnostics.

The families that satisfy the closure criteria established in the text are presented with their corresponding outputs. Branches located at the boundary of an admissible domain are identified as boundary diagnostics and are not confused with the predictions associated with internal stationary solutions.

Conclusion. The main result of the manuscript is to establish that classical mechanics on $\mathbb{S}^3$, equipped with an internal meshed structure and radial–tangent couplings, can generate a catalogue of solutions whose several properties recover major structures of known physics.

The decisive point is that these structures are not introduced separately. They appear in a single construction: localized solutions, effective masses, field charges, elementary EM-like quantum, spin, photon-like modes, interaction channels, composites, the tri–breather composite of the light neutral sector, CHSH-type correlations, effective quantum limits, relativistic kinematics, gravitational response, and global scale of the hypersphere.

HSM thus constitutes an autonomous theoretical proposal, controllable line by line and exposed to falsification. Its strength does not depend on an analogy of vocabulary with existing frameworks, but on the stability of its closure chains, the traceability of its quantitative outputs, and its ability to relate, within a single deterministic framework, the local structures of matter and the global geometry of the support.

In an HSM reading, God would not play dice; he would play with a meshed hypersphere.

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